Loan Portfolio Risk and Optimization

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Financial Crisis

- Unprecedented numbers of mortgage defaults led to the 2008 financial crisis.

- In the aftermath of the financial crisis, there is a pressing need for new models and computational methods for risk analysis of mortgages and other loans.

- Challenge: pools of loans are very large
How Large Can Pools be in Practice?

- Mortgage-backed securities (MBS) typically have thousands to hundreds of thousands of mortgages.
- Fannie Mae and Freddie Mac have credit exposure to 25 million mortgages.
- Major banks can have credit exposure to 10 million mortgages.
- Banks need to price thousands of mortgage-backed securities and hundreds of collateralized mortgage obligations (CMOs) on a daily basis.
- A credit card asset-backed security (ABS) can have tens of millions of credit cards.
Goals

1. Efficient computation of distribution of default and prepayment rates for pools of loans

2. Large-scale loan portfolio optimization
Loan-by-loan modeling of such large pools is very computationally expensive!
- **hours, days, or even weeks**

Instead, rating agencies, banks, and investors often used simplistic approaches relying only upon average features of pools (e.g., average credit score).

Pool-level characteristics can lead to inaccurate results due to ignoring the full loan-level distribution!
Why is Loan-level Analysis Needed?

- Loan-to-value (LTV) ratio = \( \frac{\text{size of loan}}{\text{value of house}} \times 100 \% \).

<table>
<thead>
<tr>
<th>LTV ratio</th>
<th>Default Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 %</td>
<td>3.1 %</td>
</tr>
<tr>
<td>50 %</td>
<td>3.3 %</td>
</tr>
<tr>
<td>90 %</td>
<td>17.9 %</td>
</tr>
</tbody>
</table>

- Pool A only has mortgages with LTV ratio 50 \%.

- Pool B has half its mortgages with LTV ratio 10 \% and half with LTV ratio 90 \%.

<table>
<thead>
<tr>
<th>Average LTV ratio</th>
<th>Default Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pool A</td>
<td>50 %</td>
</tr>
<tr>
<td>Pool B</td>
<td>50 %</td>
</tr>
</tbody>
</table>
An Efficient Monte Carlo Approximation
Asymptotically Optimal Portfolios

35% difference!

42% difference!
Example MBS pools

Distribution of LTV ratios
Mean LTV ratio
Prove weak convergence results for a broad class of models:

1. **Efficient Monte Carlo approximation** for the distribution of default and prepayment in loan pools
2. **Asymptotically optimal portfolio** (AOP) for large-scale optimization of loan portfolios

2. Numerical tests with actual mortgage data: Monte Carlo approximation is typically several orders of magnitude faster than brute-force simulation (at a similar level of accuracy).

3. Similar computational advantages using AOP.
Model Framework

The probability that the \( n \)-th loan transitions from its state \( U_{t-1}^n \) at time \( t - 1 \) to state \( u \) at time \( t \):

\[
P_\theta[U_t^n = u | \mathcal{F}_{t-1}] = h_\theta(u, U_{t-1}^n, Y^n, V_{t'<t}, H_{t'<t}^N)
\]

- \( U_t^n \in \mathcal{U} \) is the state of the \( n \)-th mortgage at time \( t \) (e.g., default, prepaid, or outstanding).
- \( Y^n \in \mathcal{Y} \) are loan-level features of the \( n \)-th mortgage (FICO score, LTV ratio, geographic location, etc.).
- \( V_t \) is a vector of common factors (such as national mortgage rate and unemployment rate).
- “Mean-field” process \( H_t^N = \frac{1}{N} \sum_{n=1}^{N} f(U_t^n, Y^n) \) (e.g., contagion)
Goal: once a model has been fitted, analyze risk for a pool of $N$ loans

The distribution of default and prepayment rates in the pool can be found via **brute-force simulation** of loans 1, \ldots, $N$.

For large pools of loans, brute-force simulation is very computationally expensive!

