How Leverage Transforms a Volatility Skew

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Joint with Ruming Wang
Leveraged volatility skew

Continuous dynamics: small $T$

Jump-diffusion dynamics: small $T$

Jump-diffusion dynamics: low/moderate/high jump-intensity

Jump-diffusion dynamics special case: low jump-intensity

Numerical results
Leveraged product

- Fix an expiry $T$.
- Consider a portfolio $L$
  
  - (examples: LETF = Leveraged ETF or “geared” ETF products) that provides $\beta$-times leverage on some underlying $S$
    
  - (examples: ETF. Stock.)

  Typical LETF $\beta \in \{-1, \pm 2, \pm 3\}$. (“Lightly levered” +1.25)

- So define $L$ to hold $\beta \times$ fully-invested number $L/S$ of shares

\[
\frac{dL_t}{L_{t-}} = \frac{\beta}{S_{t-}} dS_t \quad \text{or} \quad \frac{dL_t}{L_{t-}} = \beta \frac{dS_t}{S_{t-}}
\]

with stock $S > 0$ and LETF $L > 0$ quoted as futures/forwards. Equivalently, scale simple instantaneous returns by $\beta$. 
Leveraged product

- Fix an expiry $T$.
- Consider a portfolio $L$
  - (examples: LETF = Leveraged ETF or “geared” ETF products)
  - that provides $\beta$-times leverage on some underlying $S$
    - (examples: ETF. Stock.)
  - Typical LETF $\beta \in \{-1, \pm 2, \pm 3\}$. (“Lightly levered” $+1.25$)
- So define $L$ to hold $\beta \times$ fully-invested number $L/S$ of shares

\[
\frac{dL_t}{L_t} = \beta \frac{dS_t}{S_t}\]

with stock $S > 0$ and LETF $L > 0$ quoted as futures/forwards and $\mathcal{L}$ is stochastic logarithm.
Leveraged product: second definition

Let $S$ be a semimartingale with $S, S_\neq 0$, and let $\beta \in \mathbb{R}$.
Define the $\beta$-leveraged product on $S$, given initial value $L_0$, by

$$
L := L_0 \mathcal{E}(\beta \mathcal{L}(S))
$$

Stochastic exponential $\mathcal{E}(X)$ of semimartingale $X$ is the solution $Z$ to

$$
Z_t = 1 + \int_0^t Z_\cdot dX.
$$

Stochastic logarithm $\mathcal{L}(Z)$ of a semimartingale $Z$ with $Z, Z_\neq 0$, is the semimartingale $X$ such that $X_0 = 0$ and $Z = Z_0 \mathcal{E}(X)$; explicitly,

$$
\mathcal{L}(Z)_t = \int_0^t \frac{1}{Z_\cdot} dZ
$$

$$
\mathcal{E}(X)_t = \exp(X_t - X_0 - \frac{1}{2}[X^c]_t) \prod_{s \leq t} (1 + \Delta X_s) \exp(-\Delta X_s)
$$

(Intuition: $d \log S$ are log-returns, while $d \mathcal{L}(S)$ are simple returns)
Leveraged product: equivalence of the definitions

Stochastic exponential of $\beta$ times stochastic log of the underlying . . .

\[ L = L_0 \mathcal{E}(\beta \mathcal{L}(S)) \]
\[ \Downarrow \]
\[ \frac{L}{L_0} = 1 + \int \frac{L_-}{L_0} d(\beta \mathcal{L}(S)) \]
\[ \Downarrow \]
\[ \frac{L}{L_0} = 1 + \frac{\beta}{L_0} \int \frac{L_-}{S_-} dS \]
\[ \Downarrow \]
\[ dL_t = \beta \frac{L_{t-}}{S_{t-}} dS_t \]

. . . is equivalent to $\beta$-scaling of simple instantaneous returns $dS/S_-$
Leveraged product includes general leveraged portfolios

We are not ignoring rates/dividends/fees when assuming

\[
\frac{dL_t}{L_{t-}} = \beta \frac{dS_t}{S_{t-}}.
\]

This could arise from trading in the spot market \((S, L)\) in which the LETF portfolio invests \(\beta L_{t-}\) dollars in \(S\), borrows \((\beta - 1)L_{t-}\) at rate \(r\), receives \(S\)-proportional divs \(q_S\) and pays \(L\)-proportional fees/costs \(c_L\):

\[
dL_t = \beta \frac{L_{t-}}{S_{t-}} dS_t - r(\beta - 1)L_{t-} dt + (\beta q_S - c_L)L_{t-} dt
\]

Futures \(S_t := S_t e^{(r-q_S)(T-t)}\) and \(L_t := L_t e^{(r-c_L)(T-t)}\), so

\[
dL_t = e^{(r-c_L)(T-t)} dL_t - L_t (r - c_L) e^{(r-c_L)(T-t)} dt
\]

\[
= e^{(r-c_L)(T-t)} \left( \beta \frac{L_{t-}}{S_{t-}} dS_t - r \beta L_{t-} dt + \beta q_S L_{t-} dt \right)
\]

\[
\Rightarrow \frac{dL_t}{L_{t-}} = \beta \frac{dS_t}{S_{t-}} - r \beta dt + q_S \beta dt = \beta \frac{dS_t}{S_{t-}}
\]
Leveraged portfolio

In other words: we have assumed

- Continuous rebalancing
- Leverage $\beta$ applies on the futures.

