STATISTICAL METHODS IN FINANCE, ASSIGNMENT 2

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Exercise 1 (Performance of estimators)**.** We consider the statistical model $\mathcal{F} = \left\{\mathcal{N}(\mu, \sigma^2), \sigma^2 > 0\right\}$, for some fixed known $\mu \in \mathbb{R}$. The goal of this exercise is to determine an optimal (in some sense) estimator of the second moment σ^2. Consider a given iid sequence $(X_i)_{i=1,\dots,n}$, sampled from $\mathcal{N}(\mu, \sigma^2)$, define, for any $\alpha_n > 0$, the estimator $\widehat{\theta}_n$ as

$$\widehat{\theta}_n := \alpha_n \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2,$$

where $\overline{X} := \frac{1}{n}\sum_{i=1}^{n} X_i$ denotes the empirical mean of the sample.

(i) For which values of α_n is the sequence $(\widehat{\theta}_n)_{n\in\mathbb{N}}$ consistent?

(ii) By considering the quadratic error

$$R_n\left(\widehat{\theta}_n, \theta\right) := \mathbb{E}\left[\left|\widehat{\theta}_n - \theta\right|^2\right],$$

determine the value of α_n providing the most efficient estimator.

(iii) Consider now a different loss function, namely the so-called Stein's loss function

$$L\left(\widehat{\theta}_n, \theta\right) := \frac{\widehat{\theta}_n}{\theta} - 1 - \log\left(\frac{\widehat{\theta}_n}{\theta}\right).$$

Find the value of α_n minimising $L\left(\widehat{\theta}_n, \theta\right)$. How does this estimator compare to the previous one?

Exercise 2 (Method of moments)**.** We consider the model $Z = c + \mathrm{e}^X$, where $X \sim \mathcal{N}(\mu, \sigma^2)$.

(i) Compute $\mathbb{E}[Z]$, $\mathbb{E}[Z^2]$ and $\mathbb{E}[Z^3]$.

(ii) Deduce a Method of Moments estimator for the triplet (c, μ, σ^2).

Date: October 26, 2019.

Exercise 3 (Degenerate Likelihood)**.** Consider the function

$$f(x) = \frac{1}{6}\left(\frac{\mathbb{1}_{(0,1]}(|x|)}{\sqrt{|x|}} + \frac{\mathbb{1}_{(1,\infty)}(|x|)}{x^2}\right). \tag{0.1}$$

Show that f is a genuine density function on $\mathbb{R}$. Consider now the family $(f_\theta)_{\theta\in\mathbb{R}}$ obtained by translation $f_\theta(x) := f(x-\theta)$ for all $x \neq \theta$, which corresponds to the common distribution of an iid sample $(X_1, \ldots, X_n)$.

(i) Compute the log-likelihood function, and discuss the existence of any maximum likelihood estimator.
(ii) What about an estimator obtained by the method of moments?
(iii) Compute the cdf corresponding to f in (0.1) and show that the median is null.
(iv) By translation, determine an estimator for θ.

Exercise 4 (Maximum Likelihood)**.** Let $\theta \in (0,1)$ be the unknown parameter and consider the random variable X satisfying, for any integer n,

$$\mathbb{P}_\theta(X = n) = (n+1)(1-\theta)^2\theta^n.$$

(i) Show that, for any $\theta \in (0,1)$,
$$\mathbb{E}[X] = \frac{2\theta}{1-\theta} \qquad \text{and} \qquad \mathbb{V}[X] = \frac{2\theta}{(1-\theta)^2}.$$
(ii) Give a method of moment estimator $\widehat{\theta}_n$ for θ given an iid sample $\mathcal{X} = (X_1, \ldots, X_n)$.
(iii) Is the maximum likelihood estimator of θ given the sample $\mathcal{X}$ well defined?
(iv) Check whether $\widehat{\theta}_n$ is consistent and compute its limiting distribution when n tends to infinity.