ON THE MODULARITY OF ELLIPTIC CURVES OVER IMAGINARY QUADRATIC FIELDS

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Abstract. In this paper, we establish the modularity of every elliptic curve E/F, where F runs over infinitely many imaginary quadratic fields, including $\mathbb{Q}(\sqrt{-d})$ for $d = 1, 2, 3, 5$. More precisely, let F be imaginary quadratic and assume that the modular curve $X_0(15)$, which is an elliptic curve of rank 0 over $\mathbb{Q}$, also has rank 0 over F. Then we prove that all elliptic curves over F are modular. More generally, when $F/\mathbb{Q}$ is an imaginary CM field that does not contain a primitive 5th root of unity, we prove the modularity of elliptic curves E/F under a technical assumption on the image of the representation of $\mathrm{Gal}(\overline{F}/F)$ on $E[3]$ or $E[5]$.

The key new technical ingredient we use is a local-global compatibility theorem for the p-adic Galois representations associated to torsion in the cohomology of the relevant locally symmetric spaces. We establish this result in the crystalline case, under some technical assumptions, but allowing arbitrary dimension, arbitrarily large regular Hodge–Tate weights, and allowing p to be small and highly ramified in the imaginary CM field F.

Contents

1. Introduction

Let F be a number field. We say that an elliptic curve E/F is *modular* if either E has complex multiplication or if there exists a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ of parallel weight 2 whose associated L-function is the same as the L-function of E[1].

In this paper, we establish the modularity of every elliptic curve E/F, where F runs over infinitely many imaginary quadratic fields, including $\mathbb{Q}(\sqrt{-d})$ for $d = 1, 2, 3, 5$.

Recall that the modular curve $X_0(15)$ is an elliptic curve of rank zero over $\mathbb{Q}$ – it is the curve with Cremona label 15A1. We prove the following result.

Theorem 1.1 (Corollary 7.1.2)**.** *Let F be an imaginary quadratic field such that the Mordell–Weil group $X_0(15)(F)$ is finite. Then every elliptic curve E/F is modular.*

We can compute the ranks of $X_0(15)$ over imaginary quadratic fields of small discriminant using Sage [The22] or Magma [BCP97] and check that the theorem applies to $F = \mathbb{Q}(\sqrt{-d})$ for the above values of d. By [MN15, Theorem 3], the theorem applies to an infinite class of imaginary quadratic fields. Moreover, a celebrated conjecture of Goldfeld [Gol79], when coupled with the Birch–Swinnerton-Dyer conjecture, predicts that $X_0(15)$ should have rank 0 over 50% of quadratic fields, when these are ordered by the absolute value of the discriminant. The conjecture predicts rank 0 over slightly more than half of imaginary quadratic fields. More precisely, $X_0(15)$ is predicted to have rank 0 over 100% of those imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$ with d positive square-free and $d \bmod 15 \in \{0, 1, 2, 3, 4, 5, 8, 12\}$. This congruence condition corresponds to the global root number of $X_0(15)$ over $\mathbb{Q}(\sqrt{-d})$ being $+1$ (see for example [Dok05, Corollary 2]). Forthcoming work of Smith [Smi] will verify this prediction (since $X_0(15)$ has a rational cyclic degree 4 isogeny, the existing results of Smith [Smi22] exclude this case). In fact, Smith shows that the 2^∞-Selmer corank is 0 for 100% of discriminants satisfying these congruence conditions, which implies that the rank is 0 with no dependence on BSD.

The modularity of an elliptic curve E over a number field F implies that the associated L-function has analytic continuation to the entire complex plane. This is needed in order to formulate the Birch and Swinnerton-Dyer conjecture for E unconditionally. Furthermore, modularity has historically played a key role in progress on the BSD conjecture, going back to the use of Heegner points by Gross and Zagier for (modular) elliptic curves over $\mathbb{Q}$. Recently, Loeffler and Zerbes made significant

[1] The reason for the two cases is that if E has CM by a field which embeds in F, then it cannot be associated to a cuspidal automorphic representation.

progress on the BSD conjecture for modular elliptic curves defined over imaginary quadratic fields [LZ21], making Theorem 1.1 particularly timely.

More generally, when $F/\mathbb{Q}$ is an imaginary CM field that does not contain a primitive fifth root of unity, we prove the modularity of elliptic curves E/F under a technical assumption on the image of the representation of $\mathrm{Gal}(\overline{F}/F)$ on $E[3]$ or $E[5]$. As a consequence, we obtain the following result.

Theorem 1.2 (Corollary 6.1.2)**.** *Let F be an imaginary CM field that is Galois over $\mathbb{Q}$ and such that $\zeta_5 \notin F$. Then 100% of Weierstrass equations over F, ordered by their height, define a modular elliptic curve.*

The modularity of elliptic curves $E/\mathbb{Q}$ was pioneered by Wiles and Taylor–Wiles in [Wil95, TW95] and completed by Breuil–Conrad–Diamond–Taylor in [BCDT01]. The modularity of elliptic curves defined over *real quadratic fields* was established, more recently, in [FLHS15]. Compared to the rational case, the real quadratic case relies on the improvements to the Taylor–Wiles method due to Kisin [Kis09], on supplementing the traditional 3-5 prime switch with an ingenious 3-7 switch, and on a sophisticated analysis of quadratic points on several modular curves of small level. Further results have been obtained for more general *totally real fields*, including cubic and quartic fields [DNS20, Box22]. As another example, Thorne [Tho19] has proved the modularity of every elliptic curve defined over the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ for any prime p.

The modularity of elliptic curves defined over *imaginary CM fields* has historically been more difficult to establish. This is because the systems of Hecke eigenvalues that conjecturally match such elliptic curves contribute to the cohomology of locally symmetric spaces such as Bianchi 3-manifolds, which are not directly related to Shimura varieties. The situation has been extensively investigated numerically (for example, [Cre84, Cre92, LMF22]), and modularity of specific elliptic curves can be verified using the Faltings–Serre method [DGP10]. Inspired by a program outlined by Calegari–Geraghty in [CG18], the potential modularity of such elliptic curves was established independently in [ACC$^+$18] and in [BCGP21]. Since then, Allen–Khare–Thorne proved many instances of actual modularity in [AKT19]. More precisely, they established the modularity of a positive proportion of elliptic curves over imaginary CM fields together with strong residual modularity results modulo 3 and modulo 5.

Remark 1.2.1. In fact, [BCGP21] establish the potential modularity of elliptic curves defined over a general quadratic extension of a totally real field. A recent preprint [Whi22] by Whitmore builds on their method and on the results of [AKT19] to prove actual modularity for a positive proportion of such elliptic curves.

To prove Theorem 1.2, we combine the residual modularity results of [AKT19] with a modularity lifting theorem in the Barsotti–Tate case in the style of Kisin [Kis09]. The crucial ingredient needed to prove our Barsotti–Tate modularity lifting theorem is a *local-global compatibility* result for the Galois representations constructed by Scholze in [Sch15]. This is a result of independent interest, which we now discuss.

