(1) Let $0 \xrightarrow{\phi_0} V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} \ldots \xrightarrow{\phi_{n-1}} V_n \xrightarrow{\phi_n} 0$ be a complex of vector spaces, meaning that the $V_i$ are vector spaces and the $\phi_i$ are linear maps with $\phi_i \circ \phi_{i-1} = 0$ for $i = 1, \ldots, n$. In particular, $\ker \phi_i \supset \im \phi_{i-1}$, so it makes sense to define the quotient spaces $H_i = (\ker \phi_i)/(\im \phi_{i-1})$. Show that if all the $V_i$ are finite-dimensional, then $\sum_{i=1}^n (-1)^i \dim H_i = \sum_{i=1}^n (-1)^i \dim V_i$.

(2)

(a) We say that a sequence $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ of abelian groups and group homomorphisms is **exact at $B$** if $\ker \psi = \im \phi$, and we say that a sequence

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

is **short exact** if it is exact at $A$, $B$, and $C$ (so in particular every short exact sequence is a complex). Show that, for a short exact sequence as above, $A \cong \phi(A)$ and $C \cong B/\phi(A)$. 
(b) A short exact sequence as above \( \textit{splits} \) if there exists a homomorphism \( \rho : C \to B \) with \( \psi \circ \rho = \text{id} \). Show that the homomorphism \( A \oplus C \to B, \quad (a, c) \mapsto \phi(a) + \rho(c) \), is an isomorphism. (If you want, you can think about why \( 0 \to \mathbb{Z}_n \to \mathbb{Z}_n^2 \to \mathbb{Z}_n \to 0 \) with \( \phi(x) = nx \) doesn’t split, and/or why short exact sequences of vector spaces and linear maps always split.)

(3) Let \( U \) and \( V \) be two path-connected open subsets of \( \mathbb{R}^n \) such that \( U \cup V = \mathbb{R}^n \). Show that \( U \cap V \) is path-connected.

(4) Let \( X = S^1 \times S^1 \) be the torus and \( Y = S^1 \times B^2 \) the solid torus. Compute the induced homomorphisms on \( H_1 \) of the following two continuous maps:

(a) \( f : X \to X, \quad f(z, w) = (z^a w^b, z^c w^d) \) with \( a, b, c, d \in \mathbb{Z} \).

(b) The inclusion map \( i : X \to Y \) of \( X \) as the boundary of \( Y \).

(5) Let \( X \) be a topological space. Let \( f : X \to X \) be a homeomorphism. The \textit{mapping torus} \( M_f \) of \( f \) is the quotient \( M_f = (X \times [0, 1]) / \sim \) where \( \sim \) denotes the equivalence relation generated by \( (x, 1) \sim (f(x), 0) \) for all \( x \in X \). Introduce open sets \( U = (X \times (0, 1)) / \sim \) and \( V = (X \times ([0, \frac{1}{2}) \cup (\frac{1}{2}, 1)]) / \sim \). Argue carefully that there exists a commutative diagram

\[
\begin{array}{ccc}
H_i(U \cap V) & \xrightarrow{i_i \oplus j_2} & H_i(U) \oplus H_i(V) \\
\cong & & \cong \\
H_i(X) \oplus H_i(X) & \xrightarrow{\phi} & H_i(X) \oplus H_i(X)
\end{array}
\]

such that \( \phi(a, b) = (a + b, a + f_*(b)) \).

(6) Let \( X \) be the space formed by inserting \( n \) vertical “bars” in the sphere \( S^2 \). Compute the fundamental group and all the homology groups of \( X \).

(7) For \( i = 1, 2, 3 \) let \( L_i \subset \mathbb{R}^3 \) be three general lines, and write \( X = \mathbb{R}^3 \setminus (L_1 \cup L_2 \cup L_3) \). Compute the fundamental group and all the homology groups of \( X \).

(8) Compute the fundamental group and all the homology groups of the complement \( X \) of a (complex) line and a point not on it in \( \mathbb{P}^2(\mathbb{C}) \).
(9) Let $X$ be the complement of a small disk in a torus $T^2 = S^1 \times S^1$, and let $A = \partial X \cong S^1$ be the boundary of $X$. Compute all the relative homology groups $H_j(X, A)$.

(10) Construct a surjective map $f : S^n \to S^n$ of degree zero, for each $n \geq 1$. 