Asymptotics beats Monte Carlo: The case of correlated local vol baskets

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1  Introduction

2  Outline of our approach

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1 Introduction

2 Outline of our approach

3 Heat kernel expansions

4 Numerical examples
Methods of European option pricing

\[ u(t, S_t) = e^{-r(T-t)} E \left[ f(S_T) \mid S_t \right] \]

Example (Example treated in this work)

- \[ f(S) = \left( \sum_{i=1}^{n} w_i S_i - K \right)^+ \], at least one weight positive
- \( n \) large (e.g., \( n = 500 \) for SPX)

- PDE methods
- (Quasi) Monte Carlo method
- Fourier transform based methods
- Approximation formulas
Methods of European option pricing

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- PDE methods
  - **Pros**: fast, general
  - **Cons**: curse of dimensionality, path-dependence may or may not be easy to include

  - (Quasi) Monte Carlo method
  - Fourier transform based methods
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- PDE methods
- (Quasi) Monte Carlo method
  
  **Pros:** very general, easy to adapt, no curse of dimensionality
  
  **Cons:** slow, quasi MC may be difficult in high dimensions

- Fourier transform based methods
- Approximation formulas
Methods of European option pricing

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- PDE methods
- (Quasi) Monte Carlo method
- Fourier transform based methods

Pros: very fast to evaluate ("explicit formula")
Cons: only available for affine models, difficult to generalize, curse of dimensionality

- Approximation formulas
Methods of European option pricing

\[ u(t, S_t) = e^{-r(T-t)} \mathbb{E} [ f(S_T) | S_t ] \]

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- PDE methods
- (Quasi) Monte Carlo method
- Fourier transform based methods
- Approximation formulas
  - **Pros:** very fast evaluation
  - **Cons:** derived on case by case basis, therefore very restrictive
Methods of European option pricing

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- PDE methods
- (Quasi) Monte Carlo method
- Fourier transform based methods
- Approximation formulas
- Work horse methods: PDE methods and (in particular) (Q)MC
- Particular models allowing approximation formulas (e.g., \textit{SABR formula}) or FFT (Heston model) very popular
Outline

1 Introduction

2 Outline of our approach

3 Heat kernel expansions

4 Numerical examples
Local volatility model for forward prices

\[ dF_i(t) = \sigma_i(F_i(t))dW_i(t), \quad i = 1, \ldots, n, \]
\[ \langle dW_i(t), dW_j(t) \rangle = \rho_{ij}dt \]

Generalized spread option with payoff \((\sum_{i=1}^{n} w_i F_i - K)^+\), at least one \(w_i\) positive

Goal: fast and accurate approximation formulas, even for high \(n\)

\(n = 100\) or \(n = 500\) not uncommon (index options)

Example

- Black-Scholes model: \(\sigma_i(F_i) = \sigma_i F_i\)
- CEV model: \(\sigma_i(F_i) = \sigma_i F_i^{\beta_i}\)
Local volatility model for forward prices

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Consider the basket (index) $\sum_{i=1}^{n} w_i F_i$:

$$d \sum_{i=1}^{n} w_i F_i(t) = \sum_{i=1}^{n} w_i \sigma_i(F_i(t)) dW_i(t)$$

Ito’s formula formally implies that

Let $H_{n-1}$ be the Hausdorff measure on $E(K)$. 
Consider the basket (index) $\sum_{i=1}^{n} w_i F_i$.

Ito’s formula formally implies that

$$\left( \sum_{i=1}^{n} w_i F_i(t) - K \right)^+ = \left( \sum_{i=1}^{n} w_i F_i(0) - K \right)^+ +$$

$$+ \sum_{i=1}^{n} w_i \int_{0}^{T} 1_{\sum w_i F_i(u) > K} dF_i(u) + \frac{1}{2} \int_{0}^{T} \delta_{\sum w_i F_i(u) = K} \sigma_N^2 \mathcal{B}(F(u)) du$$

Let $H_{n-1}$ be the Hausdorff measure on $\mathcal{E}(K)$.
Basket Carr-Jarrow formula