Let $K \subset \mathrm{GL}_n(\mathbb{A}_{F,f})$ be a neat compact open subgroup and let X_K be the corresponding locally symmetric space for GL_n/F. A highest weight vector λ for $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_n$ determines a $\mathbb{Z}_p$-local system $\mathcal{V}_\lambda$ on X_K and we are interested in understanding the systems of Hecke eigenvalues occurring in $H^*(X_K, \mathcal{V}_\lambda)$. Let $\mathbb{T}$ be the usual abstract spherical Hecke algebra acting on $H^*(X_K, \mathcal{V}_\lambda)$ by correspondences, let $\mathbb{T}(K, \lambda)$ be the maximal quotient of $\mathbb{T}$ through which this action is faithful, and let $\mathfrak{m} \subset \mathbb{T}(K, \lambda)$ be a maximal ideal. When $\mathfrak{m}$ is non-Eisenstein, Scholze constructed a continuous Galois representation

$$\rho_{\mathfrak{m}} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{T}(K, \lambda)_{\mathfrak{m}}/I),$$

where $I \subset \mathbb{T}(K, \lambda)_{\mathfrak{m}}$ is an ideal satisfying $I^4 = 0$, cf. [NT16]. This p-adically interpolates the Galois representations attached to torsion classes occurring in $H^*(X_K, \mathcal{V}_\lambda)_{\mathfrak{m}}$ as well as those attached to characteristic 0 automorphic forms, first constructed by [HLTT16]. We let $\bar{\rho}_{\mathfrak{m}}$ denote the absolutely irreducible residual representation obtained by reducing $\rho_{\mathfrak{m}}$ modulo $\mathfrak{m}$.

For applications to modularity, it is extremely important to understand the properties of $\rho_{\mathfrak{m}}$, cf. [CG18, Conjecture B]. One needs to know whether $\rho_{\mathfrak{m}}$ satisfies some form of local-global compatibility: if $v \mid \ell$ is a prime of F and $G_{F_v} := \mathrm{Gal}(\overline{F}_v/F_v)$, how does the level K_v at which $\mathfrak{m}$ occurs (together with the weights λ_v if $\ell = p$) determine the ramification of $\rho_{\mathfrak{m}}|_{G_{F_v}}$? The case when $\ell = p$ is particularly subtle because it is not (a priori) clear how to formulate integral p-adic Hodge theory conditions which should be satisfied by the Galois representations $\rho_{\mathfrak{m}}$, and because the $\rho_{\mathfrak{m}}$ are constructed in [Sch15] via a p-adic interpolation argument that loses track of the weight λ and of the level K_v for $v \mid p$.

In [ACC+18], we established such a local-global compatibility result at $\ell = p$ in two restricted families of cases described by natural integral conditions: the ordinary case and certain Fontaine–Laffaille cases. In the present paper, we go much further than this and establish the desired result in the crystalline case, under some technical assumptions, but allowing arbitrary n, arbitrary weight λ, and allowing p to be small and highly ramified in F. In this generality, the formulation via integral p-adic Hodge theory is still mysterious, but the local-global compatibility conjecture can be formulated as in [GN22, Conjecture 5.1.12], using the crystalline deformation rings first constructed by Kisin [Kis08]. More precisely, we have a composition

(1.2.1)
$$\begin{array}{ccccc} R^{\square}_{\bar{\rho}_{\mathfrak{m}}|_{G_{F_v}}} & \longrightarrow & R^{\square}_{\bar{\rho}_{\mathfrak{m}}} & \longrightarrow & \mathbb{T}(K,\lambda)_{\mathfrak{m}}/I \\ & \searrow & & \nearrow & \\ & & R^{\square,\mathrm{crys}}_{\bar{\rho}_{\mathfrak{m}}|_{G_{F_v}}}(\lambda_v) & & \end{array}$$

where the first horizontal map is the usual map from the local deformation ring of $\bar{\rho}_{\mathfrak{m}}|_{G_{F_v}}$ to the global deformation ring of $\bar{\rho}_{\mathfrak{m}}$ and the second horizontal map is induced by the existence of $\rho_{\mathfrak{m}}$. When $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ is a maximal compact subgroup, the natural conjecture is that the composition (1.2.1) factors through $R^{\square,\mathrm{crys}}_{\bar{\rho}_{\mathfrak{m}}|_{G_{F_v}}}(\lambda_v)$, the crystalline deformation ring with Hodge–Tate weights determined by λ_v. We prove this conjecture in Theorem 4.2.15 under some technical assumptions - roughly, the statement is as follows.

Theorem 1.3. *Let F be an imaginary CM field that contains an imaginary quadratic field F_0 and with maximal totally real subfield F^+*[2]*. Let p be a rational prime that splits in F_0, let $\bar{v} \mid p$ be a prime of F^+, and assume the following.*

(1) *Setting $\bar{v} = v \cdot v^c$, we have $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ and $K_{v^c} = \mathrm{GL}_n(\mathcal{O}_{F_{v^c}})$.*
(2) *There exists a prime $\bar{v}' \mid p$ of F^+ distinct from $\bar{v}$ such that*

$$\sum [F^+_{\bar{v}''} : \mathbb{Q}_p] \geq \frac{1}{2}[F^+ : \mathbb{Q}],$$

where the sum runs over primes $\bar{v}'' \mid p$ of F^+ distinct from both $\bar{v}$ and $\bar{v}'$.
(3) *$\mathfrak{m}$ is a non-Eisenstein maximal ideal such that $\bar{\rho}_{\mathfrak{m}}$ is decomposed generic, cf. Definition 2.1.27.*

Then, up to possibly enlarging the nilpotent ideal I, the composition (1.2.1) factors through $R^{\square,\mathrm{crys}}_{\bar{\rho}_{\mathfrak{m}}|_{G_{F_v}}}(\lambda_v)$ as expected.

Remark 1.3.1.

(1) The method of proof is versatile enough that we expect the first assumption can be removed. Bence Hevesi is working on generalising Theorem 1.3 to the potentially semi-stable case as part of his PhD thesis. The second assumption is more serious and excludes in particular the case where $F = F_0$. The third assumption is needed in order to appeal to the results of [CS19] on unitary Shimura varieties, or alternatively to those of [Kos21].
(2) We also obtain in Theorem 4.3.1 a local-global compatibility result for the characteristic 0 Galois representations attached to regular algebraic cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$. In this setting, the local-global compatibility at $\ell \neq p$ is already known up to semi-simplification by work of Varma [Var14]. More recently, A'Campo [A'C22] proved that these automorphic Galois representations are also de Rham at all primes above p. In fact, in the latest revision of this article, A'Campo is also able to determine the Hodge–Tate weights of these representations, using Wang-Erickson's work on p-adic Hodge theoretic conditions for pseudorepresentations [WE18].
(3) Motivated by our applications to elliptic curves, we prove a slightly more general result which includes semistable ordinary representations.

There are two key new ideas that allow us to prove much stronger local-global compatibility results than in [ACC+18]. The first idea is to work with P-ordinary parts at the prime $\bar{v} \mid p$ of F^+ where we want to prove local-global compatibility. The second idea, which was suggested to us by Peter Scholze, is to increase the level at the auxiliary primes $\bar{v}'' \mid p$ of F^+ in order to simplify the analysis of the boundary of the Borel–Serre compactification in the relevant unitary Shimura varieties. Fortunately, these can be implemented simultaneously.

