(Questions marked with a ∗ are optional.)

(1) Assuming that π = 3.1415926... correct to seven decimal places, prove that the first three convergents to π are:

\[
\frac{22}{7}, \quad \frac{333}{106}, \quad \frac{355}{113}
\]

Verify that \(|\pi - 355/113| < 10^{-6}\).

(2) Find the fundamental solutions of the Pell equations \(x^2 - Ny^2 = 1\) for \(N = 5, 7, 11, 13, 17\).

(3) Find two solutions in positive integers for each of the equations \(x^2 - 21y^2 = 1\), \(x^2 - 29y^2 = 1\).

(4) Prove that the number with continued fraction \([10, 10^{2!}, 10^{3!}, \ldots]\) is transcendental.

(5) Following the examples in class, use the continued fraction algorithm to factor the numbers: 9509, 13561, 8777.

(6) Let \(M, N\) be positive integers such that \(N\) is not a square, and \(M \leq \sqrt{N}\). If \(x, y\) is a solution of the equation \(x^2 - Ny^2 = M\), prove that \(x/y\) is a convergent of \(\sqrt{N}\).

(7) Use Pollard’s \(p - 1\) method with \(k = 840\) and \(a = 2\) to try to factor \(n = 53467\). Then try with \(a = 3\).
(8*) Prove that, if \((x_n, y_n)\) for \(n = 1, 2, \ldots\) is the sequence of positive solutions of the Pell equation \(x^2 - Ny^2 = 1\) written in increasing values of \(x_n, y_n\), then \(x_n\) and \(y_n\) satisfy a recurrence relation

\[ u_{n+2} - 2au_{n+1} + u_n = 0 \]

where \(a\) is a positive integer. Find \(a\) when \(N = 7\).