- Consider the basket (index) \( \sum_{i=1}^{n} w_i F_i \).
- Ito’s formula formally implies with \( \mathcal{E}(K) = \{ F \mid \sum w_i F_i = K \} \) that

\[
C(F(0), K, T) = \left( \sum_{i=1}^{n} w_i F_i(0) - K \right)^+ + \frac{1}{2} \int_0^T E \left[ \sigma_{\mathcal{N}, \mathcal{B}}^2(F(u)) \delta_{\mathcal{E}(K)}(F(u)) \right] du
\]

- Let \( H_{n-1} \) be the Hausdorff measure on \( \mathcal{E}(K) \)
Consider the basket (index) $\sum_{i=1}^{n} w_i F_i$.

Ito’s formula formally implies with $\mathcal{E}(K) = \{ F | \sum w_i F_i = K \}$ that

$$C(F(0), K, T) = \left( \sum_{i=1}^{n} w_i F_i(0) - K \right)^+$$

$$+ \frac{1}{2} \int_{0}^{T} \int_{\mathbb{R}^n} \sigma^2_{N,\mathcal{B}}(F) \delta_{\mathcal{E}(K)}(F) p(F_0, F, u) dF du$$

Let $H_{n-1}$ be the Hausdorff measure on $\mathcal{E}(K)$.
Basket Carr-Jarrow formula

- Consider the basket (index) $\sum_{i=1}^{n} w_i F_i$.
- Ito’s formula formally implies with $\mathcal{E}(K) = \{ F | \sum w_i F_i = K \}$ and $\nu(F) := \sum_i w_i F_i$ that

$$C(F(0), K, T) = \left( \sum_{i=1}^{n} w_i F_i(0) - K \right)^+$$

$$+ \frac{1}{2|\mathbf{w}|} \int_0^T \int_{\mathbb{R}^n} |\nabla \nu(F)| \sigma_{N,B}(F) \delta_0(\nu(F) - K) p(F_0, F, u) dF du$$

- Let $H_{n-1}$ be the Hausdorff measure on $\mathcal{E}(K)$
Consider the basket (index) $\sum_{i=1}^{n} w_i F_i$.

Ito's formula formally implies with $\mathcal{E}(K) = \{ F | \sum w_i F_i = K \}$ and $\nu(F) := \sum_i w_i F_i$ that

$$C(F(0), K, T) = \left( \sum_{i=1}^{n} w_i F_i(0) - K \right)^+$$

$$+ \frac{1}{2 |w|} \int_0^T \int_{\mathbb{R}^n} |\nabla \nu(F)| \sigma^2_{N, \mathcal{B}}(F) \delta_0(\nu(F) - K) \rho(F_0, F, u) dF du$$

Let $H_{n-1}$ be the Hausdorff measure on $\mathcal{E}(K)$. Recall the co-area formula:

$$\int_{\Omega} |\nabla v(x)| g(x) dx = \int_{-\infty}^{\infty} \int_{v^{-1}(\{s\})} g(x) H_{n-1}(dx) ds$$
Consider the basket (index) $\sum_{i=1}^{n} w_i F_i$.

Ito's formula formally implies with $\mathcal{E}(K) = \{F| \sum w_i F_i = K\}$ that

$$C(F(0), K, T) = \left(\sum_{i=1}^{n} w_i F_i(0) - K\right)^+$$

$$+ \frac{1}{2|\mathbf{w}|} \int_{0}^{T} \int_{\mathbb{R}^n} |\nabla v(F)| \sigma^2_{N,\mathcal{B}}(F) \delta_0(v(F) - K)p(F_0, F, t)dFdudt$$

Let $H_{n-1}$ be the Hausdorff measure on $\mathcal{E}(K)$, then we have the Carr-Jarrow formula for the local vol baskets

$$C_{\mathcal{B}}(F_0, K, T) = \left(\sum_{i=1}^{n} w_i F_i(0) - K\right)^+$$

$$+ \frac{1}{2|\mathbf{w}|} \int_{0}^{T} \int_{-\infty}^{\infty} \delta_0(s - K) \int_{\mathcal{E}_s} \sigma^2_{N,\mathcal{B}}(F)p(F_0, F, t)H_{n-1}(dF)dsdt$$
Consider the basket (index) $\sum_{i=1}^{n} w_i F_i$.

