Density Bounds and Applications

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This talk is based on joint work with David Baños, Tilmann Blümmel and Julia Eisenberg.
Contents

1. Density Estimates

2. Applications
Formulation of the problem

In this section we always consider an Itô-process

\[ dX(t) = \beta_t \, dt + \sigma_t \, dW(t), \quad X(0) = x_0 \in \mathbb{R}^d \]

where \( W \) is a \( d \)-dimensional standard Brownian motion and \( b, \sigma \) suitable adapted integrands, i.e. \( b \) is an \( \mathbb{R}^d \)-valued progressively measurable and integrable process and \( \sigma \) is an \( \mathbb{R}^{d \times d} \)-valued progressively measurable process with

\[ \int_0^t |\sigma_{ij}(s)|^2 \, ds < \infty. \]
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Questions:

1. Does \( X(t) \) have density \( \rho_t \)? More precisely: are the natural conditions under which the density exist?

2. Is \( \rho_t \) bounded? By what?

3. Is \( \rho_t \) smooth in some sense?
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Example 1

Let $X$ be the unique strong solution of the SDE

$$dX(t) = -\text{sign}(X(t)) \, dt + dW(t)$$

where $W$ is an SBM.

Then $X(t)$ is once but not twice Malliavin differentiable.

The density of this example is known and happens to be bounded and Lipschitz-continuous.
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The density of this example is known and happens to be bounded and Lipschitz-continuous.
Bounded drift, constant diffusion coefficient

Theorem

Let \( X(t) = x + \int_{0}^{t} \beta(s) \, ds + W(t) \) where \( W \) is a \( d \)-dimensional Brownian motion, \( x \in \mathbb{R}^d \) and \( \beta \) a progressively measurable process which is bounded by some constant \( C \geq 0 \).

Then \( X(t) \) has strictly positive, globally \( \alpha \)-Hölder continuous density \( \rho_t(\cdot) \) for any \( \alpha \in (0, 1) \) and for any \( y \in \mathbb{R}^d, z > 0 \):

\[
\rho_t(y) \leq d \prod_{j=1}^{d} \eta_t(|x_j - y_j|) \eta_t(z) := C \int_{t}^{Cz} \rho_{	au}(s) \, ds
\]

If \( d = 1 \), then for any \( y \in \mathbb{R} \) we have:

\[
\max_{\beta} \rho_t(y) = \eta_t(|x - y|).
\]
Bounded drift, constant diffusion coefficient

**Theorem**

Let $X(t) = x + \int_0^t \beta_s ds + W(t)$ where $W$ is a $d$-dimensional Brownian motion, $x \in \mathbb{R}^d$ and $\beta$ a progressively measurable process which is bounded by some constant $C \geq 0$. 

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If $d = 1$, then for any $y \in \mathbb{R}$ we have $max_{\beta} \rho_t(y) = \eta_t(|x - y|)$. 

$\eta_t(z) := C \int_0^C t^2 - s, C(0) \rho_\tau Cz_0(s) ds$. 

$\rho = \sqrt{t} \phi(C \sqrt{t}) + C \Phi(C \sqrt{t})$. 

$\eta_t(0) := 1$.
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$$\rho_t(y) \leq \prod_{j=1}^d \eta_t(|x_j - y_j|)$$

$$\eta_t(0) := \frac{1}{\sqrt{t}} \varphi \left( C\sqrt{t} \right) + C \Phi \left( C\sqrt{t} \right)$$

$$\eta_t(z) := C \int_0^{tC^2} \eta_{tC^2-s,C(0)}(\tau_0 \rho_{tC^2}(s)ds$$
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**Theorem**

Let \( X(t) = x + \int_0^t \beta_s ds + W(t) \) where \( W \) is a \( d \)-dimensional Brownian motion, \( x \in \mathbb{R}^d \) and \( \beta \) a progressively measurable process which is bounded by some constant \( C \geq 0 \).

