STATISTICAL METHODS IN FINANCE
ASSIGNMENT 3 – SOLUTIONS

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Solution (Exercise 1: Confidence intervals)**.**

(i) Since each X_i is distributed as $\mathcal{N}(\mu, \sigma^2)$, then the confidence interval of level α is given by (see the lecture notes)

$$\left[\overline{X}_n - \frac{\sigma}{\sqrt{n}}q_{1-\alpha/2}, \overline{X}_n + \frac{\sigma}{\sqrt{n}}q_{1-\alpha/2}\right] = \left[\frac{b}{n} - \frac{\sigma}{\sqrt{n}}q_{1-\alpha/2}, \frac{b}{n} + \frac{\sigma}{\sqrt{n}}q_{1-\alpha/2}\right].$$

(ii) See notebook. The interval for $\sigma = 20\%$ is

$$[9.912, 10.088].$$

Solution (Exercise 2: Neyman-Pearson)**.** *See Lecture notes for the detailed solution.*

Solution (Exercise 3: Hypothesis testing 1)**.**

(1) Consider the case $n = 2$, and recall that X_i is an Exponential random variable with parameter $\lambda_i > 0$ if its probability density function reads

$$f_{X_i}(x) = \lambda_i \mathrm{e}^{-\lambda_i x} \mathbb{1}_{\{x \geq 0\}}.$$

The density of the sum $X_1 + X_2$ corresponds to a convolution and reads, if $\lambda_1 \neq \lambda_2$

$$\begin{aligned}
f_{X_1+X_2}(x) &= \int_{\mathbb{R}} f_{X_1}(z) f_{X_2}(x-z)\mathrm{d}z \\
&= \lambda_1\lambda_2 \int_0^x \mathrm{e}^{-\lambda_1 z}\mathrm{e}^{-\lambda_2(x-z)}\mathrm{d}z \\
&= \lambda_1\lambda_2 \mathrm{e}^{-\lambda_2 x}\int_0^x \mathrm{e}^{(\lambda_2-\lambda_1)z}\mathrm{d}z \\
&= \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}\mathrm{e}^{-\lambda_2 x}\left(\mathrm{e}^{(\lambda_2-\lambda_1)x} - 1\right) \\
&= \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}\left(\mathrm{e}^{-\lambda_1 x} - \mathrm{e}^{-\lambda_2 x}\right).
\end{aligned}$$

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In the case where $\lambda_1 = \lambda_2$, an analogous computation yields

$$\begin{aligned} f_{X_1+X_2}(x) &= \int_{\mathbb{R}} f_{X_1}(z) f_{X_2}(x-z)\mathrm{d}z \\ &= \lambda^2 \int_0^x \mathrm{e}^{-\lambda z}\mathrm{e}^{-\lambda(x-z)}\mathrm{d}z \\ &= \lambda^2 \int_0^x \mathrm{e}^{-\lambda x}\mathrm{d}z = \lambda^2 x \mathrm{e}^{-\lambda x}. \end{aligned}$$

We can then proceed by recursion and show that the sum $\sum_{i=1}^n X_i$ follows a Gamma distribution with parameters n and λ.

(2) *The joint density $\boldsymbol{X} = (X_1, \ldots, X_n)$, in turn, is equal to*

$$f(\boldsymbol{x}, \lambda) = \lambda^n \exp\left\{-\lambda \sum_{i=1}^n x_i\right\}, \qquad \text{for any } \boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

(3) *We can base our test on the statistics*

$$T := \frac{f(\boldsymbol{X}, \lambda_1)}{f(\boldsymbol{x}, \lambda_0)} = \left(\frac{\lambda_1}{\lambda_0}\right)^n \exp\left\{-(\lambda_1 - \lambda_0)\sum_{i=1}^n X_i\right\}.$$

Consider the case where $\lambda_0 < \lambda_1$ (a symmetric analysis can be carried out otherwise). The statistic T is a decreasing function of $S_n := \sum_{i=1}^n X_i$, so that $T \geq c$ if and only if $S_n \leq c'$. We can choose c' such that

$$\mathbb{P}_{\lambda_0}[S_n \leq c'] = \alpha.$$

Under the null hypothesis, S_n is Gamma distributed, and we can therefore compute c' directly, for each λ_0 and each α using the quantiles of this distribution.

Solution (Exercise 4: Hypothesis testing 2)**.**

(1) *The joint density of $\boldsymbol{X}$ can be written as*

$$f(\boldsymbol{x}, \theta) = \frac{1}{\theta^n} \mathbb{1}_{\max\{x_1, \ldots, x_n\} \leq \theta}.$$

This follows directly from the definition of joint distributions.

(2) *We assume here that $\theta_1 < \theta_0$. We can construct a test statistic of the form*

$$T = \frac{f(\boldsymbol{X}, \theta_1)}{f(\boldsymbol{X}, \theta_0)} = \left(\frac{\theta_0}{\theta_1}\right)^n \mathbb{1}_{\max\{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n\} \leq \theta_1}.$$

T can only take two values, zero or $(\theta_0/\theta_1)^n$. The test therefore rejects the null hypothesis when $\max\{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n\} \leq \theta_1$ *with level*

$$\alpha = \mathbb{P}_{\theta_0}[\max\{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n\} \leq \theta_1] = \left(\frac{\theta_0}{\theta_1}\right)^n,$$

and the power of the test reads

$$\mathbb{P}_{\theta_1}[\max\{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n\} \leq \theta_1] = 1.$$