Sufficient condition:

Proportional rates, dividends, fees, and trading costs

and will furthermore assume

- No bust: $S_{t-} + \beta \Delta S_t > 0$
Two volatility skews

LETF calls pay

\[ (L_T - K_L)^+. \]

Consider the implied volatility skew of \( L \), as a function of \( K_L \).

▶ Call this the **leveraged** volatility skew.

How does it relate to the implied volatility of \( S \)?

▶ Call this the **reference** volatility skew.

Simple relationships between the two volatility skews facilitate:

▶ Pricing of LETF options consistently with ETF options
▶ Detection of relative mispricings
▶ Inference about underlying dynamics
One-month volatility skews on 2014 Oct 20

Reference SPY and LETF SDS with $\beta = -2$
Scaling and shifting rules

- Stanley Zhang’s (2009) proposal: Take

\[ |\beta| \times \text{reference volatility at strike (in logs)} \text{ scaled by } \frac{1}{\beta} \]

and shifted by \( +\frac{1}{2} \times (\beta - 1) \times \text{forward-looking variance} \).

- We directly link (like above) the reference and leveraged skews, but (unlike above) we prove validity of a variety of asymptotic rules for shifting/scaling, that identify the correct shifts.

No previously known proof of any rule that identifies the shift.

- Prices/vols given an SDE: Ahn-Haugh-Jain (2012), Leung-Sircar (2012), Leung-Lorig-Pascucci (2014). Insightful heuristics for one shift (+\( \frac{1}{2} \)): L-S, Zhang (2009). One rule (+0) without identifying shift by L-L-P: “we emphasize...scaling alone is not sufficient”
Motivation for strike adjustment: stochastic volatility

Let $dS_t = \sigma_t S_t dW_t$. Then $dL_t = \beta \sigma_t L_t dW_t$ and

$$d\log S_t = -\frac{1}{2}\sigma_t^2 dt + \sigma_t dW_t$$

$$d\log L_t = -\frac{1}{2} \beta^2 \sigma_t^2 dt + \beta \sigma_t dW_t$$

So $\beta$-leverage multiplies the volatility by $\beta$ but the drift by $\beta^2$.

$$\beta \log \frac{S_T}{S_0} = \log \frac{L_T}{L_0} + \frac{1}{2}(\beta^2 - \beta)V_T$$

So $L_T/L_0$ is not just $(S_T/S_0)^\beta$. Also depends on $\frac{1}{2}(\beta - 1) \times$

$$V_T := \int_0^T \sigma_t^2 dt.$$
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$$V_T := \int_0^T \sigma_t^2 dt.$$
Motivation for strike adjustment: stochastic volatility

Let \( dS_t = \sigma_t S_t \, dt \). Then \( dL_t = \beta \sigma_t L_t \, dt \) and

\[
\begin{align*}
\text{d log } S_t &= -\frac{1}{2} \sigma_t^2 \, dt + \sigma_t \, dW_t \\
\text{d log } L_t &= -\frac{1}{2} \beta^2 \sigma_t^2 \, dt + \beta \sigma_t \, dW_t
\end{align*}
\]

So \( \beta \)-leverage multiplies the volatility by \( \beta \) but the drift by \( \beta^2 \).

\[
\beta \log \frac{S_T}{S_0} = \log \frac{L_T}{L_0} + \frac{1}{2} (\beta^2 - \beta) V_T
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Motivation for strike adjustment: stochastic volatility

Let \( dS_t = \sigma_t S_t dW_t \). Then \( dL_t = \beta \sigma_t L_t dW_t \) and

\[
\begin{align*}
d \log S_t &= -\frac{1}{2} \sigma_t^2 dt + \sigma_t dW_t \\
\frac{1}{\beta} d \log L_t &= -\frac{1}{2} \boxed{\beta} \sigma_t^2 dt + \sigma_t dW_t
\end{align*}
\]

So \( \beta \)-leverage multiplies the volatility by \( \beta \) but the drift by \( \beta^2 \).

So \( 1/\beta \)-scaling of \( \log L_t \) still leaves a factor of \( \beta \neq 1 \) in the drift.

\[
\log \frac{S_T}{S_0} = \frac{1}{\beta} \log \frac{L_T}{L_0} + \frac{1}{2} (\beta - 1) V_T
\]

So \( L_T / L_0 \) is not just \( (S_T / S_0)^\beta \). Also depends on \( + \frac{1}{2} (\beta - 1) \times \)

\[
V_T := \int_0^T \sigma_t^2 dt.
\]
Shifted scaled strike

Folk wisdom:

\[ \text{LETF IV at strike } K \approx |\beta| \times \text{reference IV at strike } K_\ast \]

where

\[
\log \frac{K_\ast}{S_0} = \frac{1}{\beta} \log \frac{K_L}{L_0} + \frac{1}{2}(\beta - 1)V_T
\]

(If \(V_T\) is random, substitute some type of implied volatility.)

Questions:

- In what sense are these heuristics valid?
- How to adjust these rules, in cases where they are not valid?

Answer both, by considering \textbf{asymptotic} limiting regimes.
Leveraged volatility skew

**Continuous dynamics:** small $T$

**Jump-diffusion dynamics:** small $T$

**Jump-diffusion dynamics:** low/moderate/high jump-intensity

**Jump-diffusion dynamics special case:** low jump-intensity

**Numerical results**
Stochastic volatility framework

Let Brownian motions $B, W$ have correlation $\rho$.

