Varadhan’s estimates, conditioned diffusions, and local volatilities

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joint work with P. Friz (TU-Berlin)

Workshop LD, Imperial College
Small-noise SDE

\[
\begin{aligned}
\left\{ \begin{array}{l}
dX_t^\varepsilon = b_\varepsilon(X_t^\varepsilon)dt + \varepsilon \sum_{j=1}^{d} \sigma_j(X_t^\varepsilon)dW_t^j, \\
X_0^\varepsilon = x_\varepsilon \in \mathbb{R}^n
\end{array} \right., \\
t \leq T,
\end{aligned}
\]

- $\sigma_j$ and $b_\varepsilon$ vector fields on $\mathbb{R}^n$
- as $\varepsilon \to 0$:
  - $x_\varepsilon \to x_0$
  - $b_\varepsilon \to b_0$ uniformly on compacts
- limit deterministic system: "$\varepsilon dW \to dh = \dot{h} dt$"
Small-noise SDE

\[
\begin{cases}
    dX_t^\varepsilon = b_\varepsilon(X_t^\varepsilon)dt + \varepsilon \sum_{j=1}^{d} \sigma_j(X_t^\varepsilon)dW_t^j, & t \leq T, \\
    X_0^\varepsilon = x_\varepsilon \in \mathbb{R}^n
\end{cases}
\]

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- as \(\varepsilon \to 0\):
  \[X_\varepsilon \to x_0\]
  \[b_\varepsilon \to b_0\] uniformly on compacts

- limit deterministic system: "\(\varepsilon dW \to dh = \dot{h} dt\)"

\[
\begin{cases}
    d\varphi_t^h = b_0(\varphi_t^h)dt + \sum_{j=1}^{d} \sigma_j(\varphi_t^h)dh_t^j \\
    \varphi_0^h = x_0
\end{cases}
\]

where

\[h \in H = \left\{ h \in AC([0, \, T]; \mathbb{R}^d), \dot{h} \in L^2([0, \, T]; \mathbb{R}^d) \right\}; \quad |h|_H^2 = \int_0^T |\dot{h}_t|^2 dt\]
The action function

- The set of controls $h$ leading from $x_0$ to $x$ in time $t$:
  \[ \mathcal{K}_t^x = \mathcal{K}_{x_0,t}^x = \{ h \in H : \varphi_0^h = x_0, \varphi_t^h = x \} \]

  ($x_0$ fixed in this presentation).

- Minimal action needed to reach $x$ from $x_0$ in time $t$:
  \[ \Lambda_t(x) = \inf \left\{ \frac{1}{2} |h|^2_H : h \in \mathcal{K}_{x_0}^x \right\} , \quad \inf \emptyset = +\infty \]

  in other words, an optimal control problem with quadratic cost and affine control system.
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in other words, an optimal control problem with quadratic cost and affine control system.

- Related to the geodesics distance on a Riemannian manifold ($b_0 = 0, \sigma \sigma^*$ invertible):

$$d(x_0, x) = \inf \left\{ \int_0^1 \sqrt{\langle \dot{\gamma}_t, \sigma \sigma^*(\gamma_t)^{-1} \dot{\gamma}_t \rangle} \, dt : \gamma_0 = x_0, \gamma_1 = x \right\}.$$ 

Have:

$$\Lambda_1(x) = \frac{1}{2} d^2(x_0, x)$$
Large Deviation Principle

\[
\begin{align*}
\left\{ \begin{array}{l}
dX_t^\varepsilon &= b_\varepsilon(X_t^\varepsilon)dt + \varepsilon \sum_{j=1}^{d} \sigma_j(X_t^\varepsilon)dW_t^j, \\
X_0^\varepsilon &= x_\varepsilon
\end{array} \right., \quad t \leq T,
\end{align*}
\]

\[b_\varepsilon, (\sigma_j)_{j=1}^{d} \text{ Lipschitz}; \ x_\varepsilon \to x_0; \ b_\varepsilon \to b_0 \text{ uniformly over compacts.}\]

**Large Deviation principle (Friedlin-Wentzell)**

For every \( t > 0 \), the family of random variables \( \{X_t^\varepsilon\}_\varepsilon \) satisfies a Large Deviation Principle (LDP) with speed \( \varepsilon^2 \) and rate function \( \Lambda_t \), i.e.

