Consistency of option prices under bid-ask spreads

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Overview

- Consistency problem: Do given call prices allow for arbitrage?
- Strassen’s theorem and some new extensions
- Application to the consistency problem under bid-ask spreads
The consistency problem

- Given a finite set of call prices
- Is there a model that generates them? Which conditions are needed?
- Carr, Madan (2005): non-negative price of calendar spreads, butterfly spreads
- Davis, Hobson (2007): model-free and model-independent arbitrage
- Cousot (2007): bid-ask spread for option prices
- We: bid-ask spread for options and the underlying
Introduction
The consistency problem

Data (frictionless case)

- Positive deterministic bank account \((B(t))_{t \in \mathcal{T}}, \ B(0) = 1\) (In this talk: usually \(B \equiv 1\))
- Strikes
  \[0 < K_{t,1} < K_{t,2} < \cdots < K_{t,N_t}, \quad t \in \mathcal{T}\]
- Corresponding call option prices (at time zero)
  \[r_{t,i} \geq 0,\]
- Price of the underlying
  \[S_0 > 0\]
Frictionless case

- For each maturity $t$ the linear interpolation $L_t$ of the points $(K_i, r_{t,i})$ has to be convex, decreasing and all slopes of $L_t$ have to be in $[-1, 0]$.
- Intuition: for every random variable $S_t$ the function $K \mapsto \mathbb{E}[(S_t - K)^+]$ has these properties.
Frictionless case: intertemporal conditions

- For all strikes $K_i$ we have that $r_{t,i} \leq r_{t+1,i}$.
- Intuition: for every martingale $S = (S_t)_{t \in \{0, \ldots, T\}}$ the function $t \mapsto \mathbb{E}[(S_t - K)^+]$ is increasing by Jensen’s inequality.
Frictionless case: necessary and sufficient conditions

For all maturities $t$

\[ 0 \geq \frac{r_{t,i+1} - r_{t,i}}{K_{i+1} - K_i} \geq \frac{r_{t,i} - r_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{1, \ldots, N-1\}, \]

and

\[ r_{t,i} = r_{t,i-1} \text{ implies } r_{t,i} = 0, \quad \text{for } i \in \{1, \ldots, N\}. \]

Note that we set $K_0 = 0$ and $r_{t,0} = S_0$ for all $t \in \{1, \ldots, T-1\}$.

For all strikes $K_i$

\[ r_{t,i} \leq r_{t+1,i}, \quad t \in \{1, \ldots, T-1\}. \]

It is possible to state arbitrage strategies if any of these conditions fails.
Frictionless case

- Main tool for the proof: Strassen’s theorem.
- Let $\mu_1$ and $\mu_2$ be two probability measures on $\mathbb{R}$ with finite mean $(\mu_1, \mu_2 \in \mathcal{M})$. Then $\mu_1$ is smaller in convex order than $\mu_2$ ($\mu_1 \leq_c \mu_2$) if

$$\int_{\mathbb{R}} \phi(x) \, d\mu_1(x) \leq \int_{\mathbb{R}} \phi(x) \, d\mu_2(x),$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$.
- It suffices to consider functions $(x - K)^+, K > 0$

Strassen’s Theorem, 1965

Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{M}$. Then there exists a martingale $(M_n)_{n \in \mathbb{N}}$ such that $M_n \sim \mu_n$ if and only if $\mu_s \leq_c \mu_t$ for all $s \leq t$. 
Data (with bid-ask spreads)

- Positive deterministic bank account \((B(t))_{t \in \mathcal{T}}, B(0) = 1\) (In this talk: usually \(B \equiv 1\))

- Strikes

\[0 < K_{t,1} < K_{t,2} < \cdots < K_{t,N_t}, \quad t \in \mathcal{T}\]

- Corresponding call option bid and ask prices (at time zero)

\[r_{t,i} > 0, \quad \bar{r}_{t,i} > 0\]

- Bid and ask of the underlying

\[0 < S_0 \leq \bar{S}_0\]
Bid-ask spread: How to define option payoff?