Let the reference price $S$ and its instantaneous volatility $\sigma$ follow

$$dS_t = \sigma_t S_t dW_t$$

$$d\sigma_t = a(\sigma_t) dt + b(\sigma_t) dB_t$$

with local regularity (Lipschitz, ellipticity) conditions on $a, b$.

Then the dynamics of $L$ can also be written in terms of $a, b, \beta, \rho$. 
Stochastic volatility framework

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$$\begin{align*}
dS_t &= \sigma_t S_t dW_t \\
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\end{align*}$$

with local regularity (Lipschitz, ellipticity) conditions on $a, b$.

Then the dynamics of $L$ can also be written in terms of $a, b, \beta, \rho$.

$$\begin{align*}
dL_t &= \sigma_{L,t} L_t dW_{L,t} \\
d\sigma_{L,t} &= a(\sigma_{L,t})dt + b(\sigma_{L,t})dB_t,
\end{align*}$$

where $a_L(\sigma) := |\beta| a(\sigma/|\beta|)$ and $b_L(\sigma) := |\beta| b(\sigma/|\beta|)$ and $\sigma_{L,0} := |\beta| \sigma_0

and $W_{L,t} := \text{sgn}(\beta) W_t$ and $dW_{L,t} dB_t = \rho_L dt := \rho \text{sgn}(\beta) dt$. 
Continuous case: small-$T$ asymptotics

Consider reference implied vol $IV(K)$.

Let $T \to 0$ and let $k := \log(K/S_0) = O(T^{1/2})$. Then

$$IV(K) = \sigma_0 + \frac{\rho b(\sigma_0)}{2\sigma_0} \log(K/S_0) + O(T)$$

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- Heston: $b$ constant $\Rightarrow$ Skew flattens as atm vol $\uparrow$, denied by data
Continuous case: small-$T$ asymptotics

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$$IV(K) = \sigma_0 + \frac{\rho b(\sigma_0)}{2\sigma_0} \log(K/S_0) + O(T)$$


Apply this formula also to $IV_L(K_L)$. Conclude that

$$IV_L(K_L) = |\beta|IV(K_*) + O(T)$$

where $\log \frac{K_*}{S_0} = \frac{1}{\beta} \log \frac{K_L}{L_0}$

Scaling with zero shift $\implies O(T)$ error.
$O(T)$ asymptotics cannot identify the correct shift

Consider reference implied vol $IV(K)$. Let $T \to 0$ and let $k := \log(K/S_0) = O(T^{1/2})$. Then

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IV(K) = \sigma_0 + \frac{\rho b(\sigma_0)}{2\sigma_0} \log(K/S_0) + O(T)
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Apply this formula also to $IV_L(K_L)$. Conclude that

\[
IV_L(K_L) = |\beta|IV(K_*) + O(T)
\]

where $\log \frac{K_0}{S_0} = \frac{1}{\beta} \log \frac{K_L}{L_0} + \psi \sigma_0^2 T$

Scaling with any shift coefficient $\psi \to O(T)$ error.
Continuous case: refined small-$T$ asymptotics

Refined expansion (MS 06 or PP 16) shows that the $O(T)$ term is

$$\left( \frac{b^2}{6\sigma_0^3} + \frac{\rho^2 bb'}{6\sigma_0^2} - \frac{5\rho^2 b^2}{12\sigma_0^3} \right) k^2 + \left( \frac{a}{2} + \frac{\rho b\sigma_0}{4} + \frac{\rho^2 b^2}{24\sigma_0} + \frac{b^2}{12\sigma_0} - \frac{\rho^2 bb'}{6} \right) T$$

Apply this to the reference skew and the leveraged skew. Then

$$IV_L(K_L) = |\beta|IV(K_*) + O(T^{3/2})$$

where

$$\log \frac{K_*}{S_0} = \frac{1}{\beta} \log \frac{K_L}{L_0} + \frac{1}{2} (\beta - 1) \sigma_0^2 T.$$ 

Also valid with ATM IV in place of $\sigma_0$. Conclude:

Scaling with shift $\frac{1}{2} (\beta - 1) \times$ variance $\implies O(T^{3/2})$ error.
Continuous case – conclusions

- Stochastic vol, no jumps, small $T$: scaling with shift $1/2$ is valid
Rough volatility

For fractional-Brownian volatility

\[ dX_t = -\frac{1}{2} g(Y_t)^2 dt + g(Y_t) (\rho dW'_t + \sqrt{1 - \rho^2} dW_t) \]

\[ Y_t = y + \varepsilon W^H_t \quad W^H_t := \int_0^t K_H(t, s) dW'_s \]

Fukasawa 2011:

\[ IV(K, T) = \sigma_0 + a T^{H-1/2} \log(K/S_0) + b T^{H+1/2} + o(\varepsilon) \]

where

\[ \sigma_0 := g(y), \quad a := \frac{\rho g'(y) c'_H}{\sigma} \varepsilon, \quad b := \frac{a \sigma^2}{2}. \]

Apply this to the reference skew and the leveraged skew.

