M3A10 Viscous Flow: Lubrication Theory – Flow in Thin Films

It is an observed fact that thin layers of fluid can prevent solid bodies from contact. The analysis of the fluids flow in thin layers is known as **lubrication theory.**

Consider a solid body with surface $z = h(x, y, t)$ close to a solid plane at $z = 0$. We can regard the gap as 'thin' provided h is small compared to the scale L of variations in the x and y directions. We write $\mathbf{u} = (u, v, w)$, where u and v have typical scales U_0 and w has typical scale W_0. Then the incompressibility condition

$$u_x + v_y + w_z = 0 \quad \Longrightarrow \quad W_0 \sim U_0 h/L \ll U_0 \ .$$

We now consider the x-component of the momentum equation,

$$\rho(u_t + uu_x + vu_y + wu_z) = -p_x + \mu(u_{xx} + u_{yy} + u_{zz})$$

Typical magnitudes of the LHS is $\rho U_0^2/L$, while the viscous term scales as $\mu U_0/h^2$, noting that the z-derivatives dominate. In lubrication theory, we assume the inertia terms are negligible, so that

$$\rho U_0^2/L \ll \mu U_0/h^2 \quad \Longrightarrow \quad R_e(h/L)^2 \ll 1 \qquad \text{where} \quad R_e = \rho U_0 L/\mu \ . \tag{4.4}$$

This suggests a pressure scale $P \sim \mu U_0 L/h^2$. If we only keep the dominant terms, we are left with the lubrication equations

$$\left.\begin{array}{l} p_x = \mu u_{zz} \\ p_y = \mu v_{zz} \\ p_z = 0 \end{array}\right\} \quad \Longrightarrow \quad \begin{cases} p = p(x, y, t) \\ u = \dfrac{p_x}{2\mu}(z^2 + Az + B) \\ v = \dfrac{p_y}{2\mu}(z^2 + Cz + D) \end{cases} \tag{4.5}$$

Imposing the solid body boundary conditions $\mathbf{u} = 0$ on $z = 0$ and $\mathbf{u} = (U_0, V_0, W_0)$ on $z = h$, we have

$$u = \frac{p_x}{2\mu}(z^2 - zh) + \frac{U_0 z}{h} \qquad v = \frac{p_y}{2\mu}(z^2 - zh) + \frac{V_0 z}{h} \ . \tag{4.6}$$

We now impose continuity of mass, by integrating $\nabla \cdot \mathbf{u} = 0$ across the layer. We first observe that

$$\frac{\partial}{\partial x}\int_0^{h(x,y,t)} u(x, y, z, t)\, dz = \int_0^h \frac{\partial u}{\partial x}\, dz + u(x, y, h, t)\frac{\partial h}{\partial x} = \int_0^h u_x\, dz + U_0 h_x \tag{4.7}$$

and a similar result holds for v and y. Thus

$$W_0 = \int_0^h w_z\, dz = -\int_0^h (u_x + v_y)\, dz = -\frac{\partial}{\partial x}\int_0^h u\, dz - \frac{\partial}{\partial y}\int_0^h v\, dz + U_0 h_x + V_0 h_y \ .$$

Substituting (4.6) into (4.7) and evaluating the z-integrals we obtain **Reynolds' Lubrication Equation:**

$$\frac{1}{12\mu}\left(\frac{\partial}{\partial x}(h^3 p_x)+\frac{\partial}{\partial y}(h^3 p_y)\right)=W_0-\tfrac{1}{2}U_0 h_x-\tfrac{1}{2}V_0 h_y \tag{4.8a}$$

We can rewrite this relation using the kinematic boundary condition, namely that on the surface $y=h$

$$0=\frac{D}{Dt}\big(z-h(x,y,t)\big)=-h_t+w-uh_x-vh_y=-h_t+W_0-U_0h_x-V_0h_y$$

to obtain

$$\nabla\cdot(h^3\nabla p)=6\mu\,(h_t+W_0) \tag{4.8b}$$

Hele-Shaw flow: If h is constant and $U_0=V_0=W_0=0$ and (4.8) reduces to $\nabla^2 p=0$ and $(\overline{u},\,\overline{v},\,0)=\nabla p/(12\mu)$, where $\overline{u}$ and $\overline{v}$ are the values of u and v averaged over z. This flow between two close rigid plates is called Hele-Shaw flow. Curiously, this highly viscous flow is the easiest way to achieve two-dimensional potential flow, beloved of inviscid theory.