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\eta_t(z) := C \int_0^{tC^2} \eta_{tC^2-s} \rho_{\tau_0 \sigma^2}(0) \rho_{\tau_0 \sigma^2}(s) ds
\]

If \( d = 1 \), then for any \( y \in \mathbb{R} \) we have \( \max_\beta \rho_t(y) = \eta_t(|x - y|) \).
Corollary

Let $X(t) = x + \int_0^t b(X(s)) \, ds + \int_0^t \sigma(X(s)) \, dW(s)$ where $W$ is a 1-dimensional Brownian motion, $x \in \mathbb{R}$ and $b, \sigma: \mathbb{R} \to \mathbb{R}$ with $b$ bounded, $\sigma$ Lipschitz-continuous, $\sigma(x) > 0$ and $b \sigma - \sigma' \leq C_0$ essentially bounded by some constant $C_0 > 0$.

Then $X(t)$ has strictly positive, globally $\alpha$-Hölder continuous density $\rho_t(\cdot)$ for any $\alpha \in (0, 1)$ and for any $y \in \mathbb{R}$ $\rho_t(y) \leq \eta_t(\left| \int y - x_1 \sigma(u) \, du \right| \sigma(y))$ where $\eta_t$ is as before.
Corollary

Let \( X(t) = x + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s) \) where \( W \) is a 1-dimensional Brownian motion, \( x \in \mathbb{R} \) and \( b, \sigma : \mathbb{R} \to \mathbb{R} \) with \( b \) bounded, \( \sigma \) Lipschitz-continuous, \( \sigma(x) > 0 \) and \( \frac{b}{\sigma} - \frac{\sigma'}{2} \) essentially bounded by some constant \( C > 0 \).
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\[
\rho_t(y) \leq \frac{\eta_t(|\int_x^y \frac{1}{\sigma(u)} \, du|)}{\sigma(y)}
\]

where \( \eta_t \) is as before.
Example 2

Let $X$ be the unique solution of the SDE

$$dX(t) = \sigma(X(t))dW(t),$$

$X(0) = 0$.

where $\sigma(x) = \begin{cases} a & x \leq 0 \\ b & x > 0 \end{cases}$

for some $a, b > 0$. Then $X(t)$ is not Malliavin differentiable and the density of $X(t)$ is given by

$$\rho_t(x) = \begin{cases} \frac{2}{b}a\sqrt{\frac{2}{\pi t}}(a+b)e^{-\frac{x^2}{2ta^2}} & x \leq 0 \\ \frac{2}{a}b\sqrt{\frac{2}{\pi t}}(a+b)e^{-\frac{x^2}{2tb^2}} & x > 0 \end{cases}.$$
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for some \( a, b > 0 \). Then \( X(t) \) is not Malliavin differentiable and the density of \( X(t) \) is given by

\[
\rho_t(x) = \begin{cases} 
    \frac{2b}{a\sqrt{2\pi t(a+b)}} \exp\left(-\frac{x^2}{2ta^2}\right) & x \leq 0, \\
    \frac{2a}{b\sqrt{2\pi t(a+b)}} \exp\left(-\frac{x^2}{2tb^2}\right) & x > 0.
\end{cases}
\]
Bounded drift, bounded elliptic diffusion

Assume that \( d = 1 \), \( \beta \) is bounded by some \( C \geq 0 \), and that \( \sigma \) is \([a, b]\)-valued for some \( 0 < a < b \). Consider \( X(t) = x + \int_0^t \beta(s) \, ds + \int_0^t \sigma(s) \, dW(s) \).

Then \( X(t) \) has a version of the density \( \rho_t \) such that
\[
\int_0^t \rho_s(y) \, ds \leq b^2 a^2 \int_0^t \eta_s(y) \, ds
\]
for any \( t > 0 \) and \( y \in \mathbb{R} \) where \( \eta_t \) is as before.
Theorem

Assume that \( d = 1 \), \( \beta \) is bounded by some \( C \geq 0 \), and that \( \sigma \) is \([a, b]\)-valued for some \( 0 < a < b \). Consider

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for any \( t > 0 \) and \( y \in \mathbb{R} \) where \( \eta_t \) is as before.
Zero drift, irregular diffusion

Lemma

Assume that $d = 1$, $\beta = 0$, $\gamma$ is a globally Lipschitz-continuous function and that $\sigma$ is $[a, b]$-valued for some $0 < a < b$ and $X(t) = x + \int_0^t \sigma_s \gamma(X(t)) \, dW(s)$, $Z(t) = x + \int_0^t b \gamma(Z(t)) \, dW(s)$.