Conclusion: scaling with shift 1/2 still valid
where $g > \delta > 0$ has bounded derivatives to order 2; and $K_H(t, s) :=$

\[
\begin{cases}
    c_H \left[ \left( \frac{t}{s}(t - s) \right)^{H - 1/2} - (H - 1/2)s^{1/2-H} \int_s^t u^{H-3/2}(u - s)^{H-1/2}du \right] \\
    c_H s^{1/2-H} \int_s^t (u - s)^{H-3/2}u^{H-1/2}du
\end{cases}
\]

where

\[
c_H := \begin{cases}
    \sqrt{\frac{2H}{(1-2H)B(1-2H,H+1/2)}} & H < 1/2 \\
    \sqrt{\frac{H(2H-1)}{B(2-2H,H-1/2)}} & H > 1/2
\end{cases}
\]

\[
c'_H := \begin{cases}
    \frac{c_H B(3/2-H,H+1/2)}{(H+3/2)(H+1/2)} & H < 1/2 \\
    \frac{c_H B(3/2-H,H-1/2)}{(H+3/2)(H+1/2)} & H > 1/2
\end{cases}
\]

where $B$ is the Euler beta function.
Leveraged volatility skew

Continuous dynamics: small $T$

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Jump-diffusion dynamics special case: low jump-intensity

Numerical results
What happens for small $T$ when we add jumps

Fix an OTM log-moneyness $k$, and let $T \to 0$.

Then $C \to 0$ and $c := |\log(C/S_0)| \to \infty$.

Gao-Lee 2014: general extreme-regime asymptotic IV. For $T \to 0$,

$$IV^2 = \frac{k^2}{2cT} \left(1 + \frac{3 \log c}{2c} + \frac{\log(16\pi/k^2) - k}{2c} + \frac{9(\log c)^2}{4c^2} + \cdots \right)$$

Truncated expansion has relative error $\to 0$ as $O(c^{-n})$ for some $n$.

**Errors small** if $C \to 0$ ($c \to \infty$) **quickly**; **large if** $\to$ **slowly**.

- In **continuous** models, $c \to \infty$ like $1/T$
  (from exponent in Gaussian kernel).

- In **jump** models, $C \to 0$ like $T$, so $c \to \infty$ like $|\log T|$. Slower
  than any power of $T$; too slow to identify the correct LETF shift.
From log call prices to implied volatility

Let $c$ denote the $|\log|$ of normalized call price $C$:

$$c := \log \left( \frac{S_0}{C} \right)$$

At an OTM log-moneyness $k$, as $T \to 0$ we have $C \to 0$ and $c \to \infty$, and implied variance has universal model-free asymptotics

$$IV^2 = \frac{k^2}{2cT} \left( 1 + \frac{3 \log c}{2c} + \frac{\log(16\pi/k^2) - k}{2c} + \frac{9(\log c)^2}{4c^2} + \ldots \right)$$

(Gao-Lee 2014 proves this and other general extreme asymptotics e.g. $|k|, T \to \infty$.) Truncating this expansion produces relative error $\to 0$ as $O(c^{-n})$ for some $n$. Formulas of short-expiry type have

- Good error bounds if $c \to \infty$ quickly.
- Bad error bounds if $c \to \infty$ slowly.
Intuition of call prices – continuous case

Lognormal density (ignoring drift)

\[ f = \frac{1}{S\sqrt{2\pi\sigma^2T}} \exp \left( -\frac{(\log S - \log S_0)^2}{2\sigma^2T} \right) \]

Call price is approximated by integrating twice by parts and ignoring the remainder terms

\[ C \approx \frac{1}{K\sqrt{2\pi\sigma^2T}} \left( \frac{K\sigma^2T}{\log K - \log S_0} \right)^2 \exp \left( -\frac{(\log K - \log S_0)^2}{2\sigma^2T} \right) \]

So in B-S case, the leading-order \(|\log C|\) behavior is

\[ c = \frac{k^2}{2\sigma^2T} - \frac{3}{2} \log T + \cdots \]

This goes to \(\infty\) like \(1/T\). Similar for 1-D diffusions but (GHLOW 12)

\[ \frac{\log K - \log S_0}{\sigma} \]

becomes \(d(\log K, \log S_0)\) where \(d(x, y) := \int_x^y \frac{1}{\sigma(z)}dz\)

- IV expansions have power-of-\(T\) errors as \(T \to 0\).
Intuition of call prices – jump case

Martingale $S$ where $\log S$ has integrable Lévy density $\nu > 0$. Then

$$\mathbb{E}(S_T - S_0 e^k)^+ = S_0 a(k) T + O(T^2)$$

$$a(k) := \int_{-\infty}^{\infty} (e^x - e^k)^+ \nu(x) dx$$

gives the short-expiry behavior of call prices for $k > 0$.

- See Carr-Wu (2003), Figueroa-Lopez/Forde (2012). Intuition:
  For OTM short expiries, need only consider jumps because impact of diffusion decays exponentially in $1/T$.
  Need only consider one jump, because $\mathbb{P}(\text{two jumps}) = O(T^2)$.
- IV expansions have power-of-$|\log T|$ errors as $T \to 0$, because

$$c = |\log T| - \log a(k) + O(T)$$

Is this helpful? It depends. For our goals, $|\log T| \to \infty$ too slowly.
Recap

- Stochastic vol, no jumps, small $T$: scaling with shift $+1/2$ is valid
- With jumps, small $T$: asymptotics too crude to identify shift
Leveraged volatility skew

Continuous dynamics: small $T$

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Numerical results
Lévy with small jumps

Let $X_{1,t}$ be a Lévy process with characteristic exponent $g_1(\varepsilon, \cdot)$

- Let $\sigma > 0$. Assume $\int_{|x|>1} e^{\delta|x|} d\nu(x) < \infty$ for some $\delta > 0$.
  For $n \geq 2$ let $\nu_n := \int x^n d\nu(x)$.

- Let $\beta \neq 0$. If $\beta > 1$ (resp. $\beta < 0$), assume the support of $\nu$ is bounded below (resp. above).

