



Properties of American option prices

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Received 21 May 2003; received in revised form 16 January 2004; accepted 12 May 2004

Available online 15 June 2004

Abstract

We investigate some properties of American option prices in the setting of time- and level-dependent volatility. The properties under consideration are convexity in the underlying stock price, monotonicity and continuity in the volatility and time decay. Some properties are direct consequences of the corresponding properties of European option prices that are already known, and some follow by writing solutions of different stochastic differential equations as time changes of the same Brownian motion.

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MSC: primary 60G40; secondary 60J60; 91B28

Keywords: Stochastic time change; Volatility; Optimal stopping; Options

1. Introduction

Consider a market consisting of a risk-free asset, the *bank account*, with deterministic price process

$$A_t = e^{rt},$$

where the risk-free rate of return $r \geq 0$ is a constant, and a risky asset with risk-neutral price process S satisfying

$$dS_t = rS_t dt + \alpha(S_t, t) d\tilde{B}_t.$$

Here \tilde{B} is a Brownian motion on some complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$, where $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ is the augmentation under Q of the filtration generated by \tilde{B} . The last possible exercise time $T \in (0, \infty)$ is a pre-specified constant, and α is a deterministic function, i.e. the only source of randomness in the volatility is in the dependence on the

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current stock price. Notice that the dynamics of S already are given under a risk-neutral probability measure, so when pricing derivatives written on the underlying S there will be no need of changing measures as is common in mathematical finance.

In the present paper we are interested in prices of American options. Recall that an American option gives the holder the right to choose an exercise time $\gamma \leq T$, not necessarily determined in advance, but rather depending on the behavior of the stock up to the exercise time. At this time the holder receives the amount $g(S_\gamma)$, where g , the *contract function* or the *pay-off function*, is a given non-negative function. Standard arbitrage theory, see [2] or [11], gives the unique arbitrage free price at time t of an American option as

$$P(s, t) = \sup_{\gamma \in \mathbb{F}[t, T]} E_{s, t} e^{-r(\gamma-t)} g(S_\gamma), \quad (1)$$

where the indices indicate that $S_t = s$ and where we have used the convention that $\mathbb{F}A$ is the set of \mathbb{F} -stopping times taking values in $A \subseteq \mathbb{R}$. If we want to emphasize the dependence on the volatility we write $P(s, t; \alpha)$.

Janson and Tysk [10] introduce a certain stochastic time change which they call volatility time, see also Hobson [9]. Janson and Tysk find general conditions for the volatility time to exist uniquely, and some general properties of European option prices are deduced. In the present paper we use the volatility time techniques to deduce the corresponding properties for options of American type. The general idea is to write the solutions of different stochastic differential equations as time changes of the same Brownian motion. Both in the article by Janson and Tysk and in the article by Hobson *discounted* stock price processes are written as time changed Brownian motions, i.e. it is assumed that $r = 0$. This cannot be done (without loss of generality) in the same way for American options. The reason is that when pricing American options, the discounting factor $e^{-r(\gamma-t)}$ in (1) depends upon γ , i.e. upon the option holder's choice of exercise policy, whereas the discounting factor $e^{-r(T-t)}$ for European options is just a constant.

The present paper is organized as follows. In Section 2 we show that in the case of a convex pay-off function, monotonicity in the volatility and convexity of American option prices follow from the corresponding properties of European option prices. In Section 3 we present the notion of volatility time as presented by Janson and Tysk, and we slightly extend the theory to be able to use it also for optimal stopping problems. Section 4 contains our main results. In Theorem 4.2 we prove option price monotonicity in the volatility when the pay-off function satisfies a certain growth condition. This condition is in particular satisfied by all decreasing pay-off functions. Thus we provide examples of not necessarily convex pay-off functions such that the corresponding American option prices increase monotonically in the volatility. We also provide an example which shows that monotonicity in the volatility is not to be expected for general contract functions except in the very special case of $r = 0$. In that case we also prove time-decay, i.e. for fixed s the function $P(s, t)$ is non-increasing in t . Moreover, we provide a continuity result that states that pointwise convergence of the volatility implies convergence of the corresponding option prices.