Then $X(t)$ has a version of the density $\rho_t$ such that $\int_0^t \rho_s(y) \, ds \leq b^2 a^2 \int_0^t \eta_s(y) \, ds$ for any $t > 0$ and $y \in \mathbb{R}$ where $\eta_t$ is the continuous density of $Z(t)$.

Otherwise put: For a bounded measurable function $f: \mathbb{R} \to \mathbb{R}^+$ we have $\int_0^t E f(X(s)) \, ds \leq b^2 a^2 \int_0^t E bW(s) \eta_s(y) \, ds$. 
Zero drift, irregular diffusion

Lemma

Assume that $d = 1$, $\beta = 0$, $\gamma$ is a globally Lipschitz-continuous function and that $\sigma$ is $[a, b]$-valued for some $0 < a < b$ and

$$X(t) = x + \int_0^t \sigma_s \gamma(X(t))dW(s), \quad Z(t) = x + \int_0^t b\gamma(Z(t))dW(s)$$
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Then $X(t)$ has a version of the density $\rho_t$ such that

$$\int_0^t \rho_s(y) ds \leq \frac{b^2}{a^2} \int_0^t \eta_s(y) ds$$

for any $t > 0$ and $y \in \mathbb{R}$ where $\eta_t$ is the continuous density of $Z(t)$.
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Assume that $d = 1$, $\beta = 0$, $\gamma$ is a globally Lipschitz-continuous function and that $\sigma$ is $[a, b]$-valued for some $0 < a < b$ and

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Then $X(t)$ has a version of the density $\rho_t$ such that

$$\int_0^t \rho_s(y)ds \leq \frac{b^2}{a^2} \int_0^t \eta_s(y)ds$$

for any $t > 0$ and $y \in \mathbb{R}$ where $\eta_t$ is the continuous density of $Z(t)$. Otherwise put: For a bounded measurable function $f : \mathbb{R} \to \mathbb{R}_+$ we have

$$\int_0^t Ef(X(s))ds \leq \frac{b^2}{a^2} \int_0^t Ef(bW(s))ds.$$
No drift, variable diffusion

Assume that \( d = 1, \beta = 0, \) that \( \sigma \) is \([a, b]\)-valued for some \( 0 < a < b \) and

\[
X(t) = x + \int_0^t \sigma_s dW(s), \quad t \geq 0.
\]
No drift, variable diffusion

Assume that $d = 1$, $\beta = 0$, that $\sigma$ is $[a, b]$-valued for some $0 < a < b$ and

$$X(t) = x + \int_0^t \sigma_s dW(s), \quad t \geq 0.$$ 

Then for any $t > 0$ the r.v. $X(t)$ has a bounded version of its density.
No drift, variable diffusion

Assume that $d = 1$, $\beta = 0$, that $\sigma$ is $[a, b]$-valued for some $0 < a < b$ and

$$X(t) = x + \int_0^t \sigma_s dW(s), \quad t \geq 0.$$ 

Then for any $t > 0$ the r.v. $X(t)$ has a bounded version of its density.

Question: Bounded by what?
Contents

1 Density Estimates

2 Applications
Consider a stochastic volatility model with $dS^\sigma(t) = \sigma(t)S^\sigma(t)dW(t)$, $\sigma$ an $[a, b]$-valued adapted process with $0 < a \leq b$. No interest rate and under the martingale measure.
Consider a stochastic volatility model with \( dS^\sigma(t) = \sigma(t)S^\sigma(t)dw(t) \), \( \sigma \) an \([a, b]\)-valued adapted process with \( 0 < a \leq b \). No interest rate and under the martingale measure.