\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(X_t^\varepsilon \in C) \leq -\Lambda_t(C)
\]

for every closed set \( C \) in \( \mathbb{R}^n \), and

\[
\liminf_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(X_t^\varepsilon \in G) \geq -\Lambda_t(G)
\]

for open set \( G \) in \( \mathbb{R}^n \), with the usual convention \( \Lambda_t(E) = \inf_{x \in \mathbb{R}} \Lambda_t(x) \).
Properties of $\Lambda_t$

1. A good rate function for large deviations: $\Lambda_t$ is lower semi-continuous with compact level sets $\{\Lambda_t \leq a\} \subset \mathbb{R}^n$.

2. Finiteness and continuity: if $\sigma_1, \ldots, \sigma_d$ smooth and satisfy strong Hörmander condition at all points:
   - $\mathcal{K}_t^x \neq \emptyset$ for every $x \in \mathbb{R}^n$ (Chow’s theorem).
   - existence of optimal controls: $h_0 \in \mathcal{K}_t^x$ such that $\frac{1}{2}|h_0|_H^2 = \Lambda_t(x)$.
   - $x \mapsto \Lambda_t(x)$ finite and continuous on $\mathbb{R}^n$. 
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Recall strong Hörmander condition at $x$:

$$\text{Lie}(\sigma_1, \ldots, \sigma_d)_x = \text{span}\{\sigma_1, \ldots, \sigma_d; [\sigma_i, \sigma_j] : 1 \leq i, j \leq d; \ldots\}|_x = \mathbb{R}^n$$

where $[\cdot, \cdot]$ the Lie-bracket $[\sigma_1, \sigma_2]^i(x) = \sigma_1 \cdot \nabla \sigma_2^i(x) - \sigma_1 \cdot \nabla \sigma_2^i(x)$. 
The Malliavin matrix

Assume $b, \sigma_j$ smooth.

- $D\varphi_t^h(x_0)$: the Fréchet differential of $\varphi_t^h(x_0)$ at $h$.

- The deterministic Malliavin matrix associated to $h$:

  $$C_{x_0}^{i,j}(h) = \langle D\varphi_t^h(x_0), D\varphi_t^h(x_0) \rangle_H$$

Bismut's (84) remark:

$$h \mapsto \varphi_t^h(x_0) \text{ is a submersion at } h \iff \text{rk}(D\varphi_t^h(x_0)) = n \iff C_{x_0}(h) \text{ is invertible}$$
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Bismut’s (84) remark:

- $h \mapsto \varphi_t^h(x_0)$ is a submersion at $h$
- $\Longleftrightarrow \text{rk}(D\varphi_t^h(x_0)) = n$
- $\Longleftrightarrow C_{x_0}(h)$ is invertible

Lemma

Assume there exists $s \in [0, t]$ such that

$$\text{span}(\sigma_1, \ldots, \sigma_d)\big|_{x_s = \varphi_s^h(x_0)} = \mathbb{R}^n,$$

then $C_{x_0}(h)$ is invertible. (⇒ local ellipticity).
Minimizing controls $h$

$$\mathcal{K}_t^x = \{ h \in H : \varphi_t^h(x_0) = x_0, \varphi_t^h(x_0) = x \} \Rightarrow \text{a constrained min. problem}$$
Minimizing controls $h$

$$K^x_t = \{ h \in H : \varphi^h_t(x_0) = x_0, \varphi^h_t(x_0) = x \} \implies \text{a constrained min. problem}$$

**Optimality condition (Pontryagin maximum principle)**

- Assume $h_0 \in K^x_t$ an optimal control and $C_{x_0}(h_0)$ invertible,

- Then there exists a unique $\bar{p}_0$ such that $\varphi_{s}^{h_0}(x_0) = x_s, s \leq t$, where $(x_s, p_s)_{s \leq t}$ solves the Hamiltonian ODEs associated to

$$H(x, p) = \langle b_0(x), p \rangle + \frac{1}{2} \sum_{j=1}^{d} \langle \sigma_j(x), p \rangle^2$$

subject to the boundary conditions:

$$x_0 = x_0 \in \mathbb{R}^n, \quad x_t = x \in \mathbb{R}^n, \quad p_0 = \bar{p}_0 \in \mathbb{R}^n.$$

- Moreover $\dot{h}_0^j(s) = \langle \sigma_j(x_s), p_s \rangle$ and $\Lambda_t(x) = \frac{1}{2} |h_0|^2_H$.