- Example: Call struck at €100
- bid-ask at maturity: $S_T = €99, \overline{S}_T = €101$. Exercise?
- Yes! Get asset for €1 less than in the market.
- No! Investing €100 gives liquidation value €99.
- Exercise cannot be decided without further assumptions.

- Typical solution in the literature: $\underline{S}_t = (1 - \varepsilon)S_t, \overline{S}_t = (1 + \varepsilon)S_t$, “mid-price” $S_t$ triggers exercise decision, then physical settlement.
Option payoff under bid-ask spreads

- Assume that options are cash-settled, using a reference price $S_t^C$
- Payoff $(S_T^C - K)^+\,$ transferred to bank account
- We do not model a limit order book, and want to avoid ad-hoc definitions of $S_t^C$
- Our approach: Any $S_t^C\,$ within the bid-ask spread will do.
- Fairly weak notion of consistency
Models with bid-ask spreads

- An arbitrage-free model consists of a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and four adapted non-negative processes $S, \overline{S}, S^C, S^*$.

- $S^C$ and $S^*$ evolve in the bid-ask spread:
  \[ S_t \leq S^C_t \leq \overline{S}_t, \quad S_t \leq S^*_t \leq \overline{S}_t \]

- $S^*$ is a martingale

- $S^C$ is not a traded asset, hence $S^C$ does not have to be a martingale.
Models with bid-ask spreads

- **Definition:** The given prices are *consistent with the absence of arbitrage*, if there is an arbitrage-free model with

\[
\mathbb{E}[(S_t^C - K_t,i)^+] \in [\underline{r}_{t,i}, \overline{r}_{t,i}], \quad 1 \leq i \leq N_t, \quad t \in T.
\]

- For each asset (underlying and options), we then have a martingale evolving in its bid-ask spread

Consistency of call prices under bid-ask spreads

- If we allow models where the bid ask can get arbitrarily large than there are no intertemporal conditions.
- For all maturities $t$ the following conditions are then necessary and sufficient for the existence of arbitrage-free models:

  \[ 0 \geq \frac{\bar{r}_{t,i+1} - \bar{r}_{t,i}}{K_{i+1} - K_i} \geq \frac{\underline{r}_{t,i} - \bar{r}_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{2, \ldots, N - 1\}, \]

  and

  \[ \underline{r}_{t,i} = \bar{r}_{t,i-1} \text{ implies } \underline{r}_{t,i} = 0, \quad \text{for } i \in \{2, \ldots, N\}. \]

- Note that the initial bid and ask price of the underlying $(S_0, \bar{S}_0)$ do not appear!
Bounded Bid-Ask Spreads

- We focus on models where the bid-ask spread is bounded by a non-negative constant:
  \[ S_t^+ - S_t^- \leq \epsilon. \]
- We then call the given prices \( \epsilon \)-consistent.
- The option prices allow us to construct measures which correspond to the law of \( S^C \).
- Strassen’s theorem is not applicable anymore since \( S^C \) does not have to be a martingale.
  But, \( S^C \) has to be close to a martingale.
Digression: Extending Strassen’s theorem

Let $d$ be a metric on $\mathcal{M}$ and $\epsilon > 0$.

Formulation 1

Given a sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\mathcal{M}$, when does there exist a martingale $(M_n)_{n \in \mathbb{N}}$ such that

$$d(\mu_n, L M_n) \leq \epsilon, \quad \text{for all } n \in \mathbb{N}?$$

Formulation 2

Given a sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\mathcal{M}$, when does there exist a sequence $(\nu_n)_{n \in \mathbb{N}}$ which is increasing in convex order (peacock) such that

$$d(\mu_n, \nu_n) \leq \epsilon, \quad \text{for all } n \in \mathbb{N}?$$
We solve this problem for different $d$:  

- Infinity Wasserstein distance  
- Modified Prokhorov distance  
- Prokhorov distance, Lévy distance, modified Lévy distance, stop-loss distance
Definitions

- The modified Prokhorov distance with parameter $p \in [0, 1]$ is the mapping $d_p^\mathcal{P} : \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty]$, defined by

$$d_p^\mathcal{P}(\mu, \nu) := \inf\left\{ h > 0 : \nu(A) \leq \mu(A^h) + p, \text{ for all closed sets } A \subseteq \mathbb{R}\right\}$$

where $A^h = \{ x \in S : \inf_{a \in A} |x-a| \leq h \}$.