Ito’s formula formally implies with $\mathcal{E}(K) = \{F : \sum w_i F_i = K\}$ that

$$C(F(0), K, T) = \left( \sum_{i=1}^{n} w_i F_i(0) - K \right)^{+}$$

$$+ \frac{1}{2 |w|} \int_{0}^{T} \int_{\mathbb{R}^n} |\nabla v(F)| \sigma_{N, B}(F) \delta_0(v(F) - K) p(F, F, u) dF du$$

Let $H_{n-1}$ be the Hausdorff measure on $\mathcal{E}(K)$, then we have the Carr-Jarrow formula

$$C(F_0, K, T) = \left( \sum_{i=1}^{n} w_i F_i(0) - K \right)^{+}$$

$$+ \frac{1}{2} \int_{0}^{T} \frac{1}{|w|} \int_{\mathcal{E}(K)} \sum_{i,j=1}^{n} w_i w_j \sigma_i(F_i) \sigma_j(F_j) \rho_{ij} p(F_0, F, u) H_{n-1}(dF) du.$$
Heat kernel expansion (to be discussed in detail later):

\[
\sigma^2_{N,B}(F)p(F_0, F, t) \approx \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{d(F_0, F)^2}{2t} - C(F_0, F)\right)
\]

By change of variables \(F_n = \frac{1}{w_n} \left(K - \sum_{i=1}^{n-1} w_i F_i\right)\) on \(E_K\):

\[
H_{n-1}(dF) = \frac{|w|}{|w_n|} dF_1 \cdots dF_{n-1}
\]

Laplace approximation: with \(F^* = \text{argmin}_{F \in E_K} d(F_0, F)\) and \(G_K = \{(F_1, \ldots, F_{n-1})| \sum_{i=1}^{n-1} w_i F_i < K\}\)

\[
\int_{G_K} e^{-\frac{d(F_0,F)^2}{2t} - C(F_0,F)} dF_1 \cdots dF_{n-1} \approx e^{-\frac{d(F_0,F^*)^2}{2t} - C(F_0,F^*)} \int_{\mathbb{R}^{n-1}} e^{-\frac{z^T Q z}{2t}} d\mathbf{z}
\]

\[
= t^{n-1/2} e^{-\frac{d(F_0,F^*)^2}{2t} - C(F_0,F^*)} \left(\frac{2\pi}{\sqrt{\det Q}}\right)^{n-1/2}
\]

We rely on the principle of not feeling the boundary.
Approximations

▶ Heat kernel expansion (to be discussed in detail later):

\[
\sigma_{N, B}^2(F) p(F_0, F, t) \approx \frac{1}{(2\pi t)^{n/2}} \exp \left( -\frac{d(F_0, F)^2}{2t} - C(F_0, F) \right)
\]

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▶ Laplace approximation: with \( F^* = \arg\min_{F \in E_K} d(F_0, F) \) and

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G_K = \{(F_1, \ldots, F_{n-1}) | \sum_{i=1}^{n-1} w_i F_i < K\}
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\int_{G_K} e^{-\frac{d(F_0, F)^2}{2t} - C(F_0, F)} dF_1 \cdots dF_{n-1} \approx e^{-\frac{d(F_0, F^*)^2}{2t} - C(F_0, F^*)} \int_{\mathbb{R}^{n-1}} e^{-\frac{z^T Q z}{2t}} d\mathbf{z}
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Heat kernel expansion (to be discussed in detail later):

\[ \sigma^2_{\mathcal{N}, \mathcal{B}}(\mathbf{F})p(\mathbf{F}_0, \mathbf{F}, t) \approx \frac{1}{(2\pi t)^{n/2}} \exp \left( - \frac{d(\mathbf{F}_0, \mathbf{F})^2}{2t} - C(\mathbf{F}_0, \mathbf{F}) \right) \]

By change of variables \( F_n = \frac{1}{w_n} \left( K - \sum_{i=1}^{n-1} w_i F_i \right) \) on \( \mathcal{E}_K \):

\[ H_{n-1}(d\mathbf{F}) = \frac{|w|}{|w_n|} dF_1 \cdots dF_{n-1} \]