Slider bearing: Consider a finite plane sliding over a stationary plane, with velocity $(U_0,\,0,\,0)$ so that $h=h_1+\alpha(x-U_0t)$ where $\alpha\ll 1$ and $V_0=W_0=0$. (4.8b) thus becomes

$$(h^3p_x)_x=-6\mu\alpha U_0 \qquad \text{or} \quad (h^3p_h)_h=-6\mu U_0/\alpha\ .$$

Integrating between the two ends of the plane $h=h_1$ and $h=h_2$ say, where we assume the pressure is atmostpheric, $p=p_a$, we find

$$p=\frac{6\mu U_0}{\alpha h}+\frac{A}{h^2}+B=p_a+\frac{6\mu U_0}{\alpha(h_1+h_2)h^2}(h-h_1)(h_2-h)\ .$$

We see that $p>p_a$ for $h_1<h<h_2$ so we expect a force to act separating the planes. We now find the force on the sliding plane. First we note that $||\mu e_{ij}||\sim\mu U_0/h\ll p\sim\mu U_0/(\alpha h)$ Thus the pressure part of the stress dominates. This acts normally to the plate, and the normal to the plane is in the direction of $\nabla(z-h)=(-\alpha,\,0,\,1)$. Thus the x-component of the force is $O(\alpha)$ times the z-component. As the motion is in the x-direction we can interpret this that the drag force is much smaller than the lift force. The total force per unit length in the y-direction is

$$\int(p-p_a)\,dx=\int_{h_1}^{h_2}\frac{(p-p_a)\,dh}{\alpha}=\frac{6\mu U_0}{\alpha^2(h_1+h_2)}\left[-2(h_2-h_1)+(h_1+h_2)\ln\frac{h_2}{h_1}\right]$$

The quantity in square brackets can be shown to be positive for $h_2>h_1$ as it should be. Note that the lift force is very large for small α. It increases with μ and U_0. Physically, fluid is being dragged into the region between the planes, keeping them apart. If U_0 were negative however, then the reverse would apply, and the solid planes would soon come into contact.

Squeeze films – viscous adhesion: Consider now a circular disc a distance $h(t)$ above the plane $z = 0$. Then $U_0 = V_0 = 0$ and $W_0 = h_t$. Using cylindrical polars $(r,\, \theta,\, z)$, (4.8a) becomes

$$\frac{h^3}{r}\left(rp_r\right)_r = 12\mu h_t \quad \Longrightarrow \quad p(r,\,t) = \frac{3\mu h_t}{h^3}(r^2 + A\ln r + B)\ .$$

If the disc includes $r = 0$ then we must have $A = 0$ and imposing $p = p_a$ at $r = a$ we have

$$p - p_a = \frac{3\mu h_t}{h^3}(r^2 - a^2)\ . \tag{4.9a}$$

The total force exerted on the disc in the z-direction is therefore

$$\int_0^a (p - p_a) 2\pi r\, dr = -\frac{3\mu\pi a^4 h_t}{2h^3}$$

The sign indicates that the thin film resists the motion, so that if $h_t > 0$ the force is in the negative z-direction. If a constant force $(0,\, 0,\, F)$ is applied to the disc for $t > 0$ when $h = h_0$, and the plane is held fixed, we can find the time T needed to separate the two.

$$FT = \tfrac{3}{2}\mu\pi a^4 \int_0^T \frac{h_t}{h^3} dt = \tfrac{3}{2}\mu\pi a^4 \int_{h_0}^{\infty} \frac{dh}{h^3} = \frac{3\mu\pi a^4}{4h_0^2}\ .$$

Once more, we see the strong influence of the thin gap $h_0 \ll a$ so that FT is large. Formally, if we try to squeeze the fluid out of the gap an infinite time is required, although in practice surface roughness (or if necessary molecular scales) limit the minimum separation.

