M4A33: Flow in curved pipes: the Dean equations

Consider flow down a slowly curved pipe. In terms of cylindrical polar coordinates 
\((r, \phi, z)\) we shall model this as a portion of a torus, \((r - b)^2 + z^2 = a^2\) where \(b \gg a\), and seek solutions independent of \(\phi\), driven by a pressure gradient in the \(\phi\)-direction.

The velocity \(u = (u_r, u_\phi, u_z)\) satisfies the Navier Stokes equations

\[
\begin{align*}
\frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{\partial u_z}{\partial z} &= 0 \\
\rho \left( \frac{Du_r}{Dt} - \frac{u^2_\phi}{r} \right) &= -\frac{\partial p}{\partial r} + \mu \left( \nabla^2 u_r - \frac{u_r}{r^2} \right) \\
\rho \left( \frac{Du_\phi}{Dt} + \frac{u_\phi u_r}{r} \right) &= G(r, z, t) + \mu \left( \nabla^2 u_\phi - \frac{u_\phi}{r^2} \right) \\
\rho \frac{Du_z}{Dt} &= -\frac{\partial p}{\partial z} + \mu \nabla^2 u_z
\end{align*}
\]

where the material derivative \(D/Dt = \partial/\partial t + u_r \partial/\partial r + u_z \partial/\partial z\). Here \(G\) is the downpipe pressure gradient, \(G = -1/r \partial p/\partial \phi\). We shall seek steady solutions to these equations. Let us first see if there is a unidirectional solution, as for the straight pipe. If we substitute \(u_r = 0 = u_z\), we find

\[
\frac{\partial p}{\partial z} = 0 \quad \text{and} \quad \frac{\partial p}{\partial r} = \frac{u^2_\phi}{r} \quad \Rightarrow \quad \frac{\partial u_\phi}{\partial z} = 0.
\]

So such a solution is only possible if \(u_\phi\) is constant on cylinders. Such a flow would be consistent with a no-slip condition only for flows between concentric cylinders. Any curved pipe-flow cannot be unidirectional.

However, if the pipe is almost straight, we might expect the flow to be almost unidirectional. Now \(r\) and \(z\) vary over the scale \(a\) and we assume

\[
b \gg a \quad \text{so that} \quad r = b + ax^* \simeq b \quad \text{and} \quad \frac{\partial}{\partial r} \sim \frac{1}{a} \gg \frac{1}{r}
\]

We scale \(z = az^*\) and let \(U_0\) be a typical scale of \(u_\phi\). Then we expect a suitable scale for the pressure to be \(p \sim \rho U_0^2 a/b\) and if we scale

\[
u_r \frac{\partial u_r}{\partial r} \sim u_z \frac{\partial u_z}{\partial z} \sim \frac{u^2_\phi}{r} \quad \Rightarrow \quad u_r \sim u_z \sim U_0 \left( \frac{a}{b} \right)^{1/2}.
\]

We therefore write

\[
u_\phi = U_0 u_\phi^* \quad u_{r,z} = U_0 \left( \frac{a}{b} \right)^{1/2} u_{x,z}^* \quad p = \rho U_0^2 \left( \frac{a}{b} \right) p^*
\]
and neglecting terms of order \((a/b)\), equations (1) become

\[
\begin{align*}
\frac{\partial u_x^*}{\partial x^*} + \frac{\partial u_z^*}{\partial z^*} &= 0 \\
\frac{\rho U_0^2}{b} \left( \frac{Du_x^*}{Dt} - \frac{u^2_\phi}{1} \right) &= -\frac{\rho U_0^2}{b} \frac{\partial p^*}{\partial x^*} + \frac{\mu U_0}{a^2} \left( \frac{a}{b} \right)^{1/2} \nabla^2 u_x^* \\
\frac{\rho U_0^2}{(ab)^{1/2}} \frac{Du_\phi^*}{Dt} &= G + \frac{\mu U_0}{a^2} \nabla^2 u_\phi^* \\
\frac{\rho U_0^2}{b} \left( \frac{Du_z^*}{Dt} \right) &= \frac{\rho U_0^2}{b} \frac{\partial p^*}{\partial z^*} + \frac{\mu U_0}{a^2} \left( \frac{a}{b} \right)^{1/2} \nabla^2 u_z^*
\end{align*}
\]

(6)

We choose the scale \(U_0\) and define a parameter \(K\) such that

\[
\frac{Ga^2}{\mu U_0} = 1 \quad \text{and} \quad K = \frac{\rho U_0 a}{\mu \left( \frac{a}{b} \right)^{1/2}}.
\]

(7)

From now on we drop the * from all the dimensionless variables to obtain the **Dean** equations.

\[
\begin{align*}
\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} &= 0 \\
K \left( \frac{Du_x}{Dt} - u^2_\phi \right) &= -K \frac{\partial p}{\partial x} + \nabla^2 u_x \\
K \frac{Du_\phi}{Dt} &= 1 + \nabla^2 u_\phi \\
K \frac{Du_z}{Dt} &= -K \frac{\partial p}{\partial z} + \nabla^2 u_z
\end{align*}
\]

(8)

These equations are essentially the two-dimensional Navier-Stokes equations with a body force \(u^2_\phi\) acting towards the inside of the bend. If we write \(\mathbf{u} = (u, v, w)\) in Cartesian coordinates \((x, y, z)\), and introduce a stream function, \(\psi(x, z)\) where \(u \equiv u_x = \partial \psi / \partial z\) and \(w \equiv u_z = -\partial \psi / \partial x\), and \(v(x, z) \equiv u_\phi\), then (8) reduce to

\[
\begin{align*}
K(\psi_z v_x - \psi_x v_z) &= 1 + \nabla^2 v \\
K(\psi_z \Omega_x - \psi_x \Omega_z) &= \nabla^2 \Omega - 2K vv_z
\end{align*}
\]

(9)

where \(\Omega = -\nabla^2 \psi\) is the downpipe vorticity and suffices now denote partial derivatives. These equations are to be solved for \(v(x, z)\) and \(\psi(x, z)\) subject to the no-slip conditions

\[
\nabla \psi = 0, \quad v = 0 \quad \text{on the pipe boundary.}
\]

(10)

There is one parameter in the problem, \(K\), which is known as the Dean number and defined in (7). It is a Reynolds number modified by the pipe curvature, \((a/b)\).