2. Convex contract functions

Definition 2.1. A function $\alpha : [0, \infty) \times [t_0, \infty) \rightarrow \mathbb{R}$ is said to be locally Hölder($\frac{1}{2}$) in the first variable on $(0, \infty) \times [t_0, \infty)$ if for every $K > 0$ there exists a constant C_K such that

$$|\alpha(x, t) - \alpha(y, t)| \leq C_K |x - y|^{1/2} \tag{2}$$

whenever $K^{-1} \leq x \leq K, K^{-1} \leq y \leq K$ and $|t| \leq K$.

Since we are interested in option pricing we restrict our attention to non-negative processes S for which 0 is an absorbing state in the sense that if $S_t = 0$ for some t , then the process remains 0 forever. We work throughout this paper under the following assumptions.

Hypothesis 2.2. $\alpha(x, t)$ is measurable on $(-\infty, \infty) \times [t_0, \infty)$, and α is locally Hölder($\frac{1}{2}$) in the first variable on $(0, \infty) \times [t_0, \infty)$. Moreover, $\alpha(x, t) = 0$ for all $t \geq t_0$ and $x \leq 0$, and there exists a constant C such that

$$|\alpha(x, t)| \leq C(1 + x) \tag{3}$$

for all $t \geq t_0$.

The hypotheses on α guarantee pathwise uniqueness and thus uniqueness in law for the solutions to the stochastic differential equation

$$dS_t = rS_t dt + \alpha(S_t, t) d\tilde{B}_t \quad S_{t_0} = s. \tag{4}$$

We also assume the following.

Hypothesis 2.3. The pay-off function g is continuous, non-negative and satisfies

$$E \left[\sup_{t_0 \leq t \leq T} g(S_t) \right] < \infty \tag{5}$$

for any choice of starting point $S_{t_0} = s$.

Janson and Tysk prove that if

$$dX_t = \alpha(X_t, t) d\tilde{B}_t, \quad X_{t_0} = x \tag{6}$$

then the European option price

$$F(x, t_0; \alpha) := Eg(X_T)$$

is convex in x provided g is convex. For options with convex contract functions they also show price monotonicity in the volatility: If

$$|\alpha_1(x, t)| \leq |\alpha_2(x, t)|$$

for all $x \geq 0$ and $t \geq t_0$, then $F(x, t_0; \alpha_1) \leq F(x, t_0; \alpha_2)$. If the underlying stock price grows according to (4) instead of (6), then it is easily seen that the discounted process $X_t := e^{-rt} S_t$ satisfies (6) with $\beta(x, t) := e^{-rt} \alpha(xe^{rt}, t)$ instead of α . Observe that β

satisfies the assumptions on α in Hypothesis 2.2 if and only if α does. If $h(x) := e^{-rT}g(e^{rT}x)$, then h is convex (provided g is convex) and

$$Ee^{-rT}g(S_T) = Ee^{-rT}g(e^{rT}X_T) = Eh(X_T)$$

showing that for European option prices both convexity and monotonicity in the volatility hold also if the underlying stock grows according to (4).

Remark. When dealing with stock prices it is customary to use volatilities σ as in

$$dS_t = rS_t dt + \sigma(S_t, t)S_t d\tilde{B}_t$$

instead of $\alpha(s, t) = s\sigma(s, t)$. Note that α is locally Hölder($\frac{1}{2}$) in the first variable on $(0, \infty) \times [t_0, \infty)$ if and only if σ is, and bound (3) holds if and only if

$$\sigma(x, t) \leq C(1 + x^{-1}).$$

Also note that the local Hölder($\frac{1}{2}$) condition is weaker than the more commonly used Lipschitz condition, and that no continuity in t is assumed.

Consider a *Bermudan option* with pay-off function g . The Bermudan option is like an American option, but the owner has the right to exercise the option only at some predetermined times $0 = t_0 < t_1 < \dots < t_M = T$. The price of a Bermudan option is

$$B(s, 0) := \sup_{\gamma \in \mathbb{F}\{t_0, t_1, \dots, t_M\}} Ee^{-r\gamma}g(S_\gamma).$$

Using the dynamic programming principle, the price also can be calculated as follows:

- (1) The price $B(s, t_M)$ at $t_M = T$ is $g(s)$.
- (2) Given the price $B(\cdot, t_m)$, the price at t_{m-1} is

$$B(s, t_{m-1}) = \max\{E_{s, t_{m-1}}e^{-r(t_m - t_{m-1})}B(S_{t_m}, t_m), g(s)\}.$$

In other words, the price $B(s, t_{m-1})$ of a Bermudan option at $t = t_{m-1}$ can be calculated inductively as the maximum of $g(s)$ and the price of a European option with expiry t_m and contract function $B(s, t_m)$. Since European option prices are convex in the underlying asset provided the contract function is convex, see Theorem 4 by Janson and Tysk [10], and since the maximum of two convex functions is again a convex function, the following lemma is immediate.