- Let $k \in \{2, 1, 0, -1, \ldots\}$ control high/mod/low jump intensity and

$$g_\beta(\varepsilon, z) := -\beta^2 \sigma^2 (z^2 + i z)/2 + h_\beta(\varepsilon, z)$$

$$h_\beta(\varepsilon, z) := -i a z + \frac{1}{\varepsilon^k} \int_{\mathbb{R}} e^{i z \xi_{\beta, \varepsilon}(x)} - 1 - i z \xi_{\beta, \varepsilon}(x) d\nu(x)$$

$$\xi_{\beta, \varepsilon}(x) := \log(1 + \beta(e^{\varepsilon x} - 1))$$

where $a := \frac{1}{\varepsilon^k} \int_{\mathbb{R}} e^{\xi_{\beta, \varepsilon}(x)} - 1 - \xi_{\beta, \varepsilon}(x) d\nu(x)$. 

Example: Compound Poisson case

We allow infinite variation, but intuition from compound Poisson case:

- Drive $S$ with Brownian $W$, independent Poisson($\lambda/\varepsilon^k$) process $N$.

Let jumps (in log) of size $\varepsilon U_i$ arrive randomly at rate $\lambda/\varepsilon^k$.

In other words, $S_t = \exp(X_t)$ where

$$X_t = -at + \beta \sigma W_t + \sum_{i=1}^{N_t} \log(1 + \beta(\exp(\varepsilon U_i) - 1))$$

where $a := \frac{\lambda}{\varepsilon^k} \beta (\mathbb{E} e^{\varepsilon U} - 1) + \beta^2 \sigma^2 / 2$ makes $S$ driftless.

- Consider the $\varepsilon \to 0$ asymptotic regime (small jumps that arrive at low/moderate/high intensity, depending on $k$).
Option price via characteristic function

Focus on $k = 2$ case, other cases similar. Let

$$IV_0 := |\beta|\sqrt{\sigma^2 + \nu_2}$$

Let $g(0, z) := -(z^2 + iz) IV_0^2 / 2$ and let

$$f(\varepsilon, z) := \exp(iz \log S_0 + T g_\beta(\varepsilon, z))$$

Note that

- $f(\varepsilon, \cdot)$ is the characteristic function of $X_{\beta,T}$
- $f(\cdot, z)$ is well-behaved in an interval $\varepsilon \in [0, \bar{\varepsilon})$

So Taylor-expand option price around the Black-Scholes at $\varepsilon = 0$. Related ideas: Jacquier-Lorig (2012)
Option price via characteristic function

For $\alpha \in (0, \delta)$ the put price is

$$C = \frac{1}{2\pi} \int_{-\infty+i\alpha}^{\infty+i\alpha} f(z) \frac{e^{(1-iz)\log K}}{-z(z + i)} \, dz$$

Therefore $C - C^{BS}(IV_0)$

$$= \frac{1}{2\pi} \int_{-\infty+i\alpha}^{\infty+i\alpha} \varepsilon \frac{\partial f}{\partial \varepsilon}(0, z) \frac{e^{(1-iz)\log K}}{-z(z + i)} \, dz + O(\varepsilon^2)$$

$$= KT \int_{-\infty}^{\infty} i\beta^2 \nu_3 \varepsilon \frac{f(0, z)(\beta z + (2\beta - 3)i)e^{-iz\log K}}{6} \, dz + O(\varepsilon^2)$$

$$= \left( \frac{\log(K/S)}{IV_0^2 T} - \frac{3}{2} + \frac{3}{\beta} \right) \frac{\beta^3 \nu_3 \varepsilon}{6IV_0} S_0 \sqrt{T} N'(d_1) + O(\varepsilon^2)$$

Divide by vega $= S_0 \sqrt{T} N'(d_1)$ to get IV perturbation around $IV_0$. 
Implied volatility

If \( k = 2 \) then implied volatility has asymptotics

\[
IV = IV_0 + \varepsilon IV_1 + O(\varepsilon^2)
\]

where

\[
IV_0 = |\beta| \sqrt{\sigma^2 + \nu_2}
\]

\[
IV_1 = \frac{\beta^3 \nu_3}{6 IV_0^3} \left( \frac{\log(K/S)}{T} + \left( \frac{3}{\beta} - \frac{3}{2} \right) IV_0^2 \right)
\]

Sanity checks:

- \{Level, slope\} of \( IV \) depend on \{second, third\} moments of jump.
- Skewness (standardized 3rd central moment) is consistent with skew (\( \partial IV/\partial k \)), as in Backus-Foresi-Wu 97, Bergomi 16.

\[
\text{Skewness} = \nu_3 T / (IV_0^2 T)^{3/2} = \text{Skew} \times 6 \sqrt{T}
\]
Implied volatility

If $k \leq 0$ then implied volatility has asymptotics

$$IV = IV_0 + \varepsilon^{2-k} IV_{2-k} + \varepsilon^{3-k} IV_{3-k} + O(\varepsilon^{4-k})$$

where

$$IV_0 = |\beta|\sigma$$

$$IV_{2-k} = \frac{\beta^2 \nu_2}{2 IV_0}$$

$$IV_{3-k} = \frac{\beta^3 \nu_3}{6 IV_0^3} \left( \frac{\log(K/S)}{T} + \left( \frac{3}{\beta} - \frac{3}{2} \right) IV_0^2 \right)$$

{Level, slope} of $IV$ depend on {second, third} moments of jump $U$. Further corrections contain higher powers of moneyness.
Implied volatility