To explain how the first new idea is useful, recall that the crystalline deformation rings $R^{\square,\mathrm{crys}}_{\bar{\rho}_{\mathfrak{m}}|_{G_{F_v}}}(\lambda_v)$ were defined by Kisin first after inverting p, and then integrally by taking Zariski closure from the generic fibre. On the other hand, the Galois representations $\rho_{\mathfrak{m}}$ could be torsion. They are constructed by congruences using a subtle argument that involves $2n$-dimensional Galois representations. If we had a characteristic 0 lift of $\rho_{\mathfrak{m}}|_{G_{F_v}}$, which we knew was crystalline at v with Hodge–Tate

[2]The field F has to satisfy some additional technical assumptions so that we can appeal to the unconditional base change results of [Shi14].

weights determined by λ_v, we would deduce that the diagram (1.2.1) factors as desired. Conversely, if the diagram factored as desired, we would expect the crystalline lift to exist by results of Tong Liu [Liu15]. It seems hard to guarantee that there is a characteristic 0 crystalline lift of the global representation $\rho_{\mathfrak{m}}$. However, by working with P-ordinary parts at $\bar{v}$ throughout, we construct for each $m \in \mathbb{Z}_{\geq 1}$ a $2n$-dimensional characteristic 0 global representation $\rho_{\widetilde{\mathfrak{m}}}$ such that

$$\rho_{\widetilde{\mathfrak{m}}}|_{G_{F_v}} \simeq \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$$

with n-dimensional diagonal blocks and such that one of these blocks is congruent to the local representation $\rho_{\mathfrak{m}}|_{G_{F_v}} \pmod{p^m}$. Moreover, we can ensure that each of these characteristic 0 lifts is crystalline with the correct Hodge–Tate weights. We expect the global representation $\rho_{\widetilde{\mathfrak{m}}}$ to be irreducible, and we do not produce characteristic 0 lifts of the global representation $\rho_{\mathfrak{m}} \pmod{p^m}$.

The second new idea is useful for the "degree-shifting" argument needed to relate the cohomology groups $H^*(X_K, \mathcal{V}_\lambda)_{\mathfrak{m}}$ to a middle degree boundary cohomology group $H^d(\partial \widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}})_{\widetilde{\mathfrak{m}}}$, of some unitary Shimura variety $\widetilde{X}_{\widetilde{K}}$. We can control the latter using the main theorem of [CS19]. However, one only has a spectral sequence of Leray–Serre type from the former cohomology groups to the latter – controlling the behaviour of this spectral sequence seems to be a tricky problem in modular representation theory. In [ACC+18], we showed that the spectral sequence degenerates if p is strictly greater than n^2 and is unramified in F. In the present paper, we increase the level at auxiliary primes $\bar{v}'' \mid p$ and, through a delicate induction argument, we keep track of the terms in the spectral sequence modulo powers of p without imposing the additional assumptions that $p > n^2$ and is unramified in F.

To prove a modularity lifting theorem in the Barsotti–Tate case and deduce Theorem 1.2, we apply Theorem 1.3 in the case when $n = 2$ and λ is trivial. For our applications, it is crucial to allow p to be small and highly ramified in F^+. (We can then ensure that the second condition of Theorem 1.3 is satisfied using an appropriate solvable base change.) This is why the local-global compatibility results of [ACC+18] in the Fontaine–Laffaille case were not strong enough and why [AKT19] appealed instead to the results in the more restrictive ordinary case. We expect Theorem 1.3 to have many more applications to modularity over CM fields in the near future.

To deduce Theorem 1.1, we analyze the imaginary quadratic points on several modular curves with small level at 3 and 5, classifying elliptic curves for which both the 3-torsion and the 5-torsion are exceptional. For a prime p, we let $\mathrm{b}p \subset \mathrm{GL}_2(\mathbb{F}_p)$ denote the upper-triangular Borel subgroup, $\mathrm{s}p \subset \mathrm{GL}_2(\mathbb{F}_p)$ denote the normalizer of the standard split Cartan subgroup and $\mathrm{ns}p \subset \mathrm{GL}_2(\mathbb{F}_p)$ denote the normalizer of the standard non-split Cartan subgroup $\mathrm{ns}p^\circ$. After some reductions using group theory, there turn out to be six modular curves of interest:

(1) $X(\mathrm{b}3, \mathrm{b}5)$ (also denoted by $X_0(15)$ above);
(2) $X(\mathrm{s}3, \mathrm{b}5)$;
(3) $X(\mathrm{ns}3^\circ, \mathrm{b}5)$;
(4) $X(\mathrm{b}3, \mathrm{ns}5)$;
(5) $X(\mathrm{s}3, \mathrm{ns}5)$;
(6) $X(\mathrm{ns}3^\circ, \mathrm{ns}5)$.

The modular curves $X(\mathrm{b}3, \mathrm{b}5)$ and $X(\mathrm{s}3, \mathrm{b}5)$ are isogenous elliptic curves of Mordell-Weil rank 0 over $\mathbb{Q}$. They are the obstruction to extending Theorem 1.1 to every imaginary quadratic field F, although we can at least understand how their torsion subgroup grows in imaginary quadratic extensions.

The modular curve $X(\mathrm{ns}3^\circ, \mathrm{b}5)$ is a genus 1 curve without a rational point. This case does not occur in the real quadratic case because $\mathrm{ns}3^\circ$ does not contain an odd element, which should represent complex conjugation. The curve contains two infinite families of imaginary quadratic points, for which, miraculously, it is still possible to prove modularity! The elliptic curves in the first family turn out to all have rational j-invariant. The elliptic curves in the second family turn out to all be $\mathbb{Q}$-curves (isogenous to their conjugates over $\overline{\mathbb{Q}}$).

The remaining cases also do not occur in the real quadratic setting. The modular curve $X(\mathrm{b}3, \mathrm{ns}5)$ is a genus 2 hyperelliptic curve and we study its imaginary quadratic points using similar methods to those of [FLHS15]. The modular curves $X(\mathrm{s}3, \mathrm{ns}5)$ and $X(\mathrm{ns}3^\circ, \mathrm{ns}5)$ are bi-elliptic curves of genus 3 whose Jacobians have Mordell-Weil rank 1. We analyze the imaginary quadratic points on these curves using the relative symmetric power Chabauty method developed by Siksek [Sik09] and Box [Box21].

Remark 1.3.2. It seems much more subtle to implement the 3-7 switch over an imaginary CM field than over a totally real field, as in [FLHS15, §7]. The modular curve with full level structure at 7 is isomorphic to the Klein quartic curve

$$x^3y + y^3z + z^3x = 0.$$

To implement the 3-7 switch, one needs to produce points on a quadratic twist of the Klein quartic that are defined over solvable CM extensions of the original CM field. In the totally real case, this can be done with a clever application of Hilbert irreducibility, obtaining rational points over a degree 4, thus solvable, totally real extension. This argument does not apply in the imaginary CM case: the direct argument gives points defined over a degree 4 extension of the original field F, but this is not necessarily a CM field. By working with Weil restrictions of scalars to the maximal totally real subfield F^+, the degree increases. One can obtain points defined over a CM extension of F but it seems hard to guarantee that this extension is always solvable.