Lemma 2.4. *If g is a convex contract function, then the Bermudan option price $B(s, t)$ is convex in the underlying s for any fixed t .*

Let

$$A_N := \{0, T2^{-N}, 2T2^{-N}, 3T2^{-N}, \dots, T\},$$

and let

$$B_N(s, 0) = \sup_{\gamma \in \mathbb{F}A_N} Ee^{-r\gamma}g(S_\gamma).$$

Lemma 2.5. *As the possible exercise times of the Bermudan option gets denser, the Bermudan option price converges to the corresponding American option price, i.e.*

$$B_N(s, 0) \rightarrow P(s, 0)$$

as $N \rightarrow \infty$.

Proof. Given a stopping time $\gamma \in \mathbb{F}[0, T]$, let

$$\gamma_N := \inf\{u \geq \gamma; u \in A_N\}.$$

Then $\gamma_N \in \mathbb{F}A_N$, and $\gamma_N \rightarrow \gamma$ almost surely as $N \rightarrow \infty$. Therefore, by dominated convergence,

$$|Ee^{-r\gamma}g(S_\gamma) - Ee^{-r\gamma_N}g(S_{\gamma_N})| \leq E|e^{-r\gamma}g(S_\gamma) - e^{-r\gamma_N}g(S_{\gamma_N})| \rightarrow 0$$

as $N \rightarrow \infty$, showing that $\liminf_{N \rightarrow \infty} B_N(s, 0) \geq P(s, 0)$. The lemma follows since $B_N(s, 0) \leq P(s, 0)$ for all N . \square

The following corollaries were proven by El Karoui et al. [6], and by Hobson [9] under somewhat different conditions.

Corollary 2.6. *In addition to Hypotheses 2.2 and 2.3, assume that the pay-off function g is convex. Then the American option price $P(s, t)$ is convex in the underlying s .*

Proof. Since the pointwise limit of a convergent sequence of convex functions is again convex, the corollary follows from Lemmas 2.4 and 2.5. \square

Corollary 2.7. *Let g be a convex pay-off function, and suppose that $\alpha_i, i = 1, 2$, both are as in Hypothesis 2.2 and that $|\alpha_1(s, t)| \leq |\alpha_2(s, t)|$ for all s and t . Then*

$$P(s, 0; \alpha_1) \leq P(s, 0; \alpha_2).$$

Proof. First observe that $B_N(s, 0; \alpha_1) \leq B_N(s, 0; \alpha_2)$. This follows from the monotonicity in the volatility for European options [10, Theorem 7] and the fact that European option prices are increasing in the contract function. Hence

$$P(s, 0; \alpha_1) = \lim_{N \rightarrow \infty} B_N(s, 0; \alpha_1) \leq \lim_{N \rightarrow \infty} B_N(s, 0; \alpha_2) = P(s, 0; \alpha_2). \quad \square$$

3. Volatility time

In this section we introduce the notion of volatility time. We also provide a new lemma which relates stopping times of different filtrations.

Definition 3.1. Let X be a solution to the stochastic differential equation

$$dX_t = \alpha(X_t, t) d\tilde{B}_t$$

with initial condition $X_{t_0} = x_0$, where \tilde{B} is a Brownian motion. Then the volatility time $\tau(t)$ of X is the quadratic variation of X , i.e.

$$\tau(t) = \int_{t_0}^t \alpha^2(X_u, u) \, du, \quad t \geq t_0.$$

It is well-known, see Section V.I in [15], that a continuous local martingale M can be represented as $M_t = B_{\langle M, M \rangle_t}$ for some Brownian motion B (possibly defined on a larger probability space). The idea by Janson and Tysk [10] is, given a *specific* Brownian motion B with $B_0 = x_0$, to find a time change $\tau(t)$ so that $X_t := B_{\tau(t)}$ is a solution to

$$X_t = x_0 + \int_{t_0}^t \alpha(X_u, u) \, d\tilde{B}_u$$

for *some* Brownian motion \tilde{B} .

