**Hydrodynamic Stability:** 2. The Stability of Plane Parallel Flow

(Some of these notes derive from material provided by Dr Robert Hunt.)

**Nondimensionalisation:** Consider a basic 2D flow \((U_*(y_*), 0, 0)\) in the \(x\)-direction in an incompressible inviscid fluid between two plane boundaries \(y_* = y_{1*}\) and \(y_{2*}\). These boundaries may be either rigid (no normal velocity) or free (constant pressure), and either of them may be at infinity. The asterisk denotes a dimensional (physical) quantity; we nondimensionalize using a length \(L\) which is characteristic of the problem (e.g. \(\frac{1}{2}(y_{2*} - y_{1*})\)) and a velocity \(V\) (e.g. \(\text{max}|U_*(y_*)|\)). Then defining

\[
x = \frac{1}{L} x_*, \quad u = \frac{1}{V} u_*, \quad t = \frac{V}{L} t_*, \quad p = \frac{1}{\rho V^2} p_* \quad \text{and} \quad U(y) = \frac{1}{V} U_*(y_*)
\]

we obtain the momentum and continuity equations with \(R_e = \rho V L / \mu\)

\[
\frac{\partial}{\partial t} u + u \cdot \nabla u = -\nabla p + \left[R_e^{-1} \nabla^2 \mathbf{u}\right], \quad \nabla \cdot \mathbf{u} = 0. \tag{2.1}
\]

To begin with we consider the inviscid limit, letting \(R_e \to \infty\). Clearly, \(u = (U(y), 0, 0)\) is then a solution when \(p = p_0\), a constant, for any profile \(U(y)\). This is called a uni-directional or parallel flow. If we include viscosity, however, the only permissible functions \(U(y)\) are quadratic, e.g. Poiseuille \(U = 1 - y^2\) or simple shear \(U = y\).

To analyse the stability, we try a small disturbance

\[
\mathbf{u} = (U, 0, 0) + \varepsilon \mathbf{u}_1, \quad p = p_0 + \varepsilon p_1, \tag{2.2}
\]

where \(\mathbf{u}_1 = (u_1, v_1, 0)\) is the disturbance velocity. (The analysis below can be performed for fully 3D disturbances with \(\mathbf{u}_1 = (u_1, v_1, w_1)\) but it may be reduced to this 2D case using Squires' theorem.) Substituting these expressions into the equations of motion and linearising (i.e., ignoring \(\mathbf{u}_1 \cdot \nabla \mathbf{u}_1\)) we obtain

\[
\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right) \mathbf{u}_1 + \left(v_1 \frac{dU}{dy}, 0, 0\right) = -\nabla p_1, \quad \nabla \cdot \mathbf{u}_1 = 0. \tag{2.3}
\]

Let \(\psi_1\) be the disturbance stream-function such that

\[
u_1 = \frac{\partial \psi_1}{\partial y}, \quad v_1 = -\frac{\partial \psi_1}{\partial x}. \tag{2.4}
\]

We take double Fourier Transforms in \(x\) and \(t\), i.e., we consider Fourier modes

\[
\psi_1 = \tilde{\psi}(k, y, \omega)e^{ikx - i\omega t}, \quad p_1 = \tilde{p}(k, y, \omega)e^{ikx - i\omega t}. \tag{2.5}
\]

The equation of motion then gives

\[
\begin{aligned}
(-i\omega + ik) \frac{d\tilde{\psi}}{dy} - ik \frac{dU}{dy} \tilde{\psi} &= -ik \tilde{p}, \\
-ik(-i\omega + ik) \tilde{\psi} &= -\frac{d\tilde{p}}{dy}.
\end{aligned} \tag{2.6}
\]
which lead to **Rayleigh’s stability equation**

$$
(U - \frac{\omega}{k}) \left( \frac{d^2 \tilde{\psi}}{dy^2} - k^2 \tilde{\psi} \right) - \frac{d^2 U}{dy^2} \tilde{\psi} = 0. \quad (2.7)
$$

Writing \( \psi \) for \( \tilde{\psi} \), Rayleigh’s equation is usually written

$$(U - c)(\psi'' - k^2 \psi) - U'' \psi = 0 \quad (2.8)$$

where \( c = \omega/k \) is the phase speed and \( \psi(y) \) is the “mode shape”. \( c \) is in general complex and we write

$$c = c_r + ic_i. \quad (2.9)$$

For instability we need \( c_i > 0 \).