The organization of the paper is as follows. In Section 2, we collect preliminaries on locally symmetric spaces and develop P-ordinary Hida theory in the setting of their Betti cohomology. In Section 3, we study the P-ordinary condition on the Galois side and record a key argument with determinants that will be used for local-global compatibility. In Section 4, we prove Theorem 1.3 and its characteristic 0 counterpart. In Section 5, we use this result together with the techniques developed in [ACC$^+$18] to prove a modularity lifting theorem over imaginary CM fields in the potentially Barsotti–Tate case. In Section 6, we combine this modularity lifting theorem with the results of [AKT19] to prove Theorem 1.2. In Section 7, we analyze the imaginary quadratic points on several modular curves of small level and prove Theorem 1.1.

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1.5. **Notation.** Our notation largely matches the one introduced in [ACC$^+$18, §1.2]. If F is a perfect field, we let $\overline{F}$ denote an algebraic closure of F and G_F denote the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$.

If F is a number field, we let $S_p(F)$ be the set of places of F above p. If S is a finite set of finite places of a number field F, we let $G_{F,S}$ denote the Galois group of the maximal extension of F that is unramified outside S. For a prime ℓ, we let ϵ_ℓ denote the ℓ-adic cyclotomic character and $\overline{\epsilon}_\ell$ denote its reduction modulo ℓ.

If π is an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A}_F)$ and $\lambda \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,\mathbb{C})}$ is dominant for the standard upper triangular Borel subgroup, we say that π is regular algebraic of weight λ if the infinitesimal character of π_∞ is the same as that of $V_\lambda^\vee$, where V_λ is the algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}_n$ of highest weight λ. See § 2.1.11 for a discussion of highest weight representations.

If K is a finite extension of $\mathbb{Q}_p$ for some prime p, we write I_K for the inertia subgroup of G_K, $\mathrm{Frob}_K \in G_K/I_K$ for the geometric Frobenius and W_K for the Weil group. We write $\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$ for the Artin map of local class field theory, normalized to take uniformizers to geometric Frobenius elements. We let rec_K denote the local Langlands correspondence of [HT01], which sends an irreducible smooth (admissible) representation π of $\mathrm{GL}_n(K)$ over $\mathbb{C}$ to a Frobenius semi-simple Weil–Deligne representation $\mathrm{rec}_K(\pi)$ of W_K, also over $\mathbb{C}$. We also write rec_K^T for the arithmetic normalization of the local Langlands correspondence, as defined for example in [CT14, §2.1]; this normalization is defined for coefficients in any field which is abstractly isomorphic to $\mathbb{C}$, such as $\overline{\mathbb{Q}}_\ell$. We define labelled Hodge–Tate weights of p-adic representations of G_K as in [ACC$^+$18, §1.2]. In particular, ϵ_p has Hodge–Tate weight -1.

If G is a locally profinite group and K is an open subgroup, we write $\mathcal{H}(G,K)$ for the $\mathbb{Z}$-algebra of compactly supported bi-K-invariant functions $f : G \to \mathbb{Z}$, cf. [NT16, Lemma 2.3].

We let $E/\mathbb{Q}_p$ be a p-adic field which will be our coefficient field, with ring of integers $\mathcal{O}$, uniformiser ϖ and finite residue field $k := \mathcal{O}/\varpi$. We let $\mathrm{CNL}_{\mathcal{O}}$ denote the category of complete, local, Noetherian $\mathcal{O}$-algebras with residue field k.

2. The cohomology of locally symmetric spaces

2.1. **Preliminaries.** In this section, we gather some preliminaries, and we largely follow [ACC$^+$18, §2] without giving complete details.

2.1.1. *Locally symmetric spaces.* Let F be a number field and G be a connected linear algebraic group over F, with a model over $\mathcal{O}_F$ that we will still denote by G. We will denote by X^{G} the *symmetric space* for $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{G}$, which is a homogeneous space for $\mathrm{G}(F \otimes_{\mathbb{Q}} \mathbb{R})$ as in [BS73, §2] and [NT16, Definition 3.1] (and which is determined by G up to isomorphism of homogeneous spaces).

Let $K_{\mathrm{G}} \subset \mathrm{G}(\mathbb{A}_{F,f})$ be a *good* compact open subgroup in the sense of [ACC+18, §2.1]: namely it is *neat* and of the form $\prod_v K_{\mathrm{G},v}$, where v runs over the finite places of F. We consider the double quotient

$$X^{\mathrm{G}}_{K_{\mathrm{G}}} := \mathrm{G}(F)\backslash X^{\mathrm{G}} \times \mathrm{G}(\mathbb{A}_{F,f})/K_{\mathrm{G}},$$

which is a smooth, orientable Riemannian manifold. We also consider the partial Borel–Serre compactification $\overline{X}^{\mathrm{G}}$ of X^{G} as in [BS73, §7.1] and form the double quotient

$$\overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}} := \mathrm{G}(F)\backslash \overline{X}^{\mathrm{G}} \times \mathrm{G}(\mathbb{A}_{F,f})/K_{\mathrm{G}},$$

which is a compact, smooth manifold with corners with interior $X^{\mathrm{G}}_{K_{\mathrm{G}}}$. We note that the spaces X^{G} are always connected; when $\mathrm{G}(\mathbb{R})$ is not connected, it is sometimes better to work with $X^{\mathrm{G}} \times \pi_0(\mathrm{G}(\mathbb{R}))$ (equivalently, replacing the isotropy subgroup in the definition of the symmetric space with its identity connected component). Since $\mathrm{G}(\mathbb{R})$ will be connected in all the cases of interest to us, this will not concern us. Finally, we consider the boundaries $\partial X^{\mathrm{G}} := \overline{X}^{\mathrm{G}} \backslash X^{\mathrm{G}}$ and $\partial X^{\mathrm{G}}_{K_{\mathrm{G}}} := \overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}} \backslash X^{\mathrm{G}}_{K_{\mathrm{G}}}$.