- The modified Prokhorov distance is not a metric in general

- The infinity Wasserstein distance $W^\infty$ is defined by

$$W^\infty(\mu, \nu) = d_0^\mathcal{P}(\mu, \nu).$$
The infinity Wasserstein distance

- For $\mu \in \mathcal{M}$ and $x \in \mathbb{R}$ we define call function resp. distribution function

$$R_\mu(x) = \int_{\mathbb{R}} (y - x)^+ \mu(dy), \quad F_\mu(x) = \mu((-\infty, x])$$

- $W^\infty$ has the following representation in terms of call functions:

$$W^\infty(\mu, \nu) = \inf \left\{ h > 0 : R'_\mu(x - h) \leq R'_\nu(x) \leq R'_\mu(x + h), \quad \forall x \in \mathbb{R} \right\}$$

$$= \inf \left\{ h > 0 : F_\mu(x - h) \leq F_\nu(x) \leq F_\mu(x + h), \quad \forall x \in \mathbb{R} \right\}$$

- Moreover:

$$W^\infty(\mu, \nu) = \inf \| X - Y \|_\infty,$$

where the inf is over all probability spaces and random pairs with marginals $(\mu, \nu)$
Minimal distance coupling

Theorem (Strassen 1965, Dudley 1968)

Given measures $\mu, \nu$ on $\mathbb{R}$, $p \in [0, 1]$, and $\epsilon > 0$ there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with random variables $X \sim \mu$ and $Y \sim \nu$ such that

$$\mathbb{P}(|X - Y| > \epsilon) \leq p,$$

if and only if

$$d_p^\mathbb{P}(\mu, \nu) \leq \epsilon.$$

Application to consistency: consider models where

$$\mathbb{P}(|S_t^C - S_t^*| > \epsilon) \leq p.$$
Strassen’s theorem for the modified Prokhorov distance

Theorem

Given a sequence \((\mu_n)_{n \in \mathbb{N}}\) in \(\mathcal{M}\), \(p \in (0, 1)\) and \(\epsilon > 0\) there always exists a peacock \((\nu_n)_{n \in \mathbb{N}}\) such that

\[
d^p_p(\mu_n, \nu_n) \leq \epsilon, \quad \text{for all } n \in \mathbb{N}.
\]
Strassen’s theorem for $W^\infty (p = 0)$, Part 1

Let $B^\infty (\mu, \epsilon)$ be the closed ball wrt. $W^\infty$ with center $\mu$ and radius $\epsilon$. Let $\mathcal{M}_m$ be the set of all probability measures on $\mathbb{R}$ with mean $m \in \mathbb{R}$.

Given $\epsilon > 0$, a measure $\mu \in \mathcal{M}$ and $m \in \mathbb{R}$ such that $B^\infty (\mu, \epsilon) \cap \mathcal{M}_m \neq \emptyset$ there exist unique measures $S(\mu), T(\mu) \in B^\infty (\mu, \epsilon) \cap \mathcal{M}_m$ such that

$$S(\mu) \leq_c \nu \leq_c T(\mu) \quad \text{for all } \nu \in B^\infty (\mu, \epsilon) \cap \mathcal{M}_m.$$