- Non-degenerate minimum and infinite dimensional Laplace method: BEN AROUS (88), leads to HKE $\sim$ see J-D. Deuschel, P. Friz talks.
From small-noise to small-time

- Consider

\[ dX_t = b(X_t)dt + \sum_{j=1}^{d} \sigma_j(X_t)dW_t^j, \quad X_0 = x_0 \]

- by Brownian scaling,

\[ X_{\varepsilon^2 T} \sim X_T^\varepsilon, \quad T > 0 \]

\[ dX_t^\varepsilon = \varepsilon^2 b(X_t^\varepsilon)dt + \varepsilon \sum_{j=1}^{d} \sigma_j(X_t^\varepsilon)dW_t^j, \quad X_0^\varepsilon = x_0 \]
From small-noise to small-time

Consider

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\[ dX_t^\varepsilon = \varepsilon^2 b(X_t^\varepsilon)dt + \varepsilon \sum_{j=1}^{d} \sigma_j(X_t^\varepsilon)dW_t^j, \quad X_0^\varepsilon = x_0 \]

\[ \lim_{\varepsilon \to 0} \varepsilon^2 b = b_0 \equiv 0 \implies \text{No drift in the limit system :} \]

\[ d\varphi_t^h = \sum_{j=1}^{d} \sigma_j(\varphi_t^h)dh_t^j, \quad \Lambda_1(x) = \inf \left\{ |h|^2_H : \varphi_0^h = x_0, \varphi_1^h = x \right\} \]

Small-time LDP from small-noise :

\[ \limsup_{t \to 0} t \log P(X_t \in C) \leq -\Lambda_1(C) \]

small-time vs. small-noise asymptotics in finance \( \rightsquigarrow \) S. Violante’s talk
Varadhan’s estimate: small-time

\[ dX_t = b(X_t)dt + \sum_{j=1}^{d} \sigma_j(X_t)dW^j_t, \quad X_0 = x_0. \]

- \( b, \sigma_j \in C_b^\infty \), \( X_t \) has a (smooth) density \( p_t(\cdot) \)
- **Varadhan (67):** if \( \sigma_1, \ldots, \sigma_d \) uniformly elliptic
  \[ \lim_{t \to 0} t \log p_t(x) = -\Lambda_1(x) \]
  with
  \[ \Lambda_1(x) = \inf \left\{ |h|^2_H : h \in \mathcal{K}^x_t \right\}; \quad d\phi_t^h = \sum_{j=1}^{d} \sigma_j(\phi_t^h)dh^j_t. \]

- **Léandre (87, 87):** same asymptotics holds under strong Hörmander condition
  \[ \text{Lie}(\sigma_1, \ldots, \sigma_d)|_x = \mathbb{R}^n, \quad \forall x \in \mathbb{R}^n. \]
Varadhan’s estimate : small-noise

\[ dX_t^\varepsilon = b_\varepsilon(X_t^\varepsilon)dt + \varepsilon \sum_{j=1}^{d} \sigma_j(X_t^\varepsilon)dW_t^i, \quad X_0 = x_0^\varepsilon \]

- \( b, \sigma_j \in C^\infty_b \), strong Hörmander condition at all points, \( b_\varepsilon \to b_0 \) together with derivatives.
- \( p_t^\varepsilon(\cdot) \) the density of \( X_t^\varepsilon \)

**Ben Arous–Léandre** (91) revisited :

\[ \limsup_{\varepsilon \to 0} \varepsilon^2 \log p_t^\varepsilon(x) \leq -\Lambda_t(x) \]

uniformly for \( x \) over compacts, and

\[ \liminf_{\varepsilon \to 0} \varepsilon^2 \log p_t^\varepsilon(x) \geq -\Lambda_{R,t}(x) \]

where now

\[ \Lambda_t(x) = \inf \left\{ |h|_H^2 : K_t^x \right\} \]

\[ \Lambda_{R,t}(x) = \inf \left\{ |h|_H^2 : K_t^x, C_{x_0}(h) \text{ is invertible} \right\} \].