We define $\mathfrak{X}_{\mathrm{G}} := \varprojlim_{K_{\mathrm{G}}} X^{\mathrm{G}}_{K_{\mathrm{G}}}$, endowed with the projective limit topology, where $K_{\mathrm{G}} \subset \mathrm{G}(\mathbb{A}_{F,f})$ runs over good compact open subgroups. We also consider the analogous spaces $\overline{\mathfrak{X}}_{\mathrm{G}}$ and $\partial\mathfrak{X}_{\mathrm{G}}$. All these spaces are equipped with a continuous action of $\mathrm{G}(\mathbb{A}_{F,f})$, which is equipped with the locally profinite topology. Note also that the spaces $\overline{\mathfrak{X}}_{\mathrm{G}}$ and $\partial\mathfrak{X}_{\mathrm{G}}$ are compact Hausdorff, being projective limits of compact Hausdorff spaces. We denote by $j : \mathfrak{X}_{\mathrm{G}} \hookrightarrow \overline{\mathfrak{X}}_{\mathrm{G}}$ the natural open immersion. As a consequence of [CGH+20, Lemma 6.2.1] and [NT16, Lemma 2.31], we see that the actions of any good subgroup K_{G} on $\overline{\mathfrak{X}}_{\mathrm{G}}$ and $\partial\mathfrak{X}_{\mathrm{G}}$ are free in the sense of [NT16, Definition 2.23]. These limits have been considered previously by Rohlfs [Roh96]. It follows from the properness of the action of arithmetic groups on the symmetric space and its compactification (cf. [Roh96, Proposition 1.9]) that we have

$$\mathfrak{X}_{\mathrm{G}} = \mathrm{G}(F)\backslash X^{\mathrm{G}} \times \mathrm{G}(\mathbb{A}_{F,f}),\ \overline{\mathfrak{X}}_{\mathrm{G}} = \mathrm{G}(F)\backslash \overline{X}^{\mathrm{G}} \times \mathrm{G}(\mathbb{A}_{F,f}),$$
$$\text{and } \partial\mathfrak{X}_{\mathrm{G}} = \mathrm{G}(F)\backslash \partial X^{\mathrm{G}} \times \mathrm{G}(\mathbb{A}_{F,f}),$$

with topologies induced by the locally profinite topology on the adelic groups. We prefer to work with these topological spaces, since they seem more natural than those used in [NT16, ACC+18] which equip the adelic groups with the discrete topology. We compare these set-ups ('topological' and 'discrete') in the next subsection.

2.1.2. *Hecke operators and coefficient systems.* If S is a finite set of finite places of F we set $\mathrm{G}^S := \mathrm{G}(\mathbb{A}^S_{F,f})$ and $\mathrm{G}_S := \mathrm{G}(\mathbb{A}_{F,S})$, and similarly $K^S_{\mathrm{G}} = \prod_{v \notin S} K_{\mathrm{G},v}$ and $K_{\mathrm{G},S} = \prod_{v \in S} K_{\mathrm{G},v}$. We write $\mathcal{H}(\mathrm{G}^S, K^S_{\mathrm{G}})$ for the global Hecke algebra over $\mathbb{Z}$ which is the restricted tensor product of the local Hecke algebras $\mathcal{H}(\mathrm{G}(F_v), K_{\mathrm{G},v})$ for v a finite place of F not contained in S.

Let R be a commutative ring and let $\mathcal{V}$ be a smooth $R[K_{\mathrm{G},S}]$-module, which is finite free as an R-module. We now explain how to obtain from it a local system $\mathcal{V}$ of R-modules on $X^{\mathrm{G}}_{K_{\mathrm{G}}}$ and how to equip the usual and compactly supported cohomology groups $R\Gamma_{(c)}(X^{\mathrm{G}}_{K_{\mathrm{G}}}, \mathcal{V})$ with an action of the Hecke algebra $\mathcal{H}(\mathrm{G}^S, K^S_{\mathrm{G}}) \otimes_{\mathbb{Z}} R$, by adapting the formalism of [NT16] to our topological setting.

Firstly, note that the $R[K_{\mathrm{G},S}]$-module $\mathcal{V}$ defines a $\mathrm{G}^S \times K_{\mathrm{G},S}$-equivariant local system, which we denote by $\mathcal{V}$ as well, on both $\mathfrak{X}_{\mathrm{G}}$ and $\overline{\mathfrak{X}}_{\mathrm{G}}$. Indeed, we first inflate $\mathcal{V}$

to a smooth $R[\mathrm{G}^S \times K_{\mathrm{G},S}]$-module, which is equivalent to a $\mathrm{G}^S \times K_{\mathrm{G},S}$-equivariant sheaf on a point by [NT16, Lemma 2.26], and then we pull back this sheaf to $\mathfrak{X}_{\mathrm{G}}$ and $\overline{\mathfrak{X}}_{\mathrm{G}}$, respectively. From now on, we consider the $\mathrm{G}^S \times K_{\mathrm{G},S}$-equivariant sheaves $\mathcal{V}$ and $j_!\mathcal{V}$ on $\overline{\mathfrak{X}}_{\mathrm{G}}$. By [Sch98, §1, Lemma 1][3], the category of $\mathrm{G}^S \times K_{\mathrm{G},S}$-equivariant sheaves on $\overline{\mathfrak{X}}_{\mathrm{G}}$ has enough injectives. By [NT16, Lemma 2.25], since $\overline{\mathfrak{X}}_{\mathrm{G}}$ is compact, the global sections of a $\mathrm{G}^S \times K_{\mathrm{G},S}$-equivariant sheaf on $\overline{\mathfrak{X}}_{\mathrm{G}}$ form a smooth $R[\mathrm{G}^S \times K_{\mathrm{G},S}]$-module. We therefore have a well-defined derived functor $R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}},\ )$, and we obtain

$$R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{V}) \text{ and } R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, j_!\mathcal{V})$$

in the bounded below derived category of smooth $R[\mathrm{G}^S \times K_{\mathrm{G},S}]$-modules. We apply the functor $R\Gamma(K_{\mathrm{G}},\ )$[4], which gives rise to objects

$$R\Gamma(K_{\mathrm{G}}, R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{V})) \text{ and } R\Gamma(K_{\mathrm{G}}, R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, j_!\mathcal{V}))$$

in the bounded below derived category of $\mathcal{H}(\mathrm{G}^S, K^S_{\mathrm{G}}) \otimes_{\mathbb{Z}} R$-modules.

On the other hand, we can also view $\mathcal{V}$ and $j_!\mathcal{V}$ as K_{G}-equivariant sheaves on $\overline{\mathfrak{X}}_{\mathrm{G}}$, using the forgetful functor. Recall that the action of K_{G} on $\overline{\mathfrak{X}}_{\mathrm{G}}$ is free and that the quotient can be identified with $\overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}}$; let $\pi : \overline{\mathfrak{X}}_{\mathrm{G}} \to \overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}}$ denote the projection map. The *descent* functor $\mathcal{F} \to (\pi_*\mathcal{F})^{K_{\mathrm{G}}}$ gives an equivalence between the category of K_{G}-equivariant sheaves on $\overline{\mathfrak{X}}_{\mathrm{G}}$ and the category of sheaves on $\overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}}$ by [NT16, Lemma 2.24]. We denote the corresponding sheaves on $\overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}}$ by $\mathcal{V}$ and $j_!\mathcal{V}$ as well.