Theorem 3.2. *Given a Brownian motion B with $B_0 = x_0$, there exists a unique (up to indistinguishability) solution τ to the pathwise stochastic differential equation*

$$\tau(t) = \int_{t_0}^t \alpha^2(B_{\tau(u)}, u) \, du, \quad t \geq t_0 \tag{7}$$

such that, for each t , $\tau(t)$ is a stopping time with respect to the completion of the filtration generated by B . Moreover, $X_t := B_{\tau(t)}$ is a solution to

$$dX_t = \alpha(X_t, t) \, d\tilde{B}_t, \quad X_{t_0} = x \tag{8}$$

for some Brownian motion \tilde{B} .

For the rather technical proof we refer to the paper by Janson and Tysk. We will also make use of Lemma 10 in that paper.

Lemma 3.3. *Let α_i , $i = 1, 2$, be as in Hypothesis 2.2, and assume that $|\alpha_1(x, t)| \leq |\alpha_2(x, t)|$ for all x and t . If B is a Brownian motion and if τ_i is the stopping time solution to*

$$\tau_i(t) = \int_{t_0}^t \alpha_i^2(B_{\tau_i(u)}, u) \, du, \quad t \geq t_0, \quad i = 1, 2$$

then $\tau_1(t) \leq \tau_2(t)$ almost surely for every $t \geq t_0$.

Below we work with some different filtrations. We let

$$\mathbb{G} = (\mathcal{G}_t)_{0 \leq t < \infty}$$

and

$$\mathbb{H} = (\mathcal{H}_t)_{0 \leq t < \infty}$$

be the completions of the filtrations generated by B and X , respectively, where B and X are the processes appearing in Theorem 3.2. We define $\tau^{-1}(t)$, the inverse of τ , as

$$\tau^{-1}(t) = \inf \{u; \tau(u) > t\}$$

with the understanding that $\inf \emptyset = \infty$. Then $\tau^{-1}(t)$ is a \mathbb{H} -stopping time for every t . Moreover, since τ is continuous, $\tau(\tau^{-1}(t)) = t \wedge \tau(\infty)$.

Lemma 3.4. *Let τ_t be the stopping time solution to (7) for some Brownian motion B . Then*

$$\mathcal{H}_t = \mathcal{G}_{\tau(t)}.$$

Remark. Intuitively, the lemma is clear. Since τ is increasing and continuous, \mathcal{H}_t and $\mathcal{G}_{\tau(t)}$ both contain all information about the Brownian motion B up to time $\tau(t)$.

Proof. We know that $X_t := B_{\tau(t)}$ is $\mathcal{G}_{\tau(t)}$ -measurable. Hence

$$\mathcal{G}_{\tau(t)} \supseteq \mathcal{H}_t.$$

To get the other inclusion we recall (see [16, Lemma 1.3.3, p. 33]), that

$$\mathcal{G}_{\tau(t)} = \sigma\{B_{\tau(t) \wedge u}; u \geq 0\} = \sigma\{X_{t \wedge \tau^{-1}(u)}; u \geq 0\} \subseteq \mathcal{H}_t$$

since $X_{t \wedge \tau^{-1}(u)}$ is \mathcal{H}_t -measurable for every u . \square

The next lemma enables us to use the volatility time techniques also for optimal stopping problems.

Lemma 3.5. *Let $\tau(t)$ be the stopping time solution to (7) for a Brownian motion B . Then $\tau(\gamma)$ is a \mathbb{G} -stopping time for every \mathbb{H} -stopping time γ .*

Conversely, every \mathbb{G} -stopping time $\rho \leq \tau(\infty)$ is of this form, i.e. $\rho = \tau(\gamma)$ for some \mathbb{H} -stopping time γ . More specifically, γ can be defined as $\gamma := \inf\{u; \tau(u) \geq \rho\}$, i.e. γ is the smallest possible random variable satisfying $\rho = \tau(\gamma)$.