Suppose now the disc has a small hole in it at $r = b \ll a$. Equation (4.7) is then

$$p - p_a = \frac{3\mu h_t}{h^3}\left[(r^2 - a^2) + (a^2 - b^2)\frac{\ln(r/a)}{\ln(b/a)}\right]\ . \tag{4.9b}$$

The total force is now multiplied by the approximate factor $(1 - 1/\ln(a/b))$ This can make quite a difference – even if $b = 0.01a$ the necessary force is reduced by a factor 0.78. The difference is that fluid can be sucked in through thw hole and spread out radially.

The journal bearing – one cylinder rotating inside another. A rotating cylinder can be supported inside a slightly larger cylinder by the high lubrication pressures. If the rotating inner cylinder has radius a, and the outer cylinder has radius $a(1 + \varepsilon)$, and the axes of the cylinders are offset by a distance $a\delta$ (where $\delta < \varepsilon$), it is not hard to show that the gap between the two varies with angle around the cylinder as

$$h(\theta) = a\big(\varepsilon - \delta\cos\theta\big) + O(\delta^2) \tag{4.10}$$

Now if $\varepsilon \ll 1$, so that the gap is much smaller than the cylinder radius, we can ignore the curvature terms in the cylindrical equations (see problem sheet 1, Q4.) Essentially, we are arguing that $\frac{\partial}{\partial r} \gg \frac{1}{r}$. That being the case, we obtain the lubrication equations with x

replaced by $a\theta$ and z replaced by $(a-r)$. If the inner cylinder rotates with angular speed Ω then. $U_0 = a\Omega$ and $V_0 = W_0 = 0$. Reynolds' lubrication equation (4.8a) becomes

$$(h^3 p_\theta)_\theta = -6a^2 \mu \Omega h_\theta \ ,$$

which integrates to give

$$p_\theta = -6a^2 \mu \Omega \frac{(h - H_0)}{h^3} \ . \tag{4.10}$$

The positive constant H_0 can be found by imposing periodicity in θ. The necessary integrals can actually be evaluated easily enough but we won't bother here. We note that as θ increases from zero (where h is minimum), the pressure increases to a maximum, and then decreases to a minimum. This large negative pressre can give rise to cavitation of the fluid, as shown in the video. It also leads to a sideways force on the rotor, which can lead to vibration. Nevertheless, rapid rotation can occur very close to a stationary support, with low drag, which is of great practical importance.

Impact of a sphere on a plane: When a sphere of radius a is a distance h_0 from the plane, we can show that at a radial distance r

$$h(r) = h_0 + \tfrac{1}{2}\frac{r^2}{a} \qquad \text{for} \quad r \ll a. \tag{4.11}$$

Once again, taking $U_0 = V_0 = 0$, $W_0 = h_t$, we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r h^3 \frac{\partial p}{\partial r}\right] = 12 \mu W_0 \quad \Longrightarrow \quad \frac{\partial p}{\partial r} = \frac{6\mu W_0 r}{h^3} \tag{4.12}$$

The integration can be done exactly, giving

$$p = p_a - \frac{3\mu a W_0}{h^2} \quad \Longrightarrow \quad \int_0^\infty (p - p_a) 2\pi r \, dr = -\frac{6\mu W_0 a^2}{h} \ .$$

Note we have integrated to $r = \infty$. By this we mean $h_0 \ll$ "∞" $\ll a$!

We can incorporate this is an equation of motion for the sphere, we find that, formally it takes an infinite time to reach the table! In practice, balls can bounce because the large pressures cause the sphere to deform.

Flows of thin films with a free surface: There are many examples of such flows – for example, see Q1 on Problem sheet 4. We have ignored gravity on this sheet, as the pressure generated are usually much larger than hydrostatic pressures. If gravity is the driving force this must clearly be included in equations (4.5).

In conclusion, recall that these viscosity dominated thin layer flows can occur even when the Reynolds number is large $R_e \gg 1$ provided $R_e(h/l)^2 \ll 1$. This might lead us to wonder what happens in layers of thickness h when $R_e(h/l)^2 \simeq 1$? This leads us in the next chapter onto the important topic of **boundary layers.**