S. De Marco (Ecole Polytechnique)
Stochastic volatility models

Stochastic volatility class \((S_t = S_0 e^{Y_t})\)

\[
dY_t = -\frac{1}{2} Z_t^2 dt + Z_t dB^1_t, \quad Y_0 = 0,
\]

\[
dZ_t = \beta(Z_t) dt + \alpha(Z_t) dB^2_t, \quad Z_0 = z_0 > 0,
\]

and \(B\) a 2-D correlated Brownian motion, i.e. \(B = \sqrt{R} W\), \(R = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\), \(W = (W^1, W^2)\).

- \(\beta, \alpha : \mathbb{R} \to \mathbb{R}\) smooth and Lipschitz functions, with \(\alpha(0) \neq 0\)

Note

\[
\sigma_1(z) = \begin{pmatrix} z \sqrt{R_{11}} \\ \alpha(z) \sqrt{R_{21}} \end{pmatrix}; \quad \sigma_2(z) = \begin{pmatrix} z \sqrt{R_{12}} \\ \alpha(z) \sqrt{R_{22}} \end{pmatrix}.
\]

\(\{z \neq 0\}\) elliptic region.

\(\{z = 0\}\) sub-elliptic set.
Local volatilities as conditional expectations

- Well-know result of Gyongy ('86) : given the stochastic volatility model \((Y_t, Z_t)_t\), a 1-D diffusion

\[
d\tilde{Y}_t = -\frac{1}{2}\sigma^2_{loc}(\tilde{Y}_t, t)dt + \sigma_{loc}(\tilde{Y}_t, t)dW_t
\]

with \(\tilde{Y}_t \sim Y_t\) (same fixed-time laws) is generated by

\[
\sigma^2_{loc}(y, t) = \mathbb{E}[Z_t^2|Y_t = y]
\]
Local volatilities as conditional expectations

- Well-know result of Gyongy ('86): given the stochastic volatility model $(Y_t, Z_t)_t$, a 1-D diffusion

$$d\tilde{Y}_t = -\frac{1}{2} \sigma_{\text{loc}}^2(\tilde{Y}_t, t) dt + \sigma_{\text{loc}}(\tilde{Y}_t, t) dW_t$$

with $\tilde{Y}_t \sim Y_t$ (same fixed-time laws) is generated by

$$\sigma_{\text{loc}}^2(y, t) = \mathbb{E}[Z_t^2 | Y_t = y]$$

- Quantification of model risk:
  - Different laws of paths of $Y$ and $\tilde{Y} \Rightarrow$ different prices of exotic options.
  - Comparison of prices under SV and LV $\leadsto$ indicators of volatility modeling risk attached to the option. (Reghai (11), Cont (06), ...
Numerical issues

- Numerical evaluation of $\sigma_{\text{loc}}^2(y, t) = \mathbb{E}[Z^2_t | Y_t = y]$ typically unstable, in particular when $|y|$ is large.

- Implementation for the Heston model (see S. Gerhold’s talk):

<table>
<thead>
<tr>
<th>Maturity</th>
<th>$\log(S_t/S_0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>min</td>
</tr>
<tr>
<td>0.5</td>
<td>$-1.10$</td>
</tr>
<tr>
<td>1</td>
<td>$-1.52$</td>
</tr>
<tr>
<td>5</td>
<td>$-4.13$</td>
</tr>
</tbody>
</table>

- Asymptotic formulae: replace the numerical evaluation of $\sigma_{\text{loc}}^2(y, t)$ when $|y|$ becomes large ($\sim$ at least provide a benchmark!)
The conditioned diffusion

- \( X_t^\varepsilon = (Y_t^\varepsilon, Z_t^\varepsilon) \) given by

\[
dX_t^\varepsilon = b_\varepsilon(X_t^\varepsilon)dt + \varepsilon \sum_{j=1}^{d} \sigma_j(X_t^\varepsilon)dW_t^j, \quad X_0 = x_0^\varepsilon
\]

- \( p_t^\varepsilon \) the density pf \( X_t^\varepsilon \), \( f_t^\varepsilon \) the density pf \( Y_t^\varepsilon \).
The conditioned diffusion

- \( X_t^\varepsilon = (Y_t^\varepsilon, Z_t^\varepsilon) \) given by

\[
dX_t^\varepsilon = b_\varepsilon(X_t^\varepsilon)dt + \varepsilon \sum_{j=1}^{d} \sigma_j(X_t^\varepsilon)dW_t^j, \quad X_0 = x_0^\varepsilon
\]

- \( p_t^\varepsilon \) the density pf \( X_t^\varepsilon \), \( f_t^\varepsilon \) the density pf \( Y_t^\varepsilon \).