Proof. The first part of the lemma follows from

$$\{\omega; \tau(\gamma) \leq s\} = \{\omega; \gamma \leq \tau^{-1}(s)\} \in \mathcal{H}_{\tau^{-1}(s)} = \mathcal{G}_{\tau(\tau^{-1}(s))} = \mathcal{G}_{s \wedge \tau(\infty)} \subseteq \mathcal{G}_s.$$

As for the second part, define

$$\gamma := \inf\{u; \tau(u) \geq \rho\}.$$

Then $\tau(\gamma) = \rho$, and

$$\{\gamma \leq s\} = \{\rho \leq \tau(s)\} \in \mathcal{G}_{\tau(s)} = \mathcal{H}_s,$$

showing that γ is a (\mathcal{H}_s) -stopping time. \square

4. Properties of American option prices

From the general theory of optimal stopping, see [5, 7, 8] or Appendix D in [12] or [6], we know that (5) implies that

$$\gamma^* := \inf\{u \geq 0; P(S_u, u) = g(S_u)\}$$

is an optimal stopping time, i.e.

$$P(s, 0) = \sup_{\gamma \in \mathbb{F}[0, T]} E_{s,0} e^{-r\gamma} g(S_\gamma) = E_{s,0} e^{-r\gamma^*} g(S_{\gamma^*}).$$

It is clear that γ^* is a stopping time not only with respect to the filtration generated by the Wiener process \tilde{B} , but also with respect to the smaller filtration generated by S itself. For future reference we state this as a lemma.

Lemma 4.1. *Let $\hat{\mathbb{H}}$ be the filtration generated by S , where S is given by (4). Then*

$$\sup_{\gamma \in \hat{\mathbb{H}}[0, T]} E e^{-r\gamma} g(S_\gamma) = \sup_{\gamma \in \mathbb{F}[0, T]} E e^{-r\gamma} g(S_\gamma).$$

4.1. Monotonicity in the volatility

Recall that if the stock price S is governed by (4) and the pay-off at T of a European option is a convex function of S_T , then the price at time 0 is monotonically increasing in the deterministic diffusion function α , see [3, 6, 9] or [10]. For European options, a convex pay-off is essential for this monotonicity result to hold. In the case of American options, however, the situation is different. As has been seen earlier (Corollary 2.7), a convex contract function g ensures monotonicity in α , but it is no longer a necessary condition. In Theorem 4.2 a new sufficient condition on the contract function is found.

Theorem 4.2. *Suppose that $\alpha_i, i = 1, 2$, are as in Hypothesis 2.2 and that $|\alpha_1(s, t)| \leq |\alpha_2(s, t)|$ for all $s \geq 0$ and for all $t_0 \leq t \leq T$. Moreover, suppose that either*

- *the interest rate r is zero*

or

- *the contract function g satisfies*

$$g(as) \leq ag(s), \quad \text{for all } a \geq 1 \quad \text{and} \quad \text{for all } s \geq 0 \tag{9}$$

holds. Then

$$P(s, t_0; \alpha_1) \leq P(s, t_0; \alpha_2).$$

Remark. Note that condition (9) on the contract function g is equivalent to requiring that $g(s)/s$ is decreasing. Geometrically, this means that for every value of s the line segment through $(s, g(s))$ and the origin lies entirely below the graph of g . Thus the monotonicity result is true for example if g is concave or if g is decreasing. In the concave case, however, the result is not very exciting as the following calculation shows. Assume that S is such that $e^{-rt}S_t$ is a martingale and that g is concave. Then,

if $u > t$,

$$\begin{aligned} e^{-rt} g(S_t) &\geq e^{-rt} e^{-r(u-t)} g(e^{r(u-t)} S_t) \\ &= e^{-ru} g(E(S_u | \mathcal{F}_t)) \\ &\geq e^{-ru} E(g(S_u) | \mathcal{F}_t), \end{aligned}$$

where we have used $g(s) \geq g(as)/a$ with $a = e^{r(u-t)}$ and Jensen’s inequality. It follows that $e^{-rt} g(S_t)$ is a non-negative supermartingale, so the Optional Sampling Theorem yields that immediate exercise of the option is always optimal. Thus the value of the option is (trivially) non-decreasing in the volatility.

