M4A33: Flow in curved pipes: the Dean equations

Consider flow down a slowly curved pipe. In terms of cylindrical polar coordinates $(r,\, \phi,\, z)$ we shall model this as a portion of a torus, $(r-b)^2+z^2=a^2$ where $b \gg a$, and seek solutions independent of ϕ, driven by a pressure gradient in the ϕ-direction.

The velocity $\mathbf{u}=(u_r,\, u_\phi,\, u_z)$ satisfies the Navier Stokes equations

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\partial u_z}{\partial z} &= 0 \\ \rho\left(\frac{Du_r}{Dt}-\frac{u_\phi^2}{r}\right) &= -\frac{\partial p}{\partial r}+\mu\left(\nabla^2 u_r-\frac{u_r}{r^2}\right) \\ \rho\left(\frac{Du_\phi}{Dt}+\frac{u_\phi u_r}{r}\right) &= G(r,z,t)+\mu\left(\nabla^2 u_\phi-\frac{u_\phi}{r^2}\right) \\ \rho\frac{Du_z}{Dt} &= -\frac{\partial p}{\partial z}+\mu\nabla^2 u_z \end{aligned}\right\} \tag{1}$$

where the material derivative $D/Dt=\partial/\partial t+u_r\partial/\partial r+u_z\partial/\partial z$. Here G is the downpipe pressure gradient, $G=-1/r\,\partial p/\partial\phi$. We shall seek steady solutions to these equations. Let us first see if there is a unidirectional solution, as for the straight pipe. If we substitute $u_r=0=u_z$, we find

$$\frac{\partial p}{\partial z}=0 \qquad \text{and} \qquad \frac{\partial p}{\partial r}=\frac{u_\phi^2}{r} \qquad \Longrightarrow \frac{\partial u_\phi}{\partial z}=0\ . \tag{2}$$

So such a solution is only possible if u_ϕ is constant on cylinders. Such a flow would be consistent with a no-slip condition only for flows between concentric cylinders. Any curved pipe-flow cannot be unidirectional.

However, if the pipe is almost straight, we might expect the flow to be almost unidirectional. Now r and z vary over the scale a and we assume

$$b \gg a \qquad \text{so that} \quad r=b+ax^*\simeq b \qquad \text{and} \quad \frac{\partial}{\partial r}\sim\frac{1}{a}\gg\frac{1}{r} \tag{3}$$

We scale $z=az^*$ and let U_0 be a typical scale of u_ϕ. Then we expect a suitable scale for the pressure to be $p\sim\rho U_0^2 a/b$ and if we scale

$$u_r\frac{\partial u_r}{\partial r}\sim u_z\frac{\partial u_r}{\partial z}\sim\frac{u_\phi^2}{r} \qquad \Longrightarrow \qquad u_r\sim u_z\sim U_0\left(\frac{a}{b}\right)^{\frac{1}{2}}\ . \tag{4}$$

We therefore write

$$u_\phi=U_0u_\phi^* \qquad u_{r,z}=U_0\left(\frac{a}{b}\right)^{\frac{1}{2}}u_{x,z}^* \qquad p=\rho U_0^2\left(\frac{a}{b}\right)p^* \tag{5}$$

and neglecting terms of order (a/b), equations (1) become

$$\left.\begin{aligned}
\frac{\partial u_x^*}{\partial x^*} + \frac{\partial u_z^*}{\partial z^*} &= 0 \\
\frac{\rho U_0^2}{b}\left(\frac{Du_x^*}{Dt} - \frac{u_\phi^{*2}}{1}\right) &= -\frac{\rho U_0^2}{b}\frac{\partial p^*}{\partial x^*} + \frac{\mu U_0}{a^2}\left(\frac{a}{b}\right)^{\frac{1}{2}}\nabla^{*2}u_x^* \\
\frac{\rho U_0^2}{(ab)^{1/2}}\frac{Du_\phi^*}{Dt} &= G + \frac{\mu U_0}{a^2}\nabla^{*2}u_\phi^* \\
\frac{\rho U_0^2}{b}\left(\frac{Du_z^*}{Dt}\right) &= \frac{\rho U_0^2}{b}\frac{\partial p^*}{\partial z^*} + \frac{\mu U_0}{a^2}\left(\frac{a}{b}\right)^{\frac{1}{2}}\nabla^{*2}u_z^*
\end{aligned}\right\} \tag{6}$$

We choose the scale U_0 and define a parameter K such that

$$\frac{Ga^2}{\mu U_0} = 1 \qquad \text{and} \quad K = \frac{\rho U_0 a}{\mu}\left(\frac{a}{b}\right)^{\frac{1}{2}} . \tag{7}$$

From now on we drop the * from all the dimensionless variables to obtain the **Dean** equations.

$$\left.\begin{aligned}
\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} &= 0 \\
K\left(\frac{Du_x}{Dt} - u_\phi^2\right) &= -K\frac{\partial p}{\partial x} + \nabla^2 u_x \\
K\frac{Du_\phi}{Dt} &= 1 + \nabla^2 u_\phi \\
K\frac{Du_z}{Dt} &= -K\frac{\partial p}{\partial z} + \nabla^2 u_z
\end{aligned}\right\} . \tag{8}$$

These equations are essentially the two-dimensional Navier-Stokes equations with a body force u_ϕ^2 acting towards the inside of the bend. If we write $\mathbf{u} = (u,\, v,\, w)$ in Cartesian coordinates $(x,\, y,\, z)$, and introduce a stream function, $\psi(x,\, z)$ where $u \equiv u_x = \partial\psi/\partial z$ and $w \equiv u_z = -\partial\psi/\partial x$, and $v(x,\, z) \equiv u_\phi$, then (8) reduce to

$$\left.\begin{aligned}
K(\psi_z v_x - \psi_x v_z) &= 1 + \nabla^2 v \\
K(\psi_z \Omega_x - \psi_x \Omega_z) &= \nabla^2\Omega - 2Kvv_z
\end{aligned}\right\} \tag{9}$$

where $\Omega = -\nabla^2\psi$ is the downpipe vorticity and suffices now denote partial derivatives. These equations are to be solved for $v(x,z)$ and $\psi(x,z)$ subject to the no-slip conditions

$$\nabla\psi = 0, \qquad v = 0 \qquad \text{on the pipe boundary.} \tag{10}$$

There is one parameter in the problem, K, which is known as the Dean number and defined in (7). It is a Reynolds number modified by the pipe curvature, (a/b).