The call functions of $S(\mu)$ and $T(\mu)$ are given by

$$R_{\mu}^{\text{min}}(x; m, \epsilon) = R_{S(\mu)}(x) = \left( m + R_{\mu}(x - \epsilon) - (\mathbb{E}_\mu + \epsilon) \right) \lor R_{\mu}(x + \epsilon),$$

$$R_{\mu}^{\text{max}}(x; m, \epsilon) = R_{T(\mu)}(x) = \text{conv} (m + R_{\mu}(\cdot + \epsilon) - (\mathbb{E}_\mu - \epsilon), R_{\mu}(\cdot - \epsilon))(x)$$
**Question**

Given a sequence \((\mu_n)_{n \in \mathbb{N}}\) in \(\mathcal{M}\) and \(\epsilon > 0\) when does there exist a peacock \((\nu_n)_{n \in \mathbb{N}}\) such that

\[
W^\infty (\mu_n, \nu_n) = d_0^P (\mu, \nu) \leq \epsilon, \quad \text{for all } n \in \mathbb{N}?
\]

**Answer:** if and only if

\[
I := \bigcap_{n \in \mathbb{N}} \left[ \mathbb{E} \mu_n - \epsilon, \mathbb{E} \mu_n + \epsilon \right] \neq \emptyset,
\]

and there exists \(m \in I\) such that for all \(N \in \mathbb{N}\), \(x_1, \ldots, x_N \in \mathbb{R}\), we have

\[
R_{\mu_1}^{\min} (x_1; m, \epsilon) + \sum_{n=2}^{N} \left( R_{\mu_n} \left( x_n + \epsilon \sigma_n \right) - R_{\mu_n} \left( x_{n-1} + \epsilon \sigma_n \right) \right) \leq R_{\mu_{N+1}}^{\max} (x_N; m, \epsilon),
\]

where \(\sigma_n = \text{sgn} (x_{n-1} - x_n)\).

If \(\epsilon = 0\) this simplifies to \(R_{\mu_1} (x) \leq R_{\mu_2} (x) \leq \cdots \leq R_{\mu_{N+1}} (x) \leq \cdots\).
Our results on the consistency problem under bid-ask spreads: overview

- Single maturity, spread bounded by $\epsilon$: Necessary and sufficient conditions
- Multiple maturities, spread bounded by $\epsilon$ with probability $1 - p$:
  - Necessary and sufficient conditions. Apply our Strassen-type thm for $d_p$
- Multiple maturities, spread bounded by $\epsilon$:
  - Necessary conditions
  - Necessary and sufficient conditions under simplified assumptions. Apply our Strassen-type thm for $W^\infty$
The following conditions are necessary and sufficient for \( \epsilon \)-consistency \( (S_t - S_t \leq \epsilon) \):

\[
0 \geq \frac{\bar{r}_{t,i+1} - \bar{r}_{t,i}}{K_{i+1} - K_i} \geq \frac{\bar{r}_{t,i} - \bar{r}_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{2, \ldots, N-1\},
\]

and

\( (*) \quad r_{t,i} = \bar{r}_{t,i-1} \) implies \( r_{t,i} = 0, \quad \text{for } i \in \{2, \ldots, N\}. \)

\[
\frac{\bar{r}_{t,2} - \bar{r}_{t,1}}{K_2 - K_1} \geq \frac{\bar{r}_{t,1} - \bar{S}_0}{K_1 - \epsilon} \quad \text{and} \quad \frac{\bar{r}_{t,1} - \bar{S}_0}{K_1 + \epsilon} \geq -1.
\]
Model-independent and weak arbitrage

- Model-independent arbitrage: Arbitrage strategy works for any model.
- Weak arbitrage: For any model, there is an arbitrage strategy (depending on the null sets of the model).
- E.g.: Use a different strategy according to whether $\mathbb{P}(S_T > K) = 0$ or not.
- Terminology from Davis and Hobson (2007)
- If the condition (✱) fails, then there is a weak arbitrage opportunity.
- If any of the other conditions is violated, then there is model-independent arbitrage.
Application of our result on the Prokhorov distance