- Want to study the conditional law

\[
\mathcal{L}(Z_t^\varepsilon|Y_t^\varepsilon = y) \quad \text{as } \varepsilon \downarrow 0
\]

- study the asymptotics of the conditional density

\[
\mathbb{P}(Z_t^\varepsilon \in dz|Y_t^\varepsilon = y) = \frac{p_t^\varepsilon(y, z)}{f_t^\varepsilon(y)} dz
\]
Convergence of the conditional law

\[ dX_t^\varepsilon = b_\varepsilon(X_t^\varepsilon)dt + \varepsilon \sum_{j=1}^{d} \sigma_j(X_t^\varepsilon)dW_t^j, \quad X_0 = x_0^\varepsilon \]

**Theorem**

Assume \( b_\varepsilon, (\sigma_j)_{j=1}^d \) smooth and bounded, bounded derivatives; \( x_0^\varepsilon \to x_0 \) and \( b_\varepsilon \to b_0 \) (with the derivatives), and strong Hörmander condition at all points. Fix \( t > 0 \) and \( y \in \mathbb{R} \) and assume

(i) there is a unique point minimizing the distance of \( x_0 \) to the affine subspace \( N_y := (y, \cdot) \)

\[ z_t^*(y) := \text{argmin}_z \Lambda_t(y, z) \]

(ii) \( C_{x_0}(h) \) is invertible for all \( h \in \mathcal{K}_t^{(y, z)} \), \( z \) in a nbhood of \( z^* \).

Then

\[ \mathcal{L}(Z_t^\varepsilon | Y_t^\varepsilon = y) \Rightarrow \delta_{z_t^*(y)} \quad \text{as} \ \varepsilon \downarrow 0 \]

i.e.

\[ \mathbb{E}[\phi(Z_t^\varepsilon) | Y_t^\varepsilon = y] \Rightarrow \phi(z_t^*(y)) \quad \text{as} \ \varepsilon \downarrow 0. \]
Extensions

- **φ with polynomial growth**: Assume φ is continuous and
  \[ φ(z) \leq C(1 + |z|^k), \] some \( C > 0 \) and \( k \in \mathbb{N} \), then

  \[ E \left[ φ \left( Z^ε_t \right) | Y^ε_t = y \right] \rightarrow φ \left( z^*_t(y) \right) \quad \text{as } ε \downarrow 0. \]

- **Extension to finitely many argmin’s**: Assume there exist finitely many
global minimizers \( z^{*,i} \) for \( Λ_t(y, ·) \), and \( C_{x_0}(h) \) is invertible for all \( h \in K_t^{(y, z)} \)
and \( z \) in a neighborhood of \( z^{*,i} \), for every \( i \). Then

  \[ E \left[ φ \left( Z^ε_t \right) | Y^ε_t = y \right] \rightarrow \sum_i α_i φ(z^{*,i}) \quad \text{as } ε \downarrow 0 \]

  some \( (α_i)_i \) with \( α_i \geq 0 \) and \( \sum_i α_i = 1 \).
Asymptotics of local volatilities

\[
\begin{align*}
\ dY_t^\varepsilon &= -\frac{1}{2}\varepsilon^\theta (Z_t^\varepsilon)^2 \ dt + \varepsilon Z_t^\varepsilon dB_1^t, \quad Y_0^\varepsilon = 0, \\
\ dZ_t^\varepsilon &= \beta_\varepsilon (Z_t^\varepsilon) \ dt + \varepsilon \alpha (Z_t^\varepsilon) dB_2^t, \quad Z_0^\varepsilon = z_0^\varepsilon.
\end{align*}
\]

\[\Box \quad \beta_\varepsilon, \beta_0, \alpha : \mathbb{R} \to \mathbb{R} \text{ are smooth and Lipschitz functions, with } \alpha(0) \neq 0;\]

\[\Box \quad z_0^\varepsilon \to z_0, \beta_\varepsilon \to \beta_0\]