Laplace approximation: with \( \mathbf{F}^* = \arg\min_{\mathbf{F} \in \mathcal{E}_K} d(\mathbf{F}_0, \mathbf{F}) \) and \( \mathcal{G}_K = \{(F_1, \ldots, F_{n-1})| \sum_{i=1}^{n-1} w_i F_i < K\} \)

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\int_{\mathcal{G}_K} e^{-\frac{d(\mathbf{F}_0, \mathbf{F})^2}{2t} - C(\mathbf{F}_0, \mathbf{F})} dF_1 \cdots dF_{n-1} \approx e^{-\frac{d(\mathbf{F}_0, \mathbf{F}^*)^2}{2t} - C(\mathbf{F}_0, \mathbf{F}^*)} \int_{\mathbb{R}^{n-1}} e^{-\frac{z^T Q z}{2t}} dz
\]

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= t^{\frac{n-1}{2}} e^{-\frac{d(\mathbf{F}_0, \mathbf{F}^*)^2}{2t} - C(\mathbf{F}_0, \mathbf{F}^*)} \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{\det Q}}
\]

We rely on the principle of not feeling the boundary.
Non-degeneracy of the optimization problem

- Assume non-degeneracy of $F^* = \arg\min_{F \in \mathcal{E}_K} d(F_0, F)$
- Generically true, but exceptional points $F_0$ or $K$ often exist.
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Generically true, but exceptional points $F_0$ or $K$ often exist.

Example: $F_1, F_2$ independent, Black-Scholes assets, $\sigma_i = 1$, $F_{0,i} = 1$, $f \ldots$ density of $F_{1,T} + F_{2,T}$. Then

$$f(K) = \begin{cases} 
\exp \left( -\frac{\Lambda(K)}{T} \right) \frac{1}{\sqrt{T}} (c_0 + O(T)), & K \neq 2e, \\
\exp \left( -\frac{\Lambda(K^*)}{T} \right) \frac{1}{T^{3/4}} (c_0 + O(T)), & K = 2e. 
\end{cases}$$
Non-degeneracy of the optimization problem

- Assume non-degeneracy of $F^* = \arg\min_{F \in \mathcal{E}_K} d(F_0, F)$
- Generically true, but exceptional points $F_0$ or $K$ often exist.
- Related concept of focality in Riemannian geometry.
Matching to implied volatilities

**Theorem**

\[
C_B(F_0, K, T) = \left( \sum_{i=1}^{n} w_i F_i(0) - K \right) + \\
+ \frac{1}{2 \sqrt{2\pi} |w_n| d(F_0, F^*)^2 \sqrt{\det Q}} e^{-C(F_0, F^*) - \frac{d(F_0, F^*)}{2T}} T^{3/2} + o(T^{3/2}), \text{ as } T \to 0.
\]

- **Bachelier implied vol** (with \( F_0 = \sum_{i=1}^{n} w_i F_{0,i} \)): \[
\sigma_B \sim \sigma_{B,0} + T \sigma_{B,1} \text{ with } \sigma_{B,0} = \frac{|F_0 - K|}{d(F_0, F^*) |F_0|}, \sigma_{B,1} = \cdots
\]

- **Black-Scholes implied vol**: \[
\sigma_{BS} \sim \sigma_{BS,0} + T \sigma_{BS,1} \text{ with } \sigma_{BS,0} = \frac{\left| \log \left( \frac{F_0}{K} \right) \right|}{d(F_0, F^*)}, \sigma_{BS,1} = \cdots
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Matching to implied volatilities

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  \]

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  \]
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**Theorem**

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- **Black-Scholes implied vol**:
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  \sigma_{BS} \sim \sigma_{BS,0} + T \sigma_{BS,1} \text{ with } \sigma_{BS,0} = \frac{\log(\overline{F}_0/K)}{d(F_0, F^*)}, \sigma_{BS,1} = \cdots
  \]
The ATM case

- Above formulas have singularities when $\overline{F}_0 = K$ (ATM)
- Resolve by l’Hopital formula or first order heat kernel expansion.