- hours, days, or even weeks
Some Previous Literature

- Parallel computing, e.g., Stein et al. (2007)
- Top-down models, e.g., Fermanian (2008)
- Top-down models with pool-level characteristics, e.g., Roll (1989)
- Limiting laws for default timing models
  - Bush, Hambly, Haworth, Jin, and Reisinger (2011)
  - Cvitanic, Ma, and Zhang (2012)
  - Giesecke, Spiliopoulos, Sowers, and Sirignano (2012)
  - Spiliopoulos, Sirignano, and Giesecke (2014)
Define the empirical measure:

\[ \mu_t^N = \frac{1}{N} \sum_{n=1}^{N} \delta(U_t^n, Y^n). \]

**Theorem**

The empirical measure \( \mu^N \xrightarrow{d} \bar{\mu} \) as \( N \rightarrow \infty \), where \( \bar{\mu} \) satisfies:

\[ \bar{\mu}_t(u, dy) = \sum_{u' \in \mathcal{U}} h_{\theta}(u, u', y, V_{t'<t}, \bar{H}_{t'<t}) \bar{\mu}_{t-1}(u', dy), \]

and \( \bar{H}_t = \sum_{u \in \mathcal{U}} \int_{\mathbb{R}^d} f(u, y) \bar{\mu}_t(u, dy). \)
Central Limit Theorem

Define the empirical fluctuation measure:

\[ \Xi^N_t = \sqrt{N}(\mu^N_t - \bar{\mu}_t) \]

Theorem

\[ \Xi^N \xrightarrow{d} \Xi \quad \text{as} \quad N \rightarrow \infty, \quad \text{where} \quad \Xi \quad \text{satisfies:} \]

\[ \Xi_t(u, dy) = \sum_{u' \in \mathcal{U}} h_\theta(u, u', y, V_{t' < t}, H_{t' < t}) \Xi_{t-1}(u', dy) \]

\[ + \sum_{u' \in \mathcal{U}} \left( \frac{\partial h_\theta}{\partial H}(u, u', y, V_{t' < t}, H_{t' < t}) \cdot E_{t' < t} \right) \bar{\mu}_{t-1}(u', dy) \]

\[ + \tilde{M}_t(u, dy), \]

where \[ \bar{E}_t = \sum_{u \in \mathcal{U}} \int_{\mathbb{R}^d} f(u, y) \Xi_t(u, dy). \]
Large Pool Approximation

- The law of large numbers and central limit theorem can be combined to form an approximation for a finite pool of $N$ mortgages:

$$\mu^N(u, dy) \approx \bar{\mu}(u, dy) + \frac{1}{\sqrt{N}} \Xi(u, dy).$$

- The approximation is conditionally Gaussian
  - easy to simulate

- Problem: curse of dimensionality when $\mathcal{Y}$ is high-dimensional.
  - sparse grids
  - low-dimensional transformation
Loan-level Data

1. Freddie Mac data set
   - 16 million prime mortgages
   - loan-level data

2. RMBS data set
   - 10 million subprime mortgages backing over 6000 MBSs
   - loan-level data
   - unique identifier for each MBS
Figure: Comparison of actual distribution with approximate distribution (using both LLN and CLT). Loss reported as fraction of pool which defaulted. The horizon is 12 months.
Ten Actual MBS pools

Figure: Comparison of actual distribution to model distribution.
Figure: Distribution of error for 99% VaR from the efficient Monte Carlo approximation across 185 actual MBS pools. The time horizon is 12 months.
### One-year time horizon

<table>
<thead>
<tr>
<th>$N$</th>
<th>Time for Brute-force Simulation</th>
<th>Time for Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,000</td>
<td>44.3 seconds</td>
<td>2.7 seconds</td>
</tr>
<tr>
<td>5,000</td>
<td>2.6 minutes</td>
<td>2.7 seconds</td>
</tr>
<tr>
<td>10,000</td>
<td>4.6 minutes</td>
<td>2.7 seconds</td>
</tr>
<tr>
<td>25,000</td>
<td>10.1 minutes</td>
<td>2.7 seconds</td>
</tr>
<tr>
<td>100,000</td>
<td>47.5 minutes</td>
<td>2.7 seconds</td>
</tr>
<tr>
<td>1,000,000</td>
<td>7.9 hours</td>
<td>2.7 seconds</td>
</tr>
<tr>
<td>10,000,000</td>
<td>79.3 hours</td>
<td>2.7 seconds</td>
</tr>
</tbody>
</table>

**Table**: Comparison of computational times (seconds) for efficient Monte Carlo approximation and brute-force Monte Carlo simulation of the pool. 1-year time horizon.
Exploit weak convergence results for optimization

- How to optimally select $N$ loans for a portfolio?

- Optimal selection of a loan portfolio is a high-dimensional nonlinear integer program.
  
  - High-dimensional: $N$ can be large
  
  - Can only choose 0 or 1 of a loan $\rightarrow$ integer program
  
  - Objective and constraint functions are nonlinear (possibly nonconvex).
  