**Theorem**

Given a sequence \((\mu_n)_{n \in \mathbb{N}}\) in \(\mathcal{M}\), \(p \in (0, 1)\) and \(\epsilon > 0\) there always exists a peacock \((\nu_n)_{n \in \mathbb{N}}\) such that

\[
d^P_p(\mu_n, \nu_n) \leq \epsilon, \quad \text{for all } n \in \mathbb{N}.
\]

**Corollary**

If we allow models where \(\mathbb{P}(\overline{S}_t - \underline{S}_t > \epsilon) \leq p\), for \(p \in (0, 1)\), then the following conditions are necessary and sufficient for the existence of arbitrage-free models:

\[
0 \geq \frac{\overline{r}_{t,i+1} - \overline{r}_{t,i}}{K_{i+1} - K_i} \geq \frac{\underline{r}_{t,i} - \overline{r}_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{2, \ldots, N_t - 1\},
\]

and

\[
\overline{r}_{t,i} = \overline{r}_{t,i-1} \text{ implies } \underline{r}_{t,i} = 0, \quad \text{for } i \in \{2, \ldots, N_t\}.
\]
Corollary: proof idea

- Necessity: From our result on unbounded bid-ask spread
- Sufficiency: Get peacock from theorem. Yields processes $S^*, S^C$ with

$$
\mathbb{P}( |S_t^* - S_t^C| \geq \epsilon ) \leq p.
$$

Define

$$
\underline{S}_t = S_t^* \land S_t^C \quad \text{and} \quad \overline{S}_t = S_t^* \lor S_t^C.
$$
ε-consistency

- What about applying our main extension of Strassen's theorem, the one with $W^\infty$?
- Should be useful for constructing models with $\overline{S}_t - \underline{S}_t \leq \epsilon$
- Necessary and sufficient conditions seem to be difficult to find (see next slide)
- We found necessary and sufficient conditions under simplified assumptions
Necessary Conditions for multiple maturities

- If we restrict ourselves to models where $\mathbb{P}(\bar{S}_t - S_t > \epsilon) = 0$ then we get the following intertemporal conditions:

- If $K_i + \epsilon < K_j - \epsilon\sigma_s < K_l + \epsilon$, $s \leq t$ and $s \leq u$ then the following conditions are necessary:

$$
\frac{\bar{r}_s^{CVB}(\sigma_s, K_j) - \bar{r}_{t,i}}{(K_j - \epsilon\sigma_s) - (K_i + \epsilon)} \leq \frac{\bar{r}_{u,l} - \bar{r}_s^{CVB}(\sigma_s, K_j)}{K_l + \epsilon - (K_s - \epsilon\sigma_s)},
$$

$$
\frac{\bar{r}_s^{CVB}(\sigma_s, K_j) - \bar{r}_{t,i}}{(K_j - \epsilon\sigma_s) - (K_i + \epsilon)} \leq 0, \quad \text{and}
$$

$$
\frac{\bar{r}_{u,l} - \bar{r}_s^{CVB}(\sigma_s, K_j)}{K_l + \epsilon - (K_s - \epsilon\sigma_s)} \geq -1
$$

where

$$
\bar{r}_s^{CVB} = \bar{r}_{1,j_1} + \sum_{t=2}^{s} (\bar{r}_{t,j_t} - \bar{r}_{t,i_{t-1}}) + 2\epsilon \mathbf{1}_{\sigma_1=-1}.
$$
Conclusion

- If there are no transaction costs on the underlying then necessary sufficient conditions can be derived from Strassen’s theorem (Carr and Madan 2005, Davis and Hobson 2007).
- If there is no bound on the bid-ask spread on the underlying, then there are no intertemporal conditions, and there is no relation between option prices and price of the underlying.
- If the bid-ask spread satisfies some boundedness conditions, we can apply our generalizations of Strassen’s theorem to derive consistency conditions.
References

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- Gerhold, Gülüm: *A variant of Strassen’s theorem: Existence of martingales within a prescribed distance.* Preprint
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