If \( k = 1 \) then implied volatility has asymptotics

\[
IV = IV_0 + \varepsilon IV_1 + \varepsilon^2 IV_2 + O(\varepsilon^3)
\]

where

\[
IV_0 = |\beta|\sigma
\]

\[
IV_1 = \frac{\beta^2 \nu_2}{2IV_0}
\]

\[
IV_2 = \frac{\beta^3 \nu_3}{6IV_0^3} \left( \frac{\log(K/S)}{T} + \left( \frac{3}{\beta} - \frac{3}{2} \right) IV_0^2 \right)
\]

\[
+ \frac{\beta^4 \nu_2^2 T}{8IV_0} \left( \frac{[\log(K/S_0)]^2}{IV_0^4T^2} - \frac{1}{IV_0^2T} - \frac{1}{4} \right)
\]

{Level, slope} of \( IV \) depend on \{second, third\} moments of jump \( U \).

Further corrections contain higher powers of moneyness
Scaling and shifting

Applying the same result to reference and leveraged skews (for \( k \neq 1 \)),

\[ IV_L(K_L) = |\beta| IV(K_*) + O(\varepsilon^2) \]

if

\[
\frac{1}{\beta} \log \frac{K_L}{L_0} + \left(3 - \frac{3}{2} \beta\right) \sigma^2 T = \log \frac{K_*}{S_0} + \frac{3}{2} \sigma^2 T
\]

or equivalently

\[
\log \frac{K_*}{S_0} = \frac{1}{\beta} \log \frac{K_L}{L_0} - \frac{3}{2} (\beta - 1) \sigma^2 T
\]

Note the shift coefficient of \(-3/2\).

Still valid with ATM IV in place of \( \sigma \).
Leveraged volatility skew

Continuous dynamics: small $T$

Jump-diffusion dynamics: small $T$

Jump-diffusion dynamics: low/moderate/high jump-intensity

Jump-diffusion dynamics special case: low jump-intensity

Numerical results
Jump-diffusion dynamics

- Let $\sigma > 0$ and let $\beta$ and $c$ satisfy $\beta(e^c - 1) > -1$ (no bust).

- Drive $S$ with Brownian $W$, independent Poisson($\lambda$) process $N$.

  Let jumps (in log) of size $c$ arrive randomly at rate $\lambda$.

In other words, $S_t = S_0 \exp(X_t)$ where

$$dX_t = -a \, dt + \sigma \, dW_t + c \, dN_t$$

where $a := \lambda(e^c - 1) + \sigma^2/2$ makes $S$ driftless.

- Now consider a family of such processes indexed by $\varepsilon$, such that

  $\lambda = \varepsilon/T$ and $c = O(\varepsilon^{1/4})$ as $\varepsilon \to 0$.

- Interpretation of $\varepsilon = \lambda T$: expected number of jumps in $[0, T]$.

- Consider the $\varepsilon \to 0$ asymptotic regime of small jump intensity.
Option price approximation

Price the call by **conditioning on how many jumps** occur in $[0, T]$.

$$C = \mathbb{P}(\text{0 jumps}) \times \mathbb{E}[(S_T - K)^+ | \text{0 jumps}]$$
$$+ \mathbb{P}(\text{1 jump}) \times \mathbb{E}[(S_T - K)^+ | \text{1 jump}]$$
$$+ \mathbb{P}(\text{2+ jumps}) \times \mathbb{E}[(S_T - K)^+ | \text{2+ jumps}]$$
Option price approximation

Price the call by conditioning on how many jumps occur in \([0, T]\).

\[
C = \mathbb{P}(0 \text{ jumps}) \times \mathbb{E}[(S_T - K)^+] \\
+ \mathbb{P}(1 \text{ jump}) \times \mathbb{E}[(S_T - K)^+] \\
+ \mathbb{P}(2+ \text{ jumps}) \times \mathbb{E}[(S_T - K)^+] \\
\quad \leftarrow \text{negligible}
\]

\[
= e^{-\varepsilon} C^{BS} \left( S_0 e^{-\varepsilon(e^c - 1)} \right) + e^{-\varepsilon} \varepsilon C^{BS} \left( S_0 e^{-\varepsilon(e^c - 1)} + c \right) + O(\varepsilon^2)
\]
Option price approximation

Price the call by conditioning on how many jumps occur in \([0, T]\).

\[
C = \mathbb{P}(\text{0 jumps}) \times \mathbb{E}[(S_T - K)^+]|\ 0 \text{ jumps} \\
+ \mathbb{P}(\text{1 jump}) \times \mathbb{E}[(S_T - K)^+]|\ 1 \text{ jump} \\
+ \mathbb{P}(\text{2+ jumps}) \times \mathbb{E}[(S_T - K)^+]|\ 2+ \text{ jumps} \quad \leftarrow \text{negligible}
\]

\[
= e^{-\varepsilon} C^{BS}(S_0 e^{-\varepsilon(e^c - 1)}) + e^{-\varepsilon} \varepsilon C^{BS}(S_0 e^{-\varepsilon(e^c - 1) + c}) + O(\varepsilon^2)
\]

\[
= C^{BS}(S_0) + \varepsilon \left( \frac{1}{2} S_0^2 (e^c - 1)^2 C^{BS}_{SS}(S_0) + \frac{1}{6} S_0^3 (e^c - 1)^3 C^{BS}_{SSS}(S_0) \right) + O(\varepsilon^2)
\]

by third-order (delta-gamma-speed) Taylor expansion around \(S_0\)
So implied volatility has asymptotics

\[ IV = \sigma + \varepsilon IV_1 + O(\varepsilon^2) \]

where price perturbation divided by vega gives the vol perturbation:

\[ IV_1 = \frac{\frac{1}{2} S_0^2 (e^{\gamma} - 1)^2 C_{SS}^{BS}(S_0) + \frac{1}{6} S_0^3 (e^{\gamma} - 1)^3 C_{SSS}^{BS}(S_0)}{C_{\sigma}^{BS}(S_0)} \]
Implied volatility approximation

