STATISTICAL METHODS IN FINANCE, ASSIGNMENT 2

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Solution (Performance of estimators)**.**

Solution (Method of moments)**.** *(i) We can write*

$$\mathbb{E}[Z] = \mathbb{E}\left[c + \mathrm{e}^X\right] = c + \mathbb{E}\left[\mathrm{e}^X\right].$$

Now, since $X \sim \mathcal{N}(\mu, \sigma^2)$, *we have*

$$\begin{aligned}
\mathbb{E}\left[\mathrm{e}^X\right] &= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^x \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \mathrm{d}x \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left\{-\frac{-2\sigma^2 x + x^2 + \mu^2 - 2x\mu}{2\sigma^2}\right\} \mathrm{d}x \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left\{-\frac{[x - (\mu+\sigma^2)]^2 - (\mu+\sigma^2)^2 + \mu^2}{2\sigma^2}\right\} \mathrm{d}x \\
&= \exp\left\{\frac{(\mu+\sigma^2)^2 - \mu^2}{2\sigma^2}\right\} = \exp\left\{\frac{\sigma^2}{2} + \mu\right\}.
\end{aligned}$$

We further have

$$\mathbb{E}\left[Z^2\right] = \mathbb{E}\left[\left(c + \mathrm{e}^X\right)^2\right] = \mathbb{E}\left[c^2 + \mathrm{e}^{2X} + 2c\mathrm{e}^X\right] = c^2 + \mathbb{E}\left[\mathrm{e}^{2X}\right] + 2c\mathbb{E}\left[\mathrm{e}^X\right],$$

and

$$\mathbb{E}\left[Z^3\right] = c^3 + 3c^2\mathbb{E}\left[\mathrm{e}^X\right] + 3c\mathbb{E}\left[\mathrm{e}^{2X}\right] + \mathbb{E}\left[\mathrm{e}^{2X}\right].$$

Now, analogous computations to above yield

$$\mathbb{E}\left[\mathrm{e}^{2X}\right] = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{2x} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \mathrm{d}x = \exp\left\{2\left(\mu + \sigma^2\right)\right\}$$

and

$$\mathbb{E}\left[\mathrm{e}^{3X}\right] = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{2x} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \mathrm{d}x = \exp\left\{3\mu + \frac{9\sigma^2}{2}\right\}.$$

We therefore obtain the first three moments of Z.

(ii) Solving the method of moments system using the first three moments above is in fact not trivial, but can be solved easily numerically.

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Solution (Degenerate Likelihood). *The function f is clearly positive, and it is easy to check that it integrates to one. It is not defined at the origin, though. The log-likelihood function reads*

$$l_n(\theta) = -n\log(6) - \frac{1}{2}\sum_{i=1}^{n}\log(|X_i-\theta|)\mathbb{1}_{(0,1]}(|X_i-\theta|) - 2\sum_{i=1}^{n}\log(|X_i-\theta|)\mathbb{1}_{(1,\infty)}(|X_i-\theta|).$$

It is clear that it diverges to $+\infty$ as soon as θ approaches any X_i, and so there is no maximum likelihood estimator. It is also easy to see that $f \notin L^1((0,\infty))$, so that the expectation does not exist, and the method of moments also fails. However, the cumulative distribution function corresponding to the density f can be written

$$F(x) = \begin{cases} -\dfrac{1}{6x}, & \text{if } x \le 1,\\ \dfrac{1}{2} - \dfrac{\sqrt{-x}}{3}, & \text{if } x \in [-1,0],\\ \dfrac{1}{2} + \dfrac{\sqrt{x}}{3}, & \text{if } x \in [0,1],\\ 1 - \dfrac{1}{6x}, & \text{if } x \ge 1. \end{cases}$$

It is clear that this function is continuous, bijective and the median is zero, so that the median of the random variable corresponding to the density f_θ is equal to θ. Therefore, an estimator of θ is simply $X_{(\lceil n/2 \rceil)}$.

Solution (Maximum Likelihood).

(i) *This is standard analysis of infinite series.*

(ii) *Let $\overline{X}_n := \frac{1}{n}\sum_{i=1}^n X_i$ denote the empirical average. Given the previous result, we deduce that*

$$\widehat{\theta}_n = \frac{\overline{X}_n}{2+\overline{X}_n}.$$

Clearly, $\widehat{\theta}_n < 1$, but we also have

$$\mathbb{P}_\theta\left(\widehat{\theta}_n = 0\right) = (1-\theta)^{2n} > 0.$$

(iii) *The log-likelihood function reads*

$$l_n(\theta) := -\sum_{i=1}^{n}\log\mathbb{P}_\theta(X=x_i) = -2n\log(1-\theta) - n\overline{X}_n\log(\theta) - \sum_{i=1}^{n}\log(1+X_i),$$

and

$$\partial_\theta l_n(\theta) = \frac{2n}{1-\theta} - \frac{n\overline{X}_n}{\theta}.$$

The derivative is negative if and only if $\theta \le \widehat{\theta}_n$. Therefore, when $\widehat{\theta}_n > 0$, then the likelihood function attains its maximum on $(0,1)$ at the point $\widehat{\theta}_n$ and the maximum likelihood estimator is equal to $\widehat{\theta}_n$. If $\widehat{\theta}_n = 0$ (which happens with strictly positive probability, as mentioned previously), then the likelihood function is strictly decreasing and the estimator is not well defined.

(iv) *The weak law of large numbers implies that $\overline{X}_n$ converges almost surely to $\mathbb{E}[X_1]$, so that $\widehat{\theta}_n$ converges almost surely to θ, and $\widehat{\theta}_n$ is consistent. The Central Limit Theorem, in turn, yields that*

$$\sqrt{n}\left(\overline{X}_n - \frac{2\theta}{1-\theta}\right) \to \mathcal{N}\left(0, \frac{2\theta}{(1-\theta)^2}\right),$$

and hence

$$\sqrt{n}\left(\widehat{\theta}_n - \theta\right) \to \mathcal{N}\left(0, \frac{\theta(1-\theta)^2}{2}\right).$$