### Theorem

As in previous theorem, assume there exists a unique \( z^* = z_t^*(y) \) minimizing \( \Lambda_t(y, \cdot) \), and invertibility of \( C_{x_0}(h) \) for all \( h \in \mathcal{K}^{(y,z)}_t \), \( z \) in a neighborhood of \( z^* \). Then for all \( \phi \in C (\mathbb{R}^{n-1}) \) with polynomial growth,

\[
E[\phi(Z_t^\varepsilon) | Y_t^\varepsilon = y] \to \phi (z_t^*(y)) \quad \text{as } \varepsilon \to 0.
\]

In particular:

\[
\sigma_{loc}^\varepsilon(t, y) = E[(Z_t^\varepsilon)^2 | Y_t^\varepsilon = y] \to (z_t^*(y))^2 \quad \text{as } \varepsilon \to 0.
\]
Elements of proof (main theorem)

Main ingredients:

(a) The uniform upper bound

\[ \limsup_{\varepsilon \to 0} \varepsilon^2 \log p_t^\varepsilon(y, z) \leq -\Lambda_t(y, z) \]

for \( z \) over compact sets of \( \mathbb{R}^{n-1} \)

(b) The pointwise lower bound for the density of \( Y_t^\varepsilon \)

\[ \liminf_{\varepsilon \to 0} \varepsilon^2 \log f_t^\varepsilon(y) \geq -\Lambda_t(y, z^*) \]

(c) Tail estimates:

Lemma

For every \( q \in (0, 1) \), there exists \( N(q) \in \mathbb{N} \) and a constant \( C_{q,t} > 0 \) such that, for every \( R < 1 \) and every \( x \in \mathbb{R}^n \)

\[ p_t^\varepsilon(x) \leq C_{q,t}(1 + R^{-N(q)}) \varepsilon^{-N(q)} \mathbb{P}(|X_t^\varepsilon - x| \leq R)^q. \]
Asymptotics of local volatility: small-time

\[ dY_t = -\frac{1}{2} Z_t^2 dt + Z_t dB_t^1, \quad Y_0 = 0, \]
\[ dZ_t = \beta(Z_t) dt + \alpha(Z_t) dB_t^2, \quad Z_0 = z_0 > 0, \]

\( B \) a correlated Brownian motion, \( \alpha(z) \neq 0 \).

- Time scaling: \((Y_{\varepsilon^2}, Z_{\varepsilon^2}) \sim (Y_1, Z_1)\), where \((Y_1, Z_1)\) solves

\[ dY_1 = -\frac{1}{2} \varepsilon^2 (Z_1^\varepsilon)^2 dt + \varepsilon Z_1^\varepsilon dB_t^1, \quad Y_0^\varepsilon = 0, \]
\[ dZ_1 = \varepsilon^2 \beta(Z_1^\varepsilon) dt + \varepsilon \alpha(Z_1^\varepsilon) dB_t^2, \quad Z_0^\varepsilon = z_0 \]
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- Time scaling : \((Y_{\varepsilon}, Z_{\varepsilon}) \sim (Y_{1\varepsilon}, Z_{1\varepsilon})\), where \((Y_{\varepsilon}, Z_{\varepsilon})\) solves

\[ dY_{\varepsilon} = -\frac{1}{2} \varepsilon^2 (Z_{\varepsilon})^2 dt + \varepsilon Z_{\varepsilon} dB_{t\varepsilon}^1, \quad Y_{0\varepsilon} = 0, \]
\[ dZ_{\varepsilon} = \varepsilon^2 \beta(Z_{\varepsilon}) dt + \varepsilon \alpha(Z_{\varepsilon}) dB_{t\varepsilon}^2, \quad Z_{0\varepsilon} = z_0 \]

- Starting point \( x_0 = (0, z_0) \rightarrow \) elliptic point \( \rightarrow C_{x_0}(h) \) invertible for all \( h \)!