So implied volatility has asymptotics

\[ IV = \sigma + \varepsilon IV_1 + O(\varepsilon^2) \]

where price perturbation divided by vega gives the vol perturbation:

\[
IV_1 = \frac{\frac{1}{2} S_0^2 (e^c - 1)^2 C_{SS}^{BS}(S_0) + \frac{1}{6} S_0^3 (e^c - 1)^3 C_{SSS}^{BS}(S_0)}{C_{\sigma}^{BS}(S_0)}
\]

\[
= \frac{1}{2\sigma T} (e^c - 1)^2 + \frac{1}{6\sigma T} (e^c - 1)^3 \left( \log\left(\frac{K}{S_0}\right) \frac{\sigma^2 T}{\sigma^2 T} - \frac{3}{2} \right)
\]

using the Black-Scholes vega, gamma, and speed

\[
C_{\sigma}^{BS} = S\sqrt{T}N'(d_1) \quad C_{SS}^{BS} = \frac{N'(d_1)}{S\sigma\sqrt{T}} \quad C_{SSS}^{BS} = \left( -\frac{\log(S/K)}{\sigma^2 T} - \frac{3}{2} \right) \frac{C_{SS}^{BS}}{S}
\]

{Level, slope} of IV depend on {second, third} moments of the jumps
Implied volatility approximation

So implied volatility has asymptotics

\[ IV = \sigma + \varepsilon IV_1 + O(\varepsilon^2) \]

where price perturbation divided by vega gives the vol perturbation:

\[
IV_1 = \frac{1}{2} S_0^2 (e^c - 1)^2 C'^{BS}_{SS}(S_0) + \frac{1}{6} S_0^3 (e^c - 1)^3 C'^{BS}_{SSS}(S_0)
\]
\[
= \frac{1}{2\sigma T} (e^c - 1)^2 + \frac{1}{6\sigma T} (e^c - 1)^3 \left( \frac{\log(K/S_0)}{\sigma^2 T} - \frac{3}{2} \right)
\]

using the Black-Scholes vega, gamma, and speed

\[
C'^{BS}_\sigma = S \sqrt{T} N'(d_1) \quad C'^{BS}_{SS} = \frac{N'(d_1)}{S\sigma \sqrt{T}} \quad C'^{BS}_{SSS} = \left( - \frac{\log(S/K)}{\sigma^2 T} - \frac{3}{2} \right) \frac{C'^{BS}_{SS}}{S}
\]

{**Level, slope**} of \( IV \) depend on {**second, third**} moments of jump
Implied volatility approximation

So implied volatility has asymptotics

\[ IV = \sigma + \varepsilon IV_1 + O(\varepsilon^2) \]

where price perturbation divided by vega gives the vol perturbation:

\[
IV_1 = \frac{\frac{1}{2} S_0^2 (e^c - 1)^2 C_{SS}^{BS}(S_0) + \frac{1}{6} S_0^3 (e^c - 1)^3 C_{SSS}^{BS}(S_0)}{C_{\sigma}^{BS}(S_0)}
\]

\[
= \frac{1}{2\sigma T} (e^c - 1)^2 + \frac{1}{6\sigma T} (e^c - 1)^3 \left( \frac{\log(K/S_0)}{\sigma^2 T} - \frac{3}{2} \right)
\]

using the Black-Scholes vega, gamma, and speed

\[
C_{\sigma}^{BS} = S \sqrt{T} N'(d_1) \quad C_{SS}^{BS} = \frac{N'(d_1)}{S \sigma \sqrt{T}} \quad C_{SSS}^{BS} = \left( - \frac{\log(S/K)}{\sigma^2 T} - \frac{3}{2} \right) \frac{C_{SS}^{BS}}{S}
\]

{Level, slope} of IV depend on {second, third} moments of jump
Implied volatility approximation

So implied volatility has asymptotics

\[ IV = \sigma + \varepsilon IV_1 + O(\varepsilon^2) \]

where price perturbation \textbf{divided by vega} gives the vol perturbation:

\[
\varepsilon IV_1 = \varepsilon \frac{\frac{1}{2} S_0^2 (e^c - 1)^2 C_{SSS}^BS(S_0) + \frac{1}{6} S_0^3 (e^c - 1)^3 C_{SSSS}^BS(S_0)}{C_{\sigma}^BS(S_0)} \\
= \frac{\lambda}{2\sigma} (e^c - 1)^2 + \frac{\lambda}{6\sigma^3} (e^c - 1)^3 \left(\frac{\log(K/S_0)}{T} - \frac{3}{2}\sigma^2\right)
\]

using the Black-Scholes vega, gamma, and speed

\[
C_{\sigma}^BS = S\sqrt{T}N'(d_1) \quad C_{SS}^BS = \frac{N'(d_1)}{S\sigma\sqrt{T}} \quad C_{SSS}^BS = \left(-\frac{\log(S/K)}{\sigma^2T} - \frac{3}{2}\right) \frac{C_{SS}^BS}{S}
\]

