(1) Calculate the greatest common divisor \( d = (a, b) \) and find integers \((x, y)\) such that \( ax + by = d \) in the following cases:

\[ (i) \ a = 841, \ b = 160 \quad (ii) \ a = 2613, \ b = 2171 \quad (iii) \ a = 8991, \ b = 3293 \]

(2) Let \( a, b \) be positive integers with \( a > b > 1 \). Let \( \lambda(a, b) \) be the number of steps (i.e. individual applications of the Euclidean algorithm) required to compute \( d = (a, b) \) via successive applications of the Euclidean algorithm. Clearly \( \lambda(a, b) < b \). Prove that

\[ \lambda(a, b) \leq 2 \frac{\log b}{\log 2} \]

(3) (i) Suppose that \( n \) is known to be the product of two primes. Show how one can determine these primes from the knowledge of \( n \) and \( \varphi(n) \).

(ii) Suppose that \( n \) is not a perfect square, and satisfies

\[ n - n^{2/3} < \varphi(n) < n - 1. \]

Deduce that \( n \) is the product of two distinct primes.

(4) Let \( p \) be a prime dividing \( b^n - 1 \), where \( b \) and \( n \) are integers \( > 1 \). Show that either \( p \equiv 1 \mod n \), or \( p|b^d - 1 \) for some divisor \( d \) of \( n \). If \( p > 2 \) and \( n \) is odd, then in the second case \( p \equiv 1 \mod 2n \). Using this, find the prime factorization of the following numbers:

\[ 2^{11} - 1 = 2047, \quad 3^{12} - 1 = 531440, \quad 2^{35} - 1 = 34359738367. \]

[Hint: If \( p|2^{11} - 1 \), for example, then \( p \equiv 1 \mod 22 \) so test \( p = 23, 67 \ldots \) You only need to test up to \( \sqrt{2047} \).]
(5)  (i) Find the smallest nonnegative integer $x$ such that

$$\begin{align*}
x &\equiv 2 \mod 3 \\
x &\equiv 3 \mod 5 \\
x &\equiv 4 \mod 11 \\
x &\equiv 5 \mod 16
\end{align*}$$

(ii) Find the smallest nonnegative integer $x$ satisfying

$$\begin{align*}
19x &\equiv 103 \mod 900 \\
10x &\equiv 511 \mod 841
\end{align*}$$

(6)  Let $A$ be the group $(\mathbb{Z}/65520\mathbb{Z})^\times$. Determine the least positive integer $n$ such that $g^n = 1$ for all $g \in A$.

(7)  Prove that $-2$ is a primitive root modulo 23. Determine all solutions to the congruences $x^7 \equiv 17 \mod 23$ and $x^{26} \equiv 10 \mod 23$.

(8)  Find a generator of $(\mathbb{Z}/p\mathbb{Z})^\times$ for $p = 5, 7, 11, 13$. Determine how many of the integers $1, 2, ..., p-1$ are generators.

(9)  Suppose that $p|2^{2^k} + 1$, where $k > 1$. Then:

1. Show that $p \equiv 1 \mod 2^{k+1}$.

2. By asking whether 2 is a quadratic residue $\mod p$, show that $p \equiv 1 \mod 2^{k+2}$.

3. Use this to show that $2^{16} + 1$ is prime.