Proposition 2.1.3. *The following diagram of derived functors is commutative*

$$\begin{array}{ccccc}
\mathrm{DSh}^+_{\mathrm{G}^S \times K_{\mathrm{G},S}}(\overline{\mathfrak{X}}_{\mathrm{G}}) & \xrightarrow{R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}},\ )} & \mathrm{D}^+_{\mathrm{sm}}(\mathrm{G}^S \times K_{\mathrm{G},S}, R) & \xrightarrow{R\Gamma(K_{\mathrm{G}},\ )} & \mathrm{D}^+(\mathcal{H}(\mathrm{G}^S, K^S) \otimes_{\mathbb{Z}} R) \\
\downarrow{\scriptstyle \text{forget}} & & & & \downarrow{\scriptstyle \text{forget}} \\
\mathrm{DSh}^+_{K_{\mathrm{G}}}(\overline{\mathfrak{X}}_{\mathrm{G}}) & \xrightarrow{\text{descent}} & \mathrm{DSh}^+(\overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}}) & \xrightarrow{R\Gamma\left(\overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}},\ \right)} & \mathrm{D}^+(R)
\end{array}$$

Proof. This is a topological version of [NT16, Prop. 2.18]. The corresponding diagram of underived functors commutes up to natural isomorphism. The forgetful functor from $\mathrm{G}^S \times K_{\mathrm{G},S}$-equivariant sheaves to K_{G}-equivariant sheaves is exact and preserves injectives by [Sch98, §3, Corollary 3]. The descent functor is also exact and preserves injectives, since it is an equivalence of categories. The functor $\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}},\ )$ preserves injectives by [NT16, Lemma 2.28]. □

Note that we have a canonical isomorphism

$$R\Gamma\left(\overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}}, \mathcal{V}\right) \xrightarrow{\sim} R\Gamma\left(X^{\mathrm{G}}_{K_{\mathrm{G}}}, \mathcal{V}\right)$$

induced by the pullback map j^*, because j is a homotopy equivalence, and that $R\Gamma(\overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}}, j_!\mathcal{V})$ precisely computes $R\Gamma_c(X^{\mathrm{G}}_{K_{\mathrm{G}}}, \mathcal{V})$. This shows how to construct morphisms

$$\mathcal{H}(\mathrm{G}^S, K^S_{\mathrm{G}}) \otimes_{\mathbb{Z}} R \to \mathrm{End}_{\mathrm{D}^+(R)}\left(R\Gamma_{(c)}(X^{\mathrm{G}}_{K_{\mathrm{G}}}, \mathcal{V})\right) [5].$$

---

[3] It is assumed in *loc. cit.* that the coefficients have characteristic 0, but this is not used in the proof.

[4] This is the derived functor of K_{G}-invariants considered with its profinite topology, so it computes the *continuous* group cohomology of K_{G}.

[5] We will only need this statement, which is slightly weaker than saying that these are objects in the bounded below derived category of $\mathcal{H}(\mathrm{G}^S, K^S_{\mathrm{G}}) \otimes_{\mathbb{Z}} R$-modules.

The same formalism also applies to $R\Gamma(\partial X^{\mathrm{G}}_{K_{\mathrm{G}}}, \mathcal{V})$.

Lemma 2.1.4. *The functor* $R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \ ) : \mathrm{DSh}^+_{\mathrm{G}^S \times K_{\mathrm{G},S}}(\overline{\mathfrak{X}}_{\mathrm{G}}) \to \mathrm{D}^+_{\mathrm{sm}}(\mathrm{G}^S \times K_{\mathrm{G},S}, R)$ *has bounded cohomological dimension.*

Proof. We can check this after applying the forgetful functor to K_{G}-equivariant sheaves. If $\mathcal{F} \in \mathrm{Sh}_{K_{\mathrm{G}}}(\overline{\mathfrak{X}}_{\mathrm{G}})$, then [NT16, Lemma 2.35] implies that $R^i\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{F})$ vanishes for $i > \dim(X^{\mathrm{G}}_{K_{\mathrm{G}}})$. □

We now compare our set-up with that of [NT16]. We set $\overline{\mathfrak{X}}^{\mathrm{dis}}_{\mathrm{G}} = \mathrm{G}(F)\backslash \overline{X}^{\mathrm{G}} \times \mathrm{G}(\mathbb{A}_{F,f})^{\mathrm{dis}}$, where the superscript indicates that we are considering $\mathrm{G}(\mathbb{A}_{F,f})$ with the discrete topology. If we have a good compact open subgroup $K_{\mathrm{G}} \subset \mathrm{G}(\mathbb{A}_{F,f})$, then $(K_{\mathrm{G}})^{\mathrm{dis}}$ acts freely on $\overline{\mathfrak{X}}^{\mathrm{dis}}_{\mathrm{G}}$ with quotient equal to $\overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}}$. Using [NT16, Lemma 2.19] to make the Hecke action on cohomology explicit, we see that, whether we use the topological or discrete set-up, we will obtain the same Hecke actions on the cohomology of $\overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}}$. We will prove something a little stronger than this, to convince the reader that the two different set-ups really are naturally equivalent.

There is a natural map $\pi_{\mathrm{dis}} : \overline{\mathfrak{X}}^{\mathrm{dis}}_{\mathrm{G}} \to \overline{\mathfrak{X}}_{\mathrm{G}}$ which induces an exact functor

$$\pi^*_{\mathrm{dis}} : \mathrm{Sh}_{\mathrm{G}^S \times K_{\mathrm{G},S}}(\overline{\mathfrak{X}}_{\mathrm{G}}) \to \mathrm{Sh}_{(\mathrm{G}^S \times K_{\mathrm{G},S})^{\mathrm{dis}}}(\overline{\mathfrak{X}}^{\mathrm{dis}}_{\mathrm{G}}).$$

Using descent to $\overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}}$ for both the topological and discrete categories, we see that pullback by π_{dis} induces an equivalence $\mathrm{Sh}_{K_{\mathrm{G}}}(\overline{\mathfrak{X}}_{\mathrm{G}}) = \mathrm{Sh}_{K^{\mathrm{dis}}_{\mathrm{G}}}(\overline{\mathfrak{X}}^{\mathrm{dis}}_{\mathrm{G}})$.

Lemma 2.1.5. *We have a natural isomorphism of functors from* $\mathrm{DSh}^+_{\mathrm{G}^S \times K_{\mathrm{G},S}}(\overline{\mathfrak{X}}_{\mathrm{G}})$ *to* $\mathrm{D}^+(\mathcal{H}(\mathrm{G}^S, K^S) \otimes_{\mathbb{Z}} R)$*:*

$$R\Gamma(K_{\mathrm{G}}, -) \circ R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, -) \cong R\Gamma(K^{\mathrm{dis}}_{\mathrm{G}}, -) \circ R\Gamma(\overline{\mathfrak{X}}^{\mathrm{dis}}_{\mathrm{G}}, -) \circ \pi^*_{\mathrm{dis}}.$$

Proof. It follows from [NT16, Lemma 2.19] that we can identify the underived functors $\Gamma(K_{\mathrm{G}}, -) \circ \Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, -) = \Gamma(K^{\mathrm{dis}}_{\mathrm{G}}, -) \circ \Gamma(\overline{\mathfrak{X}}^{\mathrm{dis}}_{\mathrm{G}}, -) \circ \pi^*_{\mathrm{dis}}$. From this, we deduce that there is a natural transformation

$$R\Gamma(K_{\mathrm{G}}, -) \circ R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, -) \to R\Gamma(K^{\mathrm{dis}}_{\mathrm{G}}, -) \circ R\Gamma(\overline{\mathfrak{X}}^{\mathrm{dis}}_{\mathrm{G}}, -) \circ \pi^*_{\mathrm{dis}}.$$