Sanity check: \textbf{Skewness} = \frac{\lambda T (e^c - 1)^3}{\sigma^3 T^{3/2}} = 6 \sqrt{T} \times \text{Skew}.
Scaling and shifting

Applying the same result also to the leveraged skew $IV_L$, we obtain

$$ IV_L(K_L) = |\beta| IV(K_*) + O(\varepsilon^2) $$

if

$$ \frac{\log(K_L/L_0)}{\beta \sigma^2 T} - \frac{3}{2} \beta = \frac{\log(K_*/S_0)}{\sigma^2 T} - \frac{3}{2} $$

or equivalently

$$ \log \frac{K_*}{S_0} = \frac{1}{\beta} \log \frac{K_L}{L_0} - \frac{3}{2} (\beta - 1) \sigma^2 T $$

Note the new \textbf{shift coefficient} of $-3/2$.

Still valid with ATM IV in place of $\sigma$.

Still valid for moderate-intensity and high-intensity jumps.
So why do jumps induce shift $-3/2$?

Intuition:

- For continuous dynamics, **second-order** effects suffice
- Introducing **jumps**, we must consider also **third-order** effects.
  (example: when pricing a variance swap in terms of log contracts, jump **skewness** matters)
So why do jumps induce shift $-3/2$?

Intuition:

- For continuous dynamics, second-order effects suffice.
- Introducing jumps, we must consider also third-order effects.
  (example: when pricing a variance swap in terms of log contracts, jump skewness matters)
- Black-Scholes third derivative (speed) includes a $-3/2$ term. It came from differentiating gamma
  
  $$\frac{e^{-d_1^2/2}}{S\sigma\sqrt{T}}$$
  
  thus combining $-1/2$ from $-d_1$ and the $-1$ exponent from $S^{-1}$. Both of which are due to log-normal (or Exp-Lévy) property.
Recap

- Stochastic vol, no jumps, small $T$: scaling with shift $1/2$ is valid
- With jumps, small $T$: asymptotics too crude to identify shift
- Small jumps, fixed $T$: scaling with shift $-3/2$ is valid
Leveraged volatility skew

Continuous dynamics: small $T$

Jump-diffusion dynamics: small $T$

Jump-diffusion dynamics: low/moderate/high jump-intensity

Jump-diffusion dynamics special case: low jump-intensity

Numerical results
Shift $+1/2$ for continuous dynamics

Heston model, expiry 1m, 3m, 6m, with $\beta = 2$
Shift $-3/2$ vs. $+1/2$ in a jump-diffusion example

Expiry 3m, $\sigma = 0.25$, $\lambda = 0.25$, $c = -0.25$, $\beta = 2$
Shift $-3/2$ vs. $+1/2$ in a jump-diffusion example

Expiry 3m, $\sigma = 0.25$, $\lambda = 0.25$, $c = 0.25$, $\beta = -2$
One-month volatility skews on 2014 Oct 20

Reference **SPY** and LETF **SDS** with $\beta = -2$
Conclusions

▶ \( \beta \)-leveraged skew = \(|\beta| \times \) reference skew, at what strike?

     At log-moneyness **scaled** by \( 1/\beta \), plus a **shift**.

▶ We show Zhang’s scaling with shift \([+1/2]\) is valid for small-\( T \)

under **continuous** dynamics; relative error \( O(T^{3/2}) \)

▶ In regimes of **low or high** intensity **jump** risk, our implied

volatility expansions prove a different scaled strike:

\[
\log \frac{K_*}{S_0} = \frac{1}{\beta} \log \frac{K_L}{L_0} - \frac{3}{2} (\beta - 1) \sigma_0^2 T
\]

The **shift coefficient** \(-3/2\) **differs** from Zhang’s \(+1/2\).

▶ For a downward-sloping reference skew and \( \beta \) positive (negative),

using \(+1/2\) will under(over)-estimate the leveraged skew here.
But what about other asymptotic regimes

- We used a stoch vol regime and a small-jump regime to compute implied volatility approximations – *not* because of accuracy (those regimes do not yield most accurate approximations).

- We used them because they reduce complexity, hence increasing the chance of getting a simple scaling/shifting formulas.

- A much more accurate implied vol expansion is in Gao-Lee (2014), which finds implied vol expansions in terms of option prices, to arbitrarily high-order, in *general extreme regimes*.
  - It sharpens small-time vol asymptotics (Roper-Rutkowski 2009).
  - It sharpens large-time vol asymptotics (Tehranchi 2009).
Large-expiry large-strike IV approximation: $T = 5$
Large-expiry large-strike IV approximation: $T = 1$
Large-expiry large-strike IV approximation: $T = 0.25$
Recap

- Stochastic vol, no jumps, small $T$: scaling with shift $1/2$ is valid
- With jumps, small $T$: asymptotics too crude to identify shift
- Small jumps: scaling with shift $-3/2$ is valid
- With jumps, large $T$, large $K$: asymptotics highly accurate, but not simple enough to admit a scaling/shifting rule
Conclusions

- In the large-strike large-expiry regime, we have explicit implied volatility approximations that are highly accurate (vs. exact Lévy model), even at strikes/expiries that are not large.
- We did not use that here. We used various small-jump regimes, due to the tradeoff of accuracy vs. simplicity.
- We always must assume away some details.
  (Life-size map is not very useful)
- Rigorous asymptotics tell us, which pricing rules result from which regimes of assumptions about what to neglect.