  - Objective and constraint functions can be computationally expensive to evaluate!
Empirical measure of the loans: \( \mu^N = (\mu^N_t)_{t=1,\ldots,T} \)

“Performance measure”: \( R^N_P = f(\mu^N, V) \)

Example: \( R^N_P \) is the return of the portfolio \( P \) of \( N \) loans.

Optimization problem:

\[
P^N,^* = \arg \min_{P^N} \mathbb{E}[g(R^N_{P^N})]
\]

\[
\text{s.t. } \mathbb{E}[\phi(R^N_{P^N})] \geq c,
\]

\[
q(P^N) \leq d.
\]

where \( P^N = (y^1, \ldots, y^N) \in \mathcal{Y} \).
Approximate empirical measure of the loans:

\[ \mu^N = (\mu^N_t)_{t=1,\ldots,T} \approx \bar{\mu}^N = (\bar{\mu}^N_t)_{t=1,\ldots,T} \]

Approximate “performance measure”:

\[ R_N^P = f(\mu^N, V) \approx \bar{R}_N^P = f(\bar{\mu}^N, V) \]

The portfolio choice can be equivalently be written as:

\[ P^N = \frac{1}{N} \sum_{n=1}^{N} \delta_{y^n} \]
1. “True” optimal portfolio:

\[ P_{N,*} = \arg \min_{P_N} \mathbb{E}[g(R_{PN})] \]
\[ \text{s.t. } \mathbb{E}[\phi(R_{PN})] \geq c, \]
\[ q(P^N) \leq d. \]

2. Asymptotically optimal portfolio (AOP):

\[ \bar{P}_{N,*} = \arg \min_{P \in \mathcal{M}(\mathcal{Y})} \mathbb{E}[g(\bar{R}_P)], \]
\[ \text{s.t. } \mathbb{E}[\phi(\bar{R}_P)] \geq c, \]
\[ q(P) \leq d. \]
The asymptotically optimal portfolio $\bar{P}^{N,*}$ converges to the true optimal portfolio $P^{N,*}$ as $N \to \infty$ in $\left(\mathcal{M}(\mathcal{Y}), \pi\right)$, where $\pi$ is the Prokhorov metric.

$$\pi\left(P^{N,*}, \bar{P}^{N,*}\right) \xrightarrow{N \to \infty} 0.$$
Instead of solving a very computationally challenging nonlinear integer program, solve for the asymptotically optimal portfolio (AOP)!

Compare computational performance of AOP with integer program solvers
- MBS equity tranche
- Mean-variance portfolio
- Log-optimal portfolio
Select $N = 250$ loans out of a pool of $N_p = 1000$ loans for portfolio which maximizes expected return of MBS equity tranche:

<table>
<thead>
<tr>
<th>Solver</th>
<th>Time</th>
<th>Exitflag</th>
<th>True objective</th>
<th>Agreement with AOP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integer Program</td>
<td>35 min</td>
<td>Maxtime</td>
<td>.05949</td>
<td>99.4 %</td>
</tr>
<tr>
<td>AOP</td>
<td>1 s</td>
<td>Min stepsize</td>
<td>.05952</td>
<td>100 %</td>
</tr>
</tbody>
</table>

Table: Performance comparison between integer program solvers and AOP. One-period model.
Select $N = 2,500$ loans out of a pool of $N_p = 10,000$ loans for portfolio which maximizes expected return of MBS equity tranche:

<table>
<thead>
<tr>
<th>Solver</th>
<th>Time</th>
<th>Exitflag</th>
<th>True objective</th>
<th>Agreement with AOP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integer Program</td>
<td>5.4 hours</td>
<td>Maxtime</td>
<td>.05856</td>
<td>99.4 %</td>
</tr>
<tr>
<td>AOP</td>
<td>1 s</td>
<td>Min stepsize</td>
<td>.05857</td>
<td>100 %</td>
</tr>
</tbody>
</table>

Table: Performance comparison between integer program solvers and AOP.
Select $N = 250$ out of a pool of $N_p = 1,000$ loans for mean variance portfolio:

**Figure:** AOP computational time: 29 seconds. Integer program computational time: 1 hr 28 min. Solutions differ on 18/1000 loans.
Select 250 loans from 1,000 subprime mortgages for log-optimal portfolio:

![Figure: AOP computational time: 34 seconds. Integer program computational time: 39 min. Solutions differ on 20/1000 loans.](image)
Summary

1. Prove weak convergence results for a broad class of models:
   - Efficient Monte Carlo approximation for the distribution of default and prepayment in loan pools
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2. Numerical tests with actual mortgage data: Monte Carlo approximation is typically several orders of magnitude faster than brute-force simulation (at a similar level of accuracy).

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