We have $F^* = F_0$ and

$$\det Q = \sigma_{N,B}^2(F_0) \det \rho^{-1} \prod_{k=1}^{n} \sigma_k(F_{0,k})^{-2} / w_n^2.$$ 

Higher order Laplace expansion required.

- $\sigma_{BS,0} = \sigma_{Bach,0} = \frac{\sigma_{N,B}(F_0)}{\overline{F}_0}$

- $\sigma_{BS,1} = \frac{\sqrt{2\pi}}{3K} \left( g^0_1 + g^1_0 \right) + \frac{\sigma_{BS,0}^3}{24} = \sigma_{Bach,1} + \frac{\sigma_{BS,0}^3}{24}$
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The ATM case

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- Higher order Laplace expansion required.

- $\sigma_{BS,0} = \sigma_{Bach,0} = \frac{\sigma_{N,B}(F_0)}{\overline{F}_0}$

- $\sigma_{BS,1} = \frac{\sqrt{2\pi}}{3K} \left(g\tilde{u}_0 + g\tilde{u}_1\right) + \frac{\sigma_{BS,0}^3}{24} = \sigma_{Bach,1} + \frac{\sigma_{BS,0}^3}{24}$
Goal: sensitivity w. r. t. model parameter $\kappa$ of the option price

\[ C_B(F_0, K, T) \approx C_{BS}(F_0, K, \sigma_{BS}, T) \]

Sensitivity:

\[ \frac{\partial}{\partial \kappa} C_{BS}(F_0, K, \sigma_{BS}, T) + \nu_{BS}(F_0, K, \sigma_{BS}, T) \frac{\partial}{\partial \sigma_{BS}} \]

Recall that $\sigma_{BS,0}, \sigma_{BS,1}$ explicit up to $F^*$

By the minimizing property: \[ \partial_{F_i}d^2(F_0, F_K(G))\big|_{G=G^*} = 0 \]

Differentiating with respect to $\kappa$ gives

\[ \frac{\partial}{\partial \kappa} \frac{\partial}{\partial F_i} d^2(F_0, F_K(G))\big|_{G=G^*} + \sum_{l=1}^{n-1} \frac{\partial}{\partial F_i} \frac{\partial}{\partial F_l} d^2(F_0, F_K(G))\big|_{G=G^*} \frac{\partial}{\partial \kappa} F^*_l = 0 \]

Up to the above system of linear equations for $\frac{\partial}{\partial \kappa} F^*$, there are explicit expression for the sensitivities of the approximate option prices.
Goal: sensitivity w. r. t. model parameter $\kappa$ of the option price

$$C_B(F_0, K, T) \approx C_{BS}(\bar{F}_0, K, \sigma_{BS}, T)$$

Sensitivity: $\partial_\kappa C_{BS}(\bar{F}_0, K, \sigma_{BS}, T) + \nu_{BS}(\bar{F}_0, K, \sigma_{BS}, T)\partial_\kappa \sigma_{BS}$

Recall that $\sigma_{BS,0}, \sigma_{BS,1}$ explicit up to $F^*$

By the minimizing property: $\partial_{F_i} d^2 (F_0, F_K(G))|_{G=G^*} = 0$

Differentiating with respect to $\kappa$ gives

$$\partial_\kappa \partial_{F_i} d^2 (F_0, F_K(G))|_{G=G^*} + \sum_{l=1}^{n-1} \partial_{F_i} \partial_{F_i} d^2 (F_0, F_K(G))|_{G=G^*} \partial_\kappa F^*_l = 0$$

Up to the above system of linear equations for $\partial_\kappa F^*$, there are explicit expression for the sensitivities of the approximate option prices.
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Heat kernels and geometry

\[ d\mathbf{X}_t = b(\mathbf{X}_t)dt + \sigma(\mathbf{X}_t)dW_t, \]

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- **Heat kernel**: fundamental solution \( p(\mathbf{x}, \mathbf{y}, t) \) of \( \frac{\partial}{\partial t} u = Lu \)
- **Transition density of** \( \mathbf{X}_t \)