Also note that when the stock price follows a geometric Brownian motion it is easy to show that (9) is sufficient to guarantee option price monotonicity (within the class of geometric Brownian motions). Moreover, the statement in this case can be sharpened as follows: Let $\alpha_i(s, t) = \sigma_i s$, $i = 1, 2$, with $\sigma_2 > \sigma_1 > 0$. Then we do not only have $P(s, 0; \alpha_1) \leq P(s, 0; \alpha_2)$ but also

$$P(s, 0; \alpha_1) \leq P(s, T - \sigma_1^2/\sigma_2^2 T; \alpha_2).$$

We leave the proof of this fact; it can be shown by letting the first of the geometric Brownian motions be driven by the Brownian motion W_t and the other one by $V_t := (\sigma_1/\sigma_2)W_{(\sigma_2^2/\sigma_1^2)t}$ which is a Brownian motion with respect to the filtration $\mathcal{F}_{(\sigma_2^2/\sigma_1^2)t}$.

We now prove Theorem 4.2.

Proof. For simplicity we prove the theorem for $t_0 = 0$. For $i = 1, 2$, define the functions

$$\beta_i(x, t) := e^{-rt} \alpha_i(xe^{rt}, t). \tag{10}$$

Let B be a Brownian motion with $B_0 = s$ and let τ_i , $i = 1, 2$, be the stopping time solutions to

$$\tau_i(t) = \int_0^t \beta_i^2(B_{\tau_i(u)}, u) du, \quad t \geq 0.$$

Recall that Lemma 3.3 tells us that $\tau_1(t) \leq \tau_2(t)$ almost surely for every $t \geq 0$. Let $\tilde{B}^{(i)}$ be Brownian motions such that $X_t^{(i)} := B_{\tau_i(t)}$ are solutions to

$$X_t^{(i)} = s + \int_0^t \beta_i(X_u^{(i)}, u) d\tilde{B}_u^{(i)},$$

and let

$$\mathbb{F}^i = (\mathcal{F}_t^i)_{t \geq 0},$$

$$\mathbb{H}^i = (\mathcal{H}_t^i)_{t \geq 0}$$

and

$$\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$$

be the completions of the five filtrations generated by $\tilde{B}^{(i)}$, $X^{(i)}$ and B , respectively. Note that the filtration generated by $X_t^{(i)}$ and the one generated by $S_t^{(i)} := e^{rt} X_t^{(i)}$

coincide. Lemma 4.1 yields that

$$\begin{aligned} P(s, 0; \alpha_1) &= \sup_{\gamma \in \mathbb{F}^1[0, T]} Ee^{-r\gamma} g(S_\gamma^{(1)}) \\ &= \sup_{\gamma \in \mathbb{H}^1[0, T]} Ee^{-r\gamma} g(S_\gamma^{(1)}) \\ &= \sup_{\gamma \in \mathbb{H}^1[0, T]} Ee^{-r\gamma} g(e^{r\gamma} B_{\tau_1(\gamma)}). \end{aligned}$$

Let $\gamma_1 \in \mathbb{H}^1[0, T]$ be optimal (recall that due to assumption (5) such an optimal stopping time exists; this is however not essential since we could deal with ε -optimal stopping times equally well). Then, according to Lemma 3.5, there exists some \mathbb{H}^2 -stopping time γ_2 such that $\tau_1(\gamma_1) = \tau_2(\gamma_2)$. Since $\tau_1(t) \leq \tau_2(t)$ almost surely for every t , and since both τ_1 and τ_2 are continuous, we know that $\tau_1(t) \leq \tau_2(t)$ for all t almost surely. Therefore $\gamma_1 \geq \gamma_2$ (recall that γ_2 can be defined as the smallest random variable satisfying $\tau_2(\gamma_2) = \tau_1(\gamma_1)$), so $\gamma_2 \in \mathbb{H}^2[0, T]$. Therefore

$$\begin{aligned} P(s, 0; \alpha_1) &= Ee^{-r\gamma_1} g(e^{r\gamma_1} B_{\tau_1(\gamma_1)}) \\ &= Ee^{-r\gamma_1} g(e^{r\gamma_1} B_{\tau_2(\gamma_2)}) \\ &\leq Ee^{-r\gamma_2} g(e^{r\gamma_2} B_{\tau_2(\gamma_2)}) \\ &\leq \sup_{\gamma \in \mathbb{H}^2[0, T]} Ee^{-r\gamma} g(e^{r\gamma} X_\gamma^{(2)}) \\ &= P(s, 0; \alpha_2), \end{aligned}$$

where we in the first inequality used (9) with $a = e^{r(\gamma_1 - \gamma_2)}$. In the case $r = 0$, this inequality reduces to an equality for any contract function g . \square

Note that the method of writing solutions of different stochastic differential equations as time changes of the same Brownian motion B does *not* directly yield the wanted monotonicity in the case of a general convex pay-off function (for the case of convex pay-off functions, see Section 2).