Theorem applies : \[
\lim_{t \to 0} \sigma_{loc}(t, y)^2 = \lim_{\varepsilon \to 0} \sigma_{loc}^\varepsilon(1, y)^2 = z_1^*(y)^2
\]

\( \sim \) Berestycki, Busca and Florent asymptotics of ‘efficient volatility’ for hypoelliptic diffusions.
Asymptotics of local volatility: extreme-strike

- Take the Stein-Stein (91) model

\[
\begin{align*}
    dY_t &= -\frac{1}{2}Z_t^2 dt + Z_t dB^1_t, \quad Y_0 = 0, \\
    dZ_t &= (a + bZ_t)dt + c dB^2_t, \quad Z_0 = z_0 > 0,
\end{align*}
\]

large-strike from small-noise (see also P. Friz's talk): \( Y_t^\epsilon = \epsilon^2 Y_t, Z_t^\epsilon = \epsilon Z_t \)
solves

\[
\begin{align*}
    dY_t^\epsilon &= -\frac{1}{2}(Z_t^\epsilon)^2 dt + \epsilon Z_t^\epsilon dW^1_t, \quad Y_0^\epsilon = 0, \\
    dZ_t^\epsilon &= (a\epsilon + bZ_t^\epsilon)dt + \epsilon c dW^2_t, \quad Z_0^\epsilon = \epsilon z_0.
\end{align*}
\]

Note that now \((Y_0^\epsilon, Z_0^\epsilon) \to (0, 0)\) (hypoelliptic point) as \(\epsilon \to 0\).
Asymptotics of local volatility: extreme-strike

- Take the Stein-Stein (91) model

$$dY_t = -\frac{1}{2}Z_t^2 dt + Z_t dB_t^1, \quad Y_0 = 0,$$

$$dZ_t = (a + bZ_t) dt + c dB_t^2, \quad Z_0 = z_0 > 0,$$

large-strike from small-noise (see also P. Friz’s talk): $Y_t^\varepsilon = \varepsilon^2 Y_t, Z_t^\varepsilon = \varepsilon Z_t$ solves

$$dY_t^\varepsilon = -\frac{1}{2} (Z_t^\varepsilon)^2 dt + \varepsilon Z_t^\varepsilon dW_t^1, \quad Y_0^\varepsilon = 0,$$

$$dZ_t^\varepsilon = (a\varepsilon + bZ_t^\varepsilon) dt + \varepsilon c dW_t^2, \quad Z_0^\varepsilon = \varepsilon z_0.$$

Note that now $(Y_0^\varepsilon, Z_0^\varepsilon) \to (0, 0)$ (hypoelliptic point) as $\varepsilon \to 0$.

- Can relate the behavior of $\sigma_{loc}(t, y), y \to \infty$, to $\sigma_{loc}^\varepsilon(t, 1), \varepsilon \to 0$:

$$\lim_{y \to \pm \infty} \frac{\sigma_{loc}^2(t, y)}{y} = \lim_{\varepsilon \to 0} \mathbb{E} \left[ (Z_t^\varepsilon)^2 \mid Y_t^\varepsilon = \pm 1 \right] = z_t^* (\pm 1)^2$$

(arrival subspace is $(\pm 1, \cdot)$: application of theorem is justified.)
Minimizing controls $h$

**Optimality condition (point-to-subspace)**

- Assume $h_0 \in \mathcal{K}_{x_0}^{(y, \cdot)}$ is a minimizing control to reach the affine subspace $N_Y = (y, \cdot)$ and $C_{x_0}(h_0)$ is invertible.

- Then, there exists a unique $\bar{p}_0$ such that $\phi^{h_0}_{s}(x_0) = x_s$, $s \leq t$ where $(x_s, p_s)_{s \leq t}$ solves the Hamiltonian ODEs subject to initial and terminal conditions:
  
  $$
  x_0 = x_0, \quad x_t = (y, \cdot) \in \mathbb{R}^l \times \mathbb{R}^{n-l} \\
  p_0 = \bar{p}_0, \quad p_t = (\cdot, 0) \in \mathbb{R}^l \times \mathbb{R}^{n-l}
  $$

- Moreover
  
  $$
  \dot{h}_0^j(s) = \langle \sigma_j(x_s), p_s \rangle, \quad \Lambda_t(N_Y) = \frac{1}{2} |h_0|^2.
  $$

  If terminal point is $x_t = (y, z_t) \in \mathbb{R}^l \times \mathbb{R}^{n-l}$, then
  
  $z_t$ minimizes $\Lambda_t(y, \cdot)$. 