"Can you hear the shape of the drum?" (Kac '66)

Take \( L = \Delta \) on a domain \( D \) and relate:

- Geometrical properties of the domain \( D \)
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- Heat kernel
- E.g. \( -\gamma_k \sim C(n)(k/ \text{vol } D)^{2/n} \) (Weyl, '46)
- E.g. (for \( n = 2 \)): \( Z = \frac{\text{area}}{4\pi t} - \frac{\text{circ.}}{\sqrt{4\pi t}} + O(1) \) (McKean & Singer, '67)

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The Riemannian metric associated to a diffusion

\[ dX_t = b(X_t)dt + \sigma(X_t)dW_t, \]

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- On \( \mathbb{R}^n \) (or a submanifold), introduce \( g^{ij} := a^{ij} \), Riemannian metric tensor \( (g^{ij}(x))_{i,j=1}^n := \left( (g^{ij}(x))_{i,j=1}^n \right)^{-1} \)

- Geodesic distance:
  \[ d(x, y) := \inf_{z(0)=x, z(1)=y} \int_0^1 \sqrt{\sum g_{ij}(z(t))\dot{z}^i(t)\dot{z}^j(t)} dt \]

- \( \inf \) attained by a smooth curve, the geodesic

- Laplace-Beltrami operator: \( \Delta_g = \left( \det(g_{ij}) \right)^{-\frac{1}{2}} \frac{\partial}{\partial x^i} \left( \det(g_{ij}) \right)^{\frac{1}{2}} g^{ij} \frac{\partial}{\partial x^j} \)

\[ L = \frac{1}{2} a^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + b^i \frac{\partial}{\partial x^i} = \frac{1}{2} \Delta_g + h^i \frac{\partial}{\partial x^i} \]
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Heat kernel expansion

\[ p_N(x_0, x, T) = \sqrt{\det(g(x)_{ij})} U_N(x_0, x, T) \frac{e^{-\frac{d^2(x_0, x)}{2T}}}{(2\pi T)^{n/2}} \]

- \( U_N(x_0, x, T) = \sum_{k=0}^{N} u_k(x_0, x) T^k \) \( \), the heat kernel coefficients
- \( u_0(x_0, x) = \sqrt{\Delta(x_0, x)} e^{\int_z \langle h(z(t)), \dot{z}(t) \rangle_g dt} \)
- \( \Delta \) is the Van Vleck-DeWitt determinant:
  \[ \Delta(x_0, x) = \frac{1}{\sqrt{\det(g(x_0)_{ij}) \det(g(x)_{ij})}} \det \left( -\frac{1}{2} \frac{\partial^2 d^2}{\partial x_0 \partial x} \right) . \]
- \( e^{\int_z \langle h(z(t)), \dot{z}(t) \rangle_g dt} \) \( \) is the exponential of the work done by the vector field \( h \) along the geodesic \( z \) joining \( x_0 \) to \( x \) with
  \[ h^i = b^i - \frac{1}{2 \sqrt{\det(g_{ij})}} \frac{\partial}{\partial x^j} \left[ \sqrt{\det(g_{ij})} g^{ij} \right] \]
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Assumption

The cut-locus of any point is empty,

Theorem (Varadhan ’67)

\( b = 0, \sigma \text{ uniformly Hölder continuous, system uniformly elliptic, then} \)
\[
\lim_{T \to 0} T \log p(x, y, T) = -\frac{1}{2} d(x, y)^2.
\]

Theorem (Yosida ’53)

On a compact Riemannian manifold, assume smooth vector fields and an ellipticity property. Then \( p(x, y, T) - p_N(x, y, T) = O(T^N) \) as \( T \to 0 \).

