(1) Show that for a space $X$ the following are equivalent:

(i) Every map $S^1 \to X$ is homotopic to a constant map, with image a point in $X$;

(ii) Every map $S^1 \to X$ extends to a map $D^2 \to X$;

(iii) $\pi_1(X, x_0) = (0)$ for all $x_0 \in X$

Deduce that a space $X$ is simply-connected if and only if all maps $S^1 \to X$ are homotopic. [N.B. In this question “homotopic” means “homotopic without regard to basepoints.”]

(2) This question is about the fundamental group of the Klein bottle $K$.

(a) Hatcher (page 51) shows two different ways of computing $\pi_1(K)$. The two presentations he obtains are $G_1 = \langle a, b | aba^{-1}b \rangle$ and $G_2 = \langle x, y | x^2y^2 \rangle$. Write a purely algebraic proof that $G_1$ and $G_2$ are isomorphic. (Hint: By the definition of “group presentation”, $G_i \cong F/N_i$, where $F = \langle u, v \rangle$ is free and $N_1, N_2$ are certain normal subgroups. Hence it suffices to
find group homomorphisms \( \phi, \psi : F \to F \) such that \( \phi \circ \psi = \psi \circ \phi = \text{id} \) and \( \phi(N_1) = N_2. \)

(b) In the first homework we saw that \( K \) can also be written as \( M \cup_f M \), where \( M \) is a Möbius strip and \( f : \partial M \to \partial M \) is a homeomorphism. Apply van Kampen to this decomposition to compute \( \pi_1(K) \) for the third time. (Hint: \( [\partial M] \) is an element of \( \pi_1(M) \). Which one?)

(3) Show very carefully that \( S^1 \) is a retract of \( S^1 \vee S^1 \), but not a deformation retract.

Construct infinitely many non-homotopic retractions \( S^1 \vee S^1 \to S^1 \).

(4) Van Kampen’s theorem talks about decompositions \( X = U \cup V \), where \( U, V \) are open and path-connected, and \( U \cap V \neq \emptyset \) is path-connected as well. Show that the assumption that both \( U \) and \( V \) are open is necessary for the theorem to hold.

(5) Suppose that a space \( Y \) is obtained from a path-connected subspace \( X \) by attaching \( n \)-cells for a fixed \( n \geq 3 \). Show that the inclusion \( X \hookrightarrow Y \) induces an isomorphism on \( \pi_1 \). Apply this to show that the complement of a discrete subspace of \( \mathbb{R}^n \) is simply-connected if \( n \geq 3 \). [N.B. a subspace \( Z \subset X \) of a topological space \( X \) is discrete if the topology on \( Z \) induced by the topology of \( X \) is the discrete topology: in other words, \( \forall z \in Z \) there is \( U \subset X \) open, \( z \in U \), \( U \cap Z = \{z\} \).]

(6) Recall the usual picture of the Klein bottle \( K \) as a subspace \( X \subset \mathbb{R}^3 \) with a circle of self-intersection (so in fact there is a continuous map \( K \to X \) identifying two circles). If one wanted a model that could actually function as a bottle, one would delete the small open disk bounded by the circle of self-intersection, producing a subspace \( Y \subset K \). Show that \( \pi_1(X) \cong \mathbb{Z} \ast \mathbb{Z} \) and that

\[
\pi_1(Y) = \langle a, b, c \mid aba^{-1}b^{-1}cb\varepsilon c^{-1} \rangle
\]

for \( \varepsilon = \pm 1 \). (Don’t worry about nailing down \( \varepsilon \).)

The space \( Y \) can be obtained from a disk with two holes by identifying the three boundary circles. Show that the other way yields a space \( Z \) with \( \pi_1(Z) \) not isomorphic to \( \pi_1(Z) \). [Hint. In fact, the abelianizations of these groups are not isomorphic.]
(7) Construct a simply-connected covering space of the space $X \subset \mathbb{R}^3$ that is the union of a sphere and a diameter. Do the same when $X$ is the union of a sphere and a circle intersecting it in two points.

(8) Draw all the connected 2-sheeted and 3-sheeted covering spaces of $S^1 \vee S^1$, up to isomorphism of covering spaces without basepoints.

(9) Find all the connected covering spaces of $\mathbb{P}^2(\mathbb{R}) \vee \mathbb{P}^2(\mathbb{R})$.

(10) For a path-connected, locally path-connected, and semilocally simply-connected space $X$, call a path-connected covering space $\tilde{X} \to X$ abelian if it is normal and has abelian deck transformation group. Show that $X$ has an abelian covering space that is a covering space of every other abelian covering space of $X$.

Draw a picture of this covering space for $X = S^1 \vee S^1$. (No proof is required, just a picture.) [Hint: paint $S^1 \vee S^1$ on the torus.]