An explicit solution of the Hamiltonian system

- **STEIN-STEIN** model (parameters $a, b, c, \rho$):
- Hamiltonian ODEs under the boundary conditions (see Deuschel, Friz, Jacquier, Violante (11), to appear)
  \[ x_0 = (0, 0); \quad x_t = (y, \cdot); \quad y \neq 0, \quad p_t = (\cdot, 0) \]

yield:

\[ z_t^{\pm}(y) = \pm \frac{q(y)c^2}{\chi_p} \sin(\chi_p t) \]

where

\[ \chi_p = \sqrt{c^2 p(p - 1) - \tilde{b}^2}; \quad \tilde{b} = b + \rho cp; \]

\[ q(y) = \frac{2}{c} \left[ \frac{2r_1^3 y}{t^3 \left( (c^2(2p - 1) - 2\rho c\tilde{b})(2r_1 - \sin(2r_1)) + 2\rho cr_1(1 - \cos(2r_1)) \right)} \right]^{1/2}; \]

$r_1$ the smaller root on $\mathbb{R}_+$ of

\[ r \cot r = (b + \rho cp(r))t, \]

\[ p(r) = \frac{1}{2(1 - \rho^2)} \left[ \left( 1 + 2\rho \frac{b}{c} \right) + \text{sign}(y) \sqrt{\left( 1 + 2\rho \frac{b}{c} \right)^2 + 4(1 - \rho^2) \left[ \frac{b^2}{c^2} + \frac{r^2}{c^2 t^2} \right]} \right]; \]

and $p = p(r_1)$. 

S. De Marco (Ecole Polytechnique)
**Convergence of** $y \mapsto \frac{\sigma^2_{\text{loc}}(y,t)}{y}$ $(y < 0)$

Blue line : $y \mapsto \frac{\sigma^2_{\text{loc}}(y,t)}{y}$. Red line : value of $(z \ast (-1))^2$.
parameters : $a = 0.0, b = -1.0, c = 1.0, \rho = -0.750$
Convergence of $y \mapsto \frac{\sigma^2_{loc}(y,t)}{y}$ ($y > 0$)

Blue line: $y \mapsto \frac{\sigma^2_{loc}(y,t)}{y}$. Red line: value of $(z^*(1))^2$.

parameters: $a = 0.0$, $b = -1.0$, $c = 1.0$, $\rho = -0.750$
Comparison with small-time Heat Kernel Expansion

- Small-time HK expansion:

\[ p_t(x) = \frac{1}{2\pi t} e^{-\frac{\Lambda(x)}{t}} (c_0(x) + O(t; x)), \quad \text{as } t \downarrow 0 \]

available under some additional conditions on \( x \) (\( \sim \) see P. Friz’s talk)
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- Application of Laplace method (cf. Henry-Labordère (08), Chap. 6):
  \[ \sigma_{loc}(t, y)^2 = \mathbb{E} \left[ Z_t^2 | Y_t = y \right] = \frac{\int z^2 p_t(y, z) dz}{\int p_t(y, z) dz} \]
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  \]
  \[
  = \int e^{-\frac{\Lambda(y, z)}{t}} (c_0(y, z) + \ldots) dz
  \]
  \[
  \sim (z^*)^2 e^{-\frac{\Lambda(y, z^*)}{t}} \sqrt{2\pi t} \left( \partial_{zz} \Lambda(y, z^*) \right)^{-1/2}
  \]
  \[
  = (z^*)^2.
  \]
  if \( z \mapsto \Lambda_t(y, z) \) has a unique minimizer \( z^* \), and \( \partial_{zz} \Lambda(y, z^*) > 0 \).

- Message of the previous theorem is: asymptotics of \( \log p_t \) is enough to establish leading order term.
Summary and conclusion

- Convergence of the law of hypoellitic diffusions (strong Hormander) conditioned to be in an affine subspace at final time.

- Application in math. finance: unified framework for small-time and (in some cases) large-strike asymptotics of local volatility functions.

- Solution of the Hamiltonian ODEs giving the minimizer $z^*$ can be performed:
  - explicitly in some cases
  - numerically $\rightarrow$ typically more stable than Monte-Carlo or Fourier inversion to compute the conditional expectation in asymptotic regimes.

Thank you!