Theorem (Azencott ’84)

For a locally elliptic system in an open set \( U \subset \mathbb{R}^n, x, y \in U \) s. t. \( d(x, y) < d(x, \partial U) + d(y, \partial U) \), we have \( p(x, y, T) - p_N(x, y, T) = O(T^N) \) as \( T \to 0 \).
The local vol case

- Domain $\mathbb{R}^n_+, dF_i(t) = \sigma_i(F_i(t))dW_i(t), \quad i = 1, \ldots, n$
- $L = \frac{1}{2} \rho_{ij} \sigma_i(x^i) \sigma_j(x^j) \frac{\partial^2}{\partial x^i \partial x^j}$
- Let $A \in \mathbb{R}^{n \times n}$ be such that $A \rho A^T = I_n$. Change variables $F \rightarrow y \rightarrow x$ according to
  
  $y_i = \int_0^{F_i} \frac{du}{\sigma_i(u)}, \quad i = 1, \ldots, n, \quad x = Ay, \quad L \rightarrow \frac{1}{2} \frac{\partial^2}{\partial x_i^2} - \frac{1}{2} A_{ik} \sigma'_k(F_k) \frac{\partial}{\partial x_i}$

- Isomorphic (up to boundary) to Euclidean geometry:
  
  $d(F_0, F) = |x_0 - x|$

- Geodesics known in closed form
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1 Introduction

2 Outline of our approach

3 Heat kernel expansions

4 Numerical examples
Implementation

- Optimization problem for $F^*$ is non-linear with a linear constraint
- With $q_i := \int_{F_0,i}^{F_i} \frac{d\sigma_i(u)}{\sigma_i(u)}$, it is a quadratic optimization problem with non-linear constraint
- Fast convergence of Newton iteration for suitable initial guess
- Given $F^*$, $C(F_0, F^*)$ is a line integral along the geodesic; this integral can be calculated in closed form in the CEV model.
- Formulas can be evaluated in less than 2 seconds for $n = 100$

Our work relies on the principle of not feeling the boundary and on non-degeneracy of the minimization problem.
Optimization problem for $F^*$ is non-linear with a linear constraint

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Fast convergence of Newton iteration for suitable initial guess

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Numerical examples

- CEV model framework
  - For CEV, the formulas are fully explicit apart from the minimizing configuration $F^*$
  - We observe very fast convergence of the iteration, but the initial guess is crucial.

- Reference values obtained using:
  - Ninomiya Victoir discretization
  - Quasi Monte Carlo based on Sobol numbers, Monte Carlo for very high dimensions ($n \approx 100$)
  - Variance (dimension) reduction using Mean value Monte Carlo based on one-dimensional Black-Scholes prices
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CEV index implied vol – three-dimensional visualization

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CEV index implied vol – three-dimensional visualization

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Normalized errors

- Approximation error supposed to depend on “dimension-free” time to maturity $\sigma^2 T$
- Use $\bar{\sigma} := \sigma_{N,B}(F_0)/\left(\sum_{i=1}^{n} w_i F_{0,i}\right)$ as proxy in local vol framework
- Normalized error: $\frac{\text{Rel. error}}{\bar{\sigma}^2 T}$

<table>
<thead>
<tr>
<th>$T$</th>
<th>Dim. 5</th>
<th>Dim. 10</th>
<th>Dim. 15</th>
<th>Dim. 100</th>
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<tr>
<td>$\bar{\sigma}$</td>
<td>0.1704</td>
<td>0.3187</td>
<td>0.1073</td>
<td>0.2964</td>
</tr>
</tbody>
</table>

Table: Normalized relative error of the zero-order asymptotic prices.
Normalized errors

- Approximation error supposed to depend on “dimension-free” time to maturity $\sigma^2 T$
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<tr>
<td>0.5</td>
<td>$-4.02 \times 10^{-4}$</td>
<td>$1.76 \times 10^{-4}$</td>
<td>$8.76 \times 10^{-3}$</td>
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</tr>
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</tr>
<tr>
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<td>$-1.63 \times 10^{-3}$</td>
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<td>5</td>
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<td>$-1.33 \times 10^{-2}$</td>
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<tr>
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<td>$-2.82 \times 10^{-2}$</td>
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<tr>
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<td>0.3187</td>
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</table>

**Table:** Normalized error of the first order asymptotic prices.
Relative errors
Objective: Compute the sensitivity (delta) w.r.t. $F_{0,3}$.

Note that the option payoff is

$$P(F) = (F_1 + F_2 - F_3 - K)^+$$
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Relative error of delta

$T = 0.5$

$T = 5$

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References