We can check that this is an isomorphism after composing with the forgetful map to $\mathrm{D}^+(R)$, and this follows from comparing Proposition 2.1.3 and [NT16, Proposition 2.18]. □

We now recall an important finiteness result:

Lemma 2.1.6. *Let* K_{G} *be a good subgroup, and let* $K'_{\mathrm{G}} \subset K_{\mathrm{G}}$ *be a normal subgroup which is also good. Let* R *be a Noetherian ring, and let* $\mathcal{V}$ *be a smooth* $R[K_{\mathrm{G}}]$*-module, finite free as* R*-module. Then* $R\Gamma_{(c)}(X^{\mathrm{G}}_{K'_{\mathrm{G}}}, \mathcal{V})$ *are perfect objects of* $\mathrm{D}^+(K_{\mathrm{G}}/K'_{\mathrm{G}}, R)$*; in other words, they are isomorphic in this category to bounded complexes of projective* $R[K_{\mathrm{G}}/K'_{\mathrm{G}}]$*-modules.*

Proof. The case of usual cohomology is essentially [ACC$^+$18, Lemma 2.1.7]: we choose a finite triangulation of $\overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}}$ and pull this back to a K_{G}-invariant triangulation of $\overline{\mathfrak{X}}_{\mathrm{G}}$, then consider the corresponding complex of simplicial chains $C_\bullet$. We notice that $\mathrm{Hom}_{\mathbb{Z}[K'_{\mathrm{G}}]}(C_\bullet, \mathcal{V})$ is isomorphic in $\mathrm{D}^+(K_{\mathrm{G}}/K'_{\mathrm{G}}, R)$ to $R\Gamma(\overline{X}^{\mathrm{G}}_{K'_{\mathrm{G}}}, \mathcal{V})$. The case of cohomology with compact support can be done in a similar way, by choosing our triangulation of $\overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}}$ in such a way that a triangulation of $\partial X^{\mathrm{G}}_{K_{\mathrm{G}}}$ is a simplicial subcomplex. Pulling back to $\overline{\mathfrak{X}}_{\mathrm{G}}$ we obtain simplicial complexes $\partial C_\bullet \to C_\bullet$. Letting $C^{\mathrm{BM}}_\bullet$ denote the cone of this map, we observe that $\mathrm{Hom}_{\mathbb{Z}[K'_{\mathrm{G}}]}\left(C^{\mathrm{BM}}_\bullet, \mathcal{V}\right)$ is isomorphic in $\mathrm{D}^+(K_{\mathrm{G}}/K'_{\mathrm{G}}, R)$ to $R\Gamma_c(X^{\mathrm{G}}_{K'_{\mathrm{G}}}, \mathcal{V})$. □

Assume now that $R = \mathcal{O}/\varpi^m$ for some $m \in \mathbb{Z}_{\geq 1}$. If $S \subseteq S_p(F)$ is a set of places of F above p, and if $\mathcal{V}$ is a smooth $\mathcal{O}/\varpi^m[K_{S_p\setminus S}]$-module, we define the *completed cohomology* at S of level K^S_{G} to be

$$R\Gamma(K^S_{\mathrm{G}}, R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{V})) \in \mathrm{D}^+_{\mathrm{sm}}(\mathrm{G}_S, \mathcal{O}/\varpi^m).$$

Similarly, we define the *completed cohomology with compact support* at S of level K^S_{G} to be

$$R\Gamma(K^S_{\mathrm{G}}, R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, j_!\mathcal{V})) \in \mathrm{D}^+_{\mathrm{sm}}(\mathrm{G}_S, \mathcal{O}/\varpi^m).$$

For a finite set of finite places $T \supseteq S_p(F)$ of F, a variant of the above formalism equips these objects with actions of $\mathcal{H}(\mathrm{G}^T, K^T_{\mathrm{G}}) \otimes_{\mathbb{Z}} \mathcal{O}/\varpi^m$. The same formalism applies to

$$R\Gamma(K^S_{\mathrm{G}}, R\Gamma(\partial\mathfrak{X}_{\mathrm{G}}, \mathcal{V})) \in \mathrm{D}^+_{\mathrm{sm}}(\mathrm{G}_S, \mathcal{O}/\varpi^m).$$

The following lemma offers a justification for the term *completed cohomology.*

Lemma 2.1.7. *For any $i \in \mathbb{Z}_{\geq 0}$, we have $\mathcal{H}(\mathrm{G}^T, K^T_{\mathrm{G}})$-equivariant isomorphisms of admissible smooth $\mathcal{O}/\varpi^m[\mathrm{G}_S]$-modules*

$$H^i\left(R\Gamma(K^S_{\mathrm{G}}, R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{V}))\right) \xrightarrow{\sim} \varinjlim_{K_{\mathrm{G},S}} H^i\left(X^{\mathrm{G}}_{K^S_{\mathrm{G}}K_{\mathrm{G},S}}, \mathcal{V}\right) \tag{2.1.1}$$

and

$$H^i\left(R\Gamma(K^S_{\mathrm{G}}, R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, j_!\mathcal{V}))\right) \xrightarrow{\sim} \varinjlim_{K_{\mathrm{G},S}} H^i_c\left(X^{\mathrm{G}}_{K^S_{\mathrm{G}}K_{\mathrm{G},S}}, \mathcal{V}\right). \tag{2.1.2}$$

Proof. In the category of compact Hausdorff spaces, we have

$$\overline{\mathfrak{X}}_{\mathrm{G}}/K^S_{\mathrm{G}} = \varprojlim_{K_{\mathrm{G},S}} \overline{X}^{\mathrm{G}}_{K^S_{\mathrm{G}}K_{\mathrm{G},S}}.$$

This shows that K^S_{G} acts freely on $\overline{\mathfrak{X}}_{\mathrm{G}}$, so we can functorially rewrite the LHS of (2.1.1) and (2.1.2) in terms of the cohomology of either $\mathcal{V}$ or $j_!\mathcal{V}$ on the quotient $\overline{\mathfrak{X}}_{\mathrm{G}}/K^S_{\mathrm{G}}$. We can functorially rewrite the terms on the RHS in terms of the cohomology of either $\mathcal{V}$ or $j_!\mathcal{V}$ on $\overline{X}^{\mathrm{G}}_{K^S_{\mathrm{G}}K_{\mathrm{G},S}}$. The result now follows from [NT16, Lemma 2.34]. Finally, admissibility of the cohomology groups follows from Lemma 2.1.6. □

Our next result will imply an important property of completed cohomology: it is, in some sense, independent of the weight $\mathcal{V}$. It will be useful to work in a little more generality, so we assume that $\mathcal{V}$ is a smooth $\mathcal{O}/\varpi^m[K_{\mathrm{G},S_p\setminus S} \times \Delta_S]$-module, flat over $\mathcal{O}/\varpi^m$, for an open submonoid $\Delta_S \subset \mathrm{G}_S$ which contains an open subgroup U_S of G_S. As above, we associate to $\mathcal{V}$ a $G^T \times K_{\mathrm{G},T\setminus S} \times U_S$-equivariant sheaf on $\overline{\mathfrak{X}}_{\mathrm{G}}$ by pulling back from a point.

