(1) Suppose that $f: X \to Y$ be a quotient map of topological spaces.

(a) Show that if $Y$ is Hausdorff, then the fibers $f^{-1}(y)$ ($y \in Y$) are closed.

(b) Is $Y$ necessarily Hausdorff if all the fibers are closed?

(2) Let $X,Y$ be topological spaces, $A \subset X$ a subspace, and $f: A \to Y$ a quotient map. Show that the two definitions of $X \sqcup_f Y$ are equivalent: in other words, $(X \sqcup Y)/R$ is homeomorphic to $X/\sim$, where $R$ is the equivalence relation on $X \sqcup Y$ generated by $x R f(x)$ for all $x \in A$, and $\sim$ is the equivalence relation on $X$ generated by $x_1 \sim x_2$ for all $x_1, x_2 \in A$ for which $f(x_1) = f(x_2)$.

(3) Show that the quotient of $\mathbb{R} \times \{0,1\}$ by the equivalence relation generated by $(x,0) \sim (\frac{1}{x}, 1)$ for all $x \neq 0$ is homeomorphic to $S^1 = \{z \in \mathbb{C} | |z| = 1\}$. 
We will take it for granted that the Klein bottle \( K \) is homeomorphic to the quotient space \( [0, 1] \times [0, 1]) / \sim \), where \( \sim \) is generated by \( (x, 0) \sim (x, 1) \) and \( (0, y) \sim (1, 1 - y) \).

Using this fact, draw pictures to convince me that \( K \) can be written as two Möbius strips \( M_1, M_2 \) attached to each other along their boundaries (formally: as the quotient of the disjoint union \( M_1 \sqcup M_2 \) by the equivalence relation generated by \( x \sim f(x) \) for \( x \in \partial M_1 \), where \( f: \partial M_1 \to \partial M_2 \) is a certain continuous map—in fact, a homeomorphism—that you don’t need to specify).

This is very similar to (4)). Show that one of the three equivalent constructions (given in class) of \( P^2(\mathbb{R}) \) is homeomorphic to a Möbius strip \( M \) attached to a disk \( D^2 \) attached along their boundaries. Here, try to be as specific as possible in defining the attachment, and the homeomorphism as well as its inverse.

We defined in class \( \mathbb{P}^n(\mathbb{C}) = (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^\times \). Briefly argue that \( \mathbb{P}^n(\mathbb{C}) = S^{2n+1} / \sim \), where \( S^{2n+1} = \{ z \in \mathbb{C}^{n+1} \mid |z_1|^2 + \ldots + |z_{n+1}|^2 = 1 \} \) and \( \sim \) refers to the usual action of the group \( S^1 \) of unit complex numbers. Write \( p_n: S^{2n+1} \to \mathbb{P}^n(\mathbb{C}) \) for the quotient map. Also denote by \( H^n_\mathbb{C} = \{ z \in S^{2n+1} \mid z_{n+1} \in [0, \infty) \} \) the complex hemisphere.

(a) Show that the restriction of \( p_n \) to \( H^n_\mathbb{C} \) is still surjective.

(b) Show that if \( z, w \in H^n_\mathbb{C} \) and \( z \sim w \), but \( z \neq w \), then \( z_{n+1} = w_{n+1} = 0 \).

(c) Show that the map \( (z_1, ..., z_n, z_{n+1}) \mapsto (z_1, ..., z_n) \) defines a homeomorphism \( H^n_\mathbb{C} \to B^{2n} \).

Remark: Question 2 shows that (a) & (b) imply that \( \mathbb{P}^n(\mathbb{C}) = H^n_\mathbb{C} \cup_f \mathbb{P}^{n-1}(\mathbb{C}) \), where \( f \) is the restriction of \( p_n \) to the complex equator \( S^{2n-1} = \{ z \in S^{2n+1} \mid z_{n+1} = 0 \} \).

Let \( f: S^1 \to X \) be a continuous map. Show that the following are equivalent:

(a) \( f \) is nullhomotopic.

(b) There exists a continuous map \( g: B^2 \to X \) such that \( g|_{\partial B^2} = f \).
(8) Let $X$ be a topological space and let $x, y, z, w \in X$. Let $f, g, h$ be paths from $x$ to $y$, $y$ to $z$, and $z$ to $w$, respectively. Show that the paths $(f \cdot g) \cdot h$ and $f \cdot (g \cdot h)$ are homotopic relative endpoints. (Remark: You need to write down an explicit homotopy.)

(9) Using the isomorphism $\Phi: \mathbb{Z} \to \pi_1(S^1, 1)$ discussed in lectures, show that every group homomorphism $\phi: \mathbb{Z} \to \mathbb{Z}$ can be written as $\phi = \Phi^{-1} \circ f_* \circ \Phi$ for some map $f: (S^1, 1) \to (S^1, 1)$. (Hint: $\phi$ is determined by its action on a generator of $\mathbb{Z}$.)

(10) The Borsuk–Ulam theorem states that if $f: S^2 \to \mathbb{R}^2$ is a continuous map, then there is a point $x \in S^2$ such that $f(x) = f(-x)$.

Can something like this be true for the torus $T^2 = S^1 \times S^1$ in place of $S^2$? I.e. is it true that for every map $f: S^1 \times S^1 \to \mathbb{R}^2$ there exists a point $(z, w)$ such that $f(z, w) = f(-z, -w)$?