Consistent yield curve modelling

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joint work with

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Challenge:

- Consistent recalibration of model parameters.

Classical approach: affine factor models for the short rate

- Main example: Vasiček model

\[ dr(t) = (b + \beta r(t)) dt + \sigma dW(t). \]

- More generally, affine multi-factor models, possibly with jumps.
Calibration to initial yield curves
Calibration to initial yield curves

Problem

• Homogeneous models: No exact fit to market yield curves.
• Therefore, inconsistency between model and market.

Solution

• Use Hull-White extensions.
• Obtain reformulations as HJM models.
Calibration to initial yield curves

Zero-coupon yields (%)

Y(0, τ)

Market

Model (b, β, σ)

Model (b̃, β, σ)

Calibration of Hull-White extensions to initial yield curves.
(Vasiček model)
Hull-White extensions

Time-dependent drift (%)

Constant drift $b$ versus time-dependent drift $\tilde{b}(t)$.
Factor models as HJM models

HJM equation

- In Vasiček models with fixed parameters \((\beta, \sigma)\), forward rates satisfy

\[
df_t = \left( \frac{\partial}{\partial \tau} f_t + \mu^{\text{HJM}}_{\beta,\sigma} \right) dt + \sigma^{\text{HJM}}_{\beta,\sigma} dW_t,
\]

where each \(f_t\) is a curve of forward rates indexed by \(\tau\).

Properties

- Finite-dimensional realisation of the HJM equation.
- Easy to simulate.
- Calibration reduces to an estimation problem because of the analytical formulas for bond prices.
Time-varying parameters
Iterative recalibration of factor models

Volatility parameter ($\%$)

Time series of calibrated Vasiček volatilities $\sigma$
(AAA rated Euro area government bonds)
Iterative recalibration of factor models

Drift parameter

Time series of calibrated Vasiček speeds of mean reversion $-\beta$
(AAA rated Euro area government bonds)
Time-varying parameters

Motivation

- Iterative recalibration results in time series of model parameters.
- The model should anticipate that parameters are subject to change.

Problem

- Introducing stochastic parameters in affine factor models destroys their good properties.

Solution (Consistent Recalibration Models)

- Make HJM parameters stochastic, but stick to HJM volatilities coming from affine factor models.
Consistent Recalibration Models
Definition

- In consistently recalibrated Vasiček models, forward rates satisfy

\[ df_t = \left( \frac{\partial}{\partial\tau} f_t + \mu_{\beta_t,\sigma_t}^{\text{HJM}} \right) dt + \sigma_{\beta_t,\sigma_t}^{\text{HJM}} dW_t. \]

- Here, \( \mu_{\beta,\sigma}^{\text{HJM}}, \sigma_{\beta,\sigma}^{\text{HJM}} \) denote the HJM drift and volatility of the Vasiček model with parameters \( \beta, \sigma \).

- \( \beta_t, \sigma_t \) are stochastic processes.
- Assume that $(\beta_t, \sigma_t)$ is piecewise constant.
- Fix $r_0$ and parameters $\tilde{b}_0, \beta_0, \sigma_0$ calibrated to the market.
- Simulate $r_1$ starting from $r_0$ using these parameters.
Choose new parameters $\beta_1, \sigma_1$.

Calibrate $\tilde{b}_1$ to the yield curve at $t = 1$ of the model with old parameters and repeat.
A semigroup perspective

The simulation algorithm as a splitting scheme

- Assume that the parameter process \((\beta_t, \sigma_t)\) is Markovian.
- Then the simulation scheme with piecewise constant parameter process \((\beta_t, \sigma_t)\) is an *exponential Euler splitting schemes* for the joint evolution of \((f_t, \beta_t, \sigma_t)\).

Convergence of the simulation scheme

- By semigroup methods, one obtains convergence to solutions of

\[
    df_t = \left( \frac{\partial}{\partial \tau} f_t + \mu_{HJM}^{\beta_t, \sigma_t} \right) dt + \sigma_{HJM}^{\beta_t, \sigma_t} dW_t.
\]
Foliations of the space of forward rate curves

- Each choice of $(\beta, \sigma)$ corresponds to a foliation of the space of forward rate curves.
- In factor models, forward rate curves evolve on single leaves of the foliation.
- In CRC models, forward rate evolutions are tangent to the foliation corresponding to $(\beta_t, \sigma_t)$, at all $t$. 
Realised covariations

- Consider the $10 \times 10$ matrix of realised covariations (on time-windows $[t, t+1]$ of one year) between yields of maturities $\tau_i, \tau_j \in \{1, \ldots, 10\}$.
- On the Euro-area government bond market, this matrix had ranks $7–10$ over the years 2005–2013.
- In the Vasiček and CIR model, this matrix has rank $1$.
- In Vasiček CRC models, where $\beta$ is updated stochastically every week, this matrix has rank $7–9$, in our simulations.
- In the continuous-time limit of the model, the matrix has full rank.
Conclusion

Advantages of HJM models...

- Exact fits to initial yield curves can be achieved.
- Dynamics can be specified independently of the initial fit.
- Time-dependent parameters pose no problem.
- No arbitrage.

...combined with advantages of factor models

- Simulation is easy.
- Analytical bond pricing formulas hold.
- Calibration reduces to an estimation problem.
Thank you


Term structure of interest rates

Zero-coupon bond yields (%)

Y(t, t+τ)

Stochastic evolution of yields with fixed maturities

Consistent yield curve modelling

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Term structure of interest rates

Zero-coupon bond yields (%)

Y(0, τ)

Term structure of yields on a fixed day
The short rate, as obtained by approximation by 3-month yields.
The instantaneous volatility $\sigma = \sqrt{a}$ of the short rate, obtained by path-wise estimation.
The mean reversion parameter $\beta < 0$, estimated from the covariation of yields.
The estimated $\beta$ depends on the times to maturity of the yields used in the estimation.