Rayleigh’s equation must be solved subject to boundary conditions at \( y = y_1 \) and \( y_2 \); in the case of rigid boundaries, \( \psi = 0 \) there. This is an eigenvalue problem and will only have solutions for particular values of \( \omega \) and \( k \), leading to a dispersion relation \( f(k, \omega) = 0 \).

Note that when \( k \) is real, if \( \psi(y) \) is a solution corresponding to \( \omega \) then \( \psi^* \) is also a solution corresponding to \( \omega^* \), where * denotes a complex conjugate.

We have assumed that the velocity profile \( U(y) \) is twice differentiable in deriving (2.9). If this is not the case, for example if \( U(y) \) or \( U'(y) \) is discontinuous at some value, say \( y = y_0 \), we should solve (2.8) separately in \( y < y_0 \) and \( y > y_0 \), and then ensure that the pressure \( \tilde{p} \) is continuous at \( y = y_0 \). From (2.6) we have

$$\tilde{p} = \psi U' - (U - c) \psi' \quad \text{is continuous everywhere.} \quad (2.10)$$

If \( U \) is continuous we must have \( \psi \) continuous at \( y = y_0 \), but if \( U \) is discontinuous we must derive the kinematic condition as we did before. Perturbing the vortex sheet to \( y = y_0 + \varepsilon h_0 e^{ik(x-ct)} \) we find on both sides of the sheet

$$\frac{h_0}{ik} = \frac{\psi}{U - c} \implies \frac{\psi}{U - c} \quad \text{must be continuous everywhere.} \quad (2.11)$$

**Necessary conditions for instability**

As discussed earlier, we can determine whether the flow is unstable by searching real \( k \) for a corresponding wave speed \( c \) with \( c_i > 0 \).

**Rayleigh’s inflection-point theorem:** If the flow is unstable then \( U(y) \) has an inflection point.

**Proof:** Since the flow is unstable, we have for some real \( k \) with \( c_i > 0 \),

$$\psi'' - k^2 \psi - \frac{U''}{U - c} \psi = 0 . \quad (2.12)$$

Multiplying by the complex conjugate \( \psi^* \) and integrating gives

$$\int_{y_1}^{y_2} \left( |\psi'|^2 + k^2 |\psi|^2 + \frac{2U''}{U - c} |\psi|^2 \right) dy = 0 \quad (2.13)$$

after an integration by parts for the first term. Taking imaginary parts,

$$\int_{y_1}^{y_2} \frac{U''}{|U - c|^2} |\psi|^2 dy = 0 \implies c_i \int_{y_1}^{y_2} \left| \frac{\psi}{U - c} \right|^2 dy = 0. \quad (2.14)$$

Hence \( U'' \) must change sign (at least once).
Fjørtoft’s theorem: If the flow is unstable then \( U''(U - U_s) < 0 \) for some value of \( y \) in \((y_1, y_2)\), where \( y_s \) is a point at which \( U''(y_s) = 0 \), and \( U_s = U(y_s) \).

Proof: The real part of (2.13) gives

\[
\int_{y_1}^{y_2} \left( |\psi'|^2 + k^2 |\psi|^2 + U''(U - c_r) \left| \frac{\psi}{U - c} \right|^2 \right) dy = 0. \tag{2.15}
\]

Now \( \int_{y_1}^{y_2} U'' |\psi/(U - c)|^2 dy = 0 \), so adding \( (c_r - U_s) \) times this gives

\[
\int_{y_1}^{y_2} \left( |\psi'|^2 + k^2 |\psi|^2 + U''(U - U_s) \left| \frac{\psi}{U - c} \right|^2 \right) dy = 0 \tag{2.16}
\]

from which the result follows.

Examples:
(i) Here \( U'' < 0 \) everywhere, so the flow is necessarily stable by Rayleigh’s theorem.
(ii) Here \( U'' = 0 \) at \( y_s \); but \( U''(U - U_s) \geq 0 \) everywhere so the flow is stable by Fjørtoft’s theorem.
(iii) Here \( U'' = 0 \) at \( y_s \), but \( U''(U - U_s) \leq 0 \) everywhere. This flow might be unstable.

Note that both theorems give only necessary, not sufficient, conditions for instability.

\( U = \sin y \) can be shown to satisfy the conditions but to be stable if \( y_2 - y_1 < \pi \).

The Viscous Case: The Orr-Sommerfeld equation.