Remark. Monotonicity in the volatility is not to be expected for general contract functions. Indeed, let the contract function $g \in C^2[0, \infty)$ be such that $g''(s_0) < 0$ and $s_0 g'(s_0) - g(s_0) > 0$ for some stock value s_0 . The American option price P is known to satisfy the variational inequality

$$\max\{g(s) - P(s, t), P_t(s, t) + \alpha^2(s, t)P_{ss}(s, t)/2 + rsP_s(s, t) - rP(s, t)\} = 0$$

with terminal condition $P(s, T) = g(s)$. Thus it is easy to see that if $|\alpha_1(s)| \leq |\alpha_2(s)|$ for all s , and if

$$-(rs_0 g'(s_0) - rg(s_0)) < \frac{\alpha_2^2(s_0)}{2} g''(s_0) < \frac{\alpha_1^2(s_0)}{2} g''(s_0)$$

then $P_t(s_0, T; \alpha_1) < P_t(s_0, T; \alpha_2) < 0$. Thus $P(s, t; \alpha_1) > P(s, t; \alpha_2)$ in a neighborhood of (s_0, T) .

Remark. Of course condition (9) or the convexity of the contract function g are both enough to guarantee option price monotonicity also for perpetual options ($T = \infty$). Moreover, if one only considers time-homogenous models, then it is possible to show price monotonicity in the volatility for any perpetual option, see [1].

If the stock pays a continuous dividend yield $\delta > 0$, then the stock price S^δ evolves (under the martingale measure) according to

$$dS_t^\delta = (r - \delta)S_t^\delta dt + \alpha(S_t^\delta, t) d\tilde{B}_t.$$

Let P^δ denote the price of an American option written on the dividend paying stock, i.e.

$$P^\delta(s, t; \alpha) = \sup_{\gamma \in \mathbb{F}[t, T]} E_{s, t} e^{-r(\gamma-t)} g(S_\gamma^\delta).$$

We then have the following generalization of Theorem 4.2. The proof is almost identical to the proof of Theorem 4.2 and is therefore omitted.

Theorem 4.3. *Let $0 \leq \delta < r$. Suppose that α_1 and α_2 are as in Theorem 4.2, and that the contract function g satisfies*

$$g(as) \leq a^\xi g(s) \quad \text{for all } a \geq 1,$$

where $\xi = r/(r - \delta)$. Then

$$P^\delta(s, t; \alpha_1) \leq P^\delta(s, t; \alpha_2).$$

4.2. Time-decay of option prices

It is obvious that the American option price is an increasing function of T (since when increasing T , the set of possible stopping times to exercise the option increases). In the famous Black and Scholes formula, the dependence on time is through the quantity $T - t$, the remaining time to maturity. The same holds of course for American option prices in a time-homogenous model, i.e. when the diffusion coefficient $\alpha = \alpha(s)$ does not depend on time. Thus, in contrast to European option prices, American options prices decrease in all time-homogenous models.

When dealing with time-inhomogenous models (as we do), matters are not quite as simple. The question we ask is if decreasing t_0 , the present, necessarily implies an increase in the American option price. It is rather easy to see that the answer is negative in general, but in the special case when the interest rate $r = 0$, the answer is affirmative.

Example (Option prices are not always decreasing in time). We only outline the example. Define

$$\sigma(t) := \begin{cases} 0 & \text{if } 0 \leq t < T_0, \\ \sigma & \text{if } T_0 \leq t \leq T, \end{cases}$$

where $0 < T_0 < T$ and $\sigma > 0$, and let S be defined by

$$dS_t = rS_t dt + \sigma(t)S_t d\tilde{B}_t \quad S_0 = s.$$

Let $P(s, t)$ be the price of an American put option with pay-off function $g(s) = (K - s)^+$ written on S . Now it can be shown that

$$P(K, 0) = e^{-rT_0} P(Ke^{rT_0}, T_0) < P(Ke^{rT_0}, T_0) \leq P(K, T_0).$$

Proposition 4.4. *Suppose that the American option price $P(s, t; \alpha)$ is monotone increasing in the diffusion coefficient α (for this to hold it suffices that $r = 0$ or that g is convex or satisfies (9)). Then, for any $T_0 \in [0, T]$,*

$$P(s, 0) \geq e^{-rT_0} P(se^{rT_0}, T_0).$$

Proof. Given $\alpha(s, t)$, define

$$\alpha_0(s, t) := \begin{cases} 0 & \text{if } 0 \leq t < T_0, \\ \alpha(s, t) & \text{if } T_0 \leq t \leq T. \end{cases}$$