We have

\[
\int_0^t \mathbb{E}[f(S^\sigma(s))] ds \leq \frac{b^2}{a^2} \int_0^t \mathbb{E}[f(S^b(s))] ds.
\]
Suboptimal stochastic control

Let $A := \{ \beta : \| \beta \| \leq a \}$ and consider the control problem

$$\sup_{\beta \in A} \mathbb{E} \left[ f \left( x + \int_{t}^{T} \beta_s ds + W(T) - W(t) \right) \right] = V(t, x), \quad x \in \mathbb{R}, 0 \leq t \leq T.$$
Suboptimal stochastic control

Let \( A := \{ \beta : \| \beta \| \leq a \} \) and consider the control problem

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\]

Pick feedback strategy \( u \in A \) and assume that

\[
V^u(t, x) := \mathbb{E} \left[ f \left( x + \int_t^T u_s ds + W(T) - W(t) \right) \mid \int_0^t u_s ds + W(t) = x \right]
\]

is in \( C^{1,2} \).
Suboptimal stochastic control

Let $\mathcal{A} := \{\beta : \|\beta\| \leq a\}$ and consider the control problem

$$
\sup_{\beta \in \mathcal{A}} \mathbb{E} \left[ f \left( x + \int_t^T \beta_s ds + W(T) - W(t) \right) \right] = V(t, x), \quad x \in \mathbb{R}, 0 \leq t \leq T.
$$

Pick feedback strategy $u \in \mathcal{A}$ and assume that $V^u(t, x) := \mathbb{E} \left[ f \left( x + \int_t^T u_s ds + W(T) - W(t) \right) \right]$ is in $C^{1,2}$. Then for any other strategy $\beta \in \mathcal{A}$ we have

$$
\mathbb{E}[V^u(T, X^\beta(T))] = V^u(0, x) + \int_0^T \mathbb{E}[V^u_1(t, X^\beta(t)) + V^u_2(t, X^\beta(t))\beta(t)] + \frac{1}{2} dt
$$

$$
= V^u(0, x) + \int_0^T \mathbb{E}[V^u_2(t, X^\beta(t))(\beta(t) - u(t))] dt
$$

$$
\leq V^u(0, x) + 2a \int_0^T \int_{\mathbb{R}} |V^u_2(t, x)| \eta_t(x) dx dt
$$
References

Thank you for your attention!
Related results

Assume that \( c(x) \in \mathbb{R}^d \) is invertible f.a. \( x \in \mathbb{R}^d \) and consider the SDE

\[
dX(t) = b(X(t))dt + c(X(t))dW(t).
\]

- [B. Levy '54, \( d = 1 \)] [Friedman '71, \( d \geq 1 \)] If \( b, c \in C^k \) with all derivatives bounded for some \( k \geq 2 \), then the density of \( X(t) \) exists and it is in \( C^k \).
- [Malliavin '78] If \( b, c \) are in \( C^\infty \) with all derivatives bounded, then the density of \( X(t) \) is in \( C^\infty \) and all its derivatives are bounded.
- [Nualart '89] If \( b, c \) are in \( C^n \) with all those derivatives bounded, then \( X(t) \) is \( n \)-times Malliavin differentiable.
- [Nualart '89] If \( X(t) \) is \( n \)-times Malliavin differentiable, \( t > 0 \) and \( n > d \), then the density of \( X(t) \) is in \( C^{n-d-1} \).
- [Bally, Caramellino '11] If \( X(t) \) is \( 2 \)-times Malliavin differentiable and \( t > 0 \), then \( X(t) \) has \( \alpha \)-Hölder continuous density for any \( \alpha \in (0, 1) \).
- [De Marco '11] If \( b, c \) are measurable, have linear growths and \( b, c \) are \( C^\infty \) on an open set \( U \subseteq \mathbb{R}^d \), then \( X(t) \) has \( C^\infty \) density on \( U \).
- [Baños, Nilssen '15] If \( b \) is Lipschitz continuous and \( c \) is constant, then \( X(t) \) is twice Malliavin differentiable.