For a viscous fluid subject to a basic flow which is either plane Couette flow \((U = y)\) or plane Poiseuille flow \((U = 1 - y^2)\), an analysis similar to that above leads to the Orr–Sommerfeld equation

\[
(U - c)(\psi''' - k^2 \psi) - U'' \psi = \frac{1}{ikR_e} \left( \psi''' + 2k^2 \psi'' + k^4 \psi \right) \tag{2.17}
\]

where \( R_e = VL/\nu \) is the Reynolds number. It leads to a dispersion relation of the form \( F(k, c; kR_e) = 0 \) which depends on the value of \( R \). We are interested in the values of \( k \) and \( R_e \) for which \( c_1 > 0 \) giving instability. Usually we draw the neutral stability curve, where \( c_1 = 0 \) in the parametric plane \((k, kR_e)\). Typically, there is a minimum value \( R_{c_1} \) of \( R_e \) above which some mode becomes unstable. The theory then predicts instability will set in for \( R_e > R_{c_1} \).

The Orr-Sommerfeld equation can also be used to analyse the stability of boundary layers, for example to predict the behaviour of small disturbances in the Blasius layer on a semi-infinite flat plate. Although the flow in the boundary layer is not exactly parallel, it can be argued that correct results will be obtained by analysing the stability of a unidirectional flow which agrees with the \( U(y) \)-profile at some distance \( x \) along the layer. For the Blasius boundary layer, because of its self-similar structure and a layer thickness which increases as \( x^{1/2} \), the critical Reynolds number can be interpreted as a critical distance along the plate at which instability will commence.
Squires' Theorem:
If a 3D-mode $e^{ikx+ilz}$ becomes unstable at a particular value of $Re$, then there is an equivalent 2D-mode which is unstable at a smaller value of $Re$. It therefore suffices to consider only 2-D modes.

Proof: Consider the 3-D perturbation

$$u = (U(y), 0, 0) + \varepsilon(u(y), v(y), w(y)) e^{ikx+ilz+st}.$$ (2.18)

Then the perturbed Navier-Stokes equations are

$$iku + v' + ilw = 0$$

$$(s + ikU)u + vU' = -ikp + Re^{-1}(u'' - k^2u - l^2u)$$

$$(s + ikU)v = -p' + Re^{-1}(v'' - k^2v - l^2v)$$

$$(s + ikU)w = -ilp + Re^{-1}(w'' - k^2w - l^2w)$$

(2.19)

We now define variables $\bar{u}$ and $\bar{p}$ such that

$$\bar{u} = (ku + lw)/k, \quad \bar{p} = kp/k \quad \text{where} \quad \bar{k}^2 = (k^2 + l^2).$$ (2.20)

Then the first equation becomes $i\bar{k}\bar{u} + v' = 0$ while adding $k$ times the 2nd equation to $l$ times the 4th we obtain

$$(s + ikU)\bar{u} + kvU' = -i\bar{k}^2 p + Re^{-1}(u'' - \bar{k}^2 u)$$

(2.21)

or writing $\bar{R} = kRe/\bar{k}$, and $\bar{s} = sk/k$

$$\bar{\bar{\sigma}} = -i\bar{k}c$$

$$\bar{\bar{\sigma}} + i\bar{k}U)\bar{u} + vU' = -i\bar{\bar{\sigma}}p + \bar{\bar{R}}^{-1}(\bar{u''} - \bar{k}^2 \bar{u})$$

$$\bar{\bar{\sigma}} + i\bar{k}U)v = -\bar{\bar{\sigma}}' + \bar{\bar{R}}^{-1}(\bar{v''} - \bar{k}^2 \bar{v})$$

$$i\bar{k}\bar{u} + v' = 0$$

(2.22)

Comparing (2.22) with (2.19), we see that the 3D problem to find $\bar{\sigma}$ in terms of $\bar{k}$ and $\bar{R}$ is mathematically identical to the 2D problem obtained by setting $l = 0$ in (2.19). Furthermore, if we write $s = -ikc$, then $\bar{s} = -i\bar{k}c$.

Consider first the inviscid problem, setting $(Re^{-1} = 0$. Then if a mode $(k,l)$ is unstable with growth rate $Re(s)$, then the mode $(\bar{k},0)$ must also be unstable with growth rate $Re(\bar{s})$. But as $k \geq k$, the 2D disturbance grows faster.

If $Re$ is finite, then the 3D mode $(k,l)$ may go unstable at some critical Reynolds number $Re = Re_c$, corresponding to $\bar{R} = \bar{Re}_c \equiv kRe_c/\bar{k}$. Then by the mathematical similarity, the 2-D mode $(\bar{k},0)$ will go unstable at $Re = \bar{Re}_c$. But as $\bar{Re}_c \leq Re_c$, an equivalent 2-D mode will go unstable at a lower Reynolds number. We deduce that 2-D modes are the first to go unstable as $Re$ increases, and it is sufficient to consider these only.