Observe that if S satisfies

$$dS = rS dt + \alpha_0(S_t, t) d\tilde{B}, \quad S_0 = s$$

then S grows deterministically in the time interval $[0, T_0]$. It follows that $P(s, 0; \alpha_0) \geq e^{-rT_0} P(se^{rT_0}, T_0; \alpha_0)$. Therefore, using the monotonicity we get

$$\begin{aligned} P(s, 0; \alpha) &\geq P(s, 0; \alpha_0) \\ &\geq e^{-rT_0} P(se^{rT_0}, T_0; \alpha_0) \\ &= e^{-rT_0} P(se^{rT_0}, T_0; \alpha). \quad \square \end{aligned}$$

Remark. Note that if $r = 0$, then we have time-decay of option prices. This case can also be derived in a similar way as Theorem 4.2 using volatility times and Lemma 7 in [3]. Note also that under some regularity assumptions on α and g , the option price is known to satisfy the Black–Scholes equation

$$P_t(s, t) + \frac{\alpha^2(s, t)}{2} P_{ss}(s, t) + rsP_s(s, t) - rP(s, t) = 0 \tag{11}$$

in the continuation region $\mathcal{C} := \{(s, t) : P(s, t) > g(s)\}$. Proposition 4.4 implies that

$$\begin{aligned} \frac{P(s, t + \varepsilon) - P(s, t)}{\varepsilon} &\leq \frac{e^{r\varepsilon} P(se^{-r\varepsilon}, t) - P(s, t)}{\varepsilon} \\ &= e^{r\varepsilon} \frac{P(se^{-r\varepsilon}, t) - P(s, t)}{\varepsilon} + P(s, t) \frac{e^{r\varepsilon} - 1}{\varepsilon}. \end{aligned}$$

Taking limits we obtain

$$\frac{\partial P(s, t)}{\partial t} + rs \frac{\partial P(s, t)}{\partial s} - rP(s, t) \leq 0,$$

i.e. in the continuation region \mathcal{C} the option price satisfies

$$\frac{\alpha^2(s,t)}{2} \frac{\partial^2 P(s,t)}{\partial s^2} \geq 0.$$

El Karoui et al. [6] show that convexity of the price in s implies monotonicity in the volatility (both for European and American options). Since a similar result as in Proposition 4.4 holds for *European* options, and since the Black and Scholes equation (11) holds everywhere in that case, the above calculations show that monotonicity in the volatility implies convexity in s for European options. For American options one can only conclude convexity of the value function *locally* in the continuation region. This is however enough to conclude that an option holder who underestimates the volatility will subreplicate an American option, for details see [6].

4.3. Continuity in the volatility

Convergence of American option prices and Snell envelopes have been studied by several authors, see for example [13, 14]. Theorem 4.5 provides conditions under which American option prices are continuous in the volatility. A proof of Theorem 4.5 based on the notion of volatility time can be found in [4].

Theorem 4.5. *Suppose that α and $\alpha_1, \alpha_2, \dots$ satisfy the conditions of Hypothesis 2.2 uniformly, i.e. with the same constants C_K and C in (2) and (3). Assume that $\alpha_n(s,t) \rightarrow \alpha(s,t) \neq 0$ as $n \rightarrow \infty$ for all $s \geq 0$ and $t \geq 0$, and that α is such that S never reaches 0. Moreover, assume that the contract function $g(s) \leq C_1(1+s)^k$ for some $C_1 > 0$, $k > 0$, and that g is Hölder(p)-continuous for some $p > 0$. Then*

$$\lim_{n \rightarrow \infty} P(s, 0; \alpha_n) = P(s, 0; \alpha).$$

Acknowledgements

The author thanks Johan Tysk for many helpful discussions and comments. The author also thanks Christer Borell for careful reading and valuable comments.

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