Lemma 2.1.8. *We have canonical isomorphisms*

$$R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{O}/\varpi^m) \otimes \mathcal{V} \xrightarrow{\sim} R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{V})$$

and

$$R\Gamma(\partial\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{O}/\varpi^m) \otimes \mathcal{V} \xrightarrow{\sim} R\Gamma(\partial\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{V})$$

in $D^+_{\mathrm{sm}}(G^T \times K_{\mathrm{G},T\setminus S} \times U_S, \mathcal{O}/\varpi^m)$.

Proof. We explain the case of $\overline{\mathfrak{X}}_{\mathrm{G}}$, the case of $\partial\overline{\mathfrak{X}}_{\mathrm{G}}$ is the same. Let $f : \overline{\mathfrak{X}}_{\mathrm{G}} \to *$ be the $\mathrm{G}(\mathbb{A}_{F,f})$-equivariant projection to a point. We set $H := G^T \times K_{\mathrm{G},T\setminus S} \times U_S$. There is a pair of adjoint functors (f^*, Rf_*) between $D^+(\mathrm{Sh}_H(*), \mathcal{O}/\varpi^m) \simeq D^+_{\mathrm{sm}}(H, \mathcal{O}/\varpi^m)$ and $D^+(\mathrm{Sh}_H(\overline{\mathfrak{X}}_{\mathrm{G}}), \mathcal{O}/\varpi^m)$. There is also a natural isomorphism

$$f^*(Rf_*(\mathcal{O}/\varpi^m) \otimes \mathcal{V}) \xrightarrow{\sim} f^*Rf_*(\mathcal{O}/\varpi^m) \otimes f^*\mathcal{V}$$

and hence by adjunction a natural map

$$f^*(Rf_*(\mathcal{O}/\varpi^m) \otimes \mathcal{V}) \to f^*\mathcal{V}.$$

We therefore have a morphism

$$R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{O}/\varpi^m) \otimes \mathcal{V} = Rf_*(\mathcal{O}/\varpi^m) \otimes \mathcal{V} \longrightarrow Rf_*f^*\mathcal{V} = R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{V}) \tag{2.1.3}$$

in $D^+_{\mathrm{sm}}(H, \mathcal{O}/\varpi^m)$. It is enough to show that this is an isomorphism after forgetting the equivariant structure. By [Sch98, §1, Corollary 3], if we forget the equivariant structure for the sheaves on $\overline{\mathfrak{X}}_{\mathrm{G}}$, the resulting derived functors compute cohomology with compact support. Since $\overline{\mathfrak{X}}_{\mathrm{G}}$ is a compact Hausdorff space, we can apply [KS94, Prop. 2.6.6], which implies that the morphism in (2.1.3) is an isomorphism (since $\mathcal{V}$ is flat over $\mathcal{O}/\varpi^m$, the assumption in *loc. cit.* that the coefficient ring has finite weak global dimension is not necessary). □

We can use this lemma to *define* an object

$$R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{V}) := R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{O}/\varpi^m) \otimes \mathcal{V} \in D^+_{\mathrm{sm}}(G^T \times K_{\mathrm{G},T\setminus S} \times \Delta_S, \mathcal{O}/\varpi^m).$$

It is independent of the choice of U_S, by [Sch98, §1, Corollary 3]. When $K_{\mathrm{G},S}$ is a compact open subgroup of Δ_S, we obtain $R\Gamma(K_{\mathrm{G}}, R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{V})) \in D^+(\mathcal{O}/\varpi^m)$ with an action of $\mathcal{H}(\mathrm{G}^T, K^T_{\mathrm{G}}) \otimes \mathcal{H}(\Delta_S, K_{\mathrm{G},S})$.

In §4.1, we will need a variant of this lemma with a coefficient system in a derived category. When we apply this lemma, we will just have group actions, not monoids, so we now assume $\mathcal{V} \in D^b_{\mathrm{sm}}(K_{\mathrm{G},S_p\setminus S}, \mathcal{O}/\varpi^m)$. After inflation and pullback from a point, we get a corresponding object $\mathcal{V} \in D^b(\mathrm{Sh}_{\mathrm{G}^T \times K_{\mathrm{G},T}}(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{O}/\varpi^m))$. To state the lemma, we need the derived tensor product functor $R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{O}/\varpi^m) \otimes^{\mathbb{L}} -$. Although $\mathrm{Mod}_{\mathrm{sm}}(G^T \times K_{\mathrm{G},T}, \mathcal{O}/\varpi^m)$ does not have enough projectives, every object has a surjection from a $\mathcal{O}/\varpi^m$-flat object. Indeed, smoothness implies that there is a surjection from a direct sum of copies of compact inductions of trivial representations of compact open subgroups on $\mathcal{O}/\varpi^m$[6]. This gives enough acyclic objects to compute derived tensor products on $D^-_{\mathrm{sm}}(G^T \times K_{\mathrm{G},T}, \mathcal{O}/\varpi^m)$. Since the functor $R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, -)$ on $D^+(\mathrm{Sh}_{\mathrm{G}^T \times K_{\mathrm{G},T}}(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{O}/\varpi^m))$ has bounded cohomological dimension (Lemma 2.1.4), it takes bounded objects to bounded objects.

Lemma 2.1.9. *Let* $\mathcal{V} \in D^b_{\mathrm{sm}}(K_{\mathrm{G},S_p\setminus S}, \mathcal{O}/\varpi^m)$. *We have a canonical isomorphism*

$$R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{O}/\varpi^m) \otimes^{\mathbb{L}} \mathcal{V} \xrightarrow{\sim} R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{V})$$

in $D^b_{\mathrm{sm}}(\mathrm{G}^T \times K_{\mathrm{G},T}, \mathcal{O}/\varpi^m)$.

[6] This is the usual proof of 'enough projectives' over a characteristic 0 field. The problem here is that the trivial representation with coefficients in $\mathcal{O}/\varpi^m$ of a non-trivial compact p-adic group is not projective.

Proof. As in the proof of Lemma 2.1.8, we set $H = \mathrm{G}^T \times K_{\mathrm{G},T}$ and consider $f : \overline{\mathfrak{X}}_{\mathrm{G}} \to *$ the map to the point. By Lemma 2.1.4, we have a pair of adjoint functors (f^*, Rf_*) between unbounded derived categories $D_{\mathrm{sm}}(H, \mathcal{O}/\varpi^m)$ and $D(\mathrm{Sh}_H(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{O}/\varpi^m))$. Since f^* is exact, it is easy to see that we have a natural isomorphism

$$f^*(Rf_*(\mathcal{O}/\varpi^m) \otimes^{\mathbb{L}} \mathcal{V}) \xrightarrow{\sim} f^*Rf_*(\mathcal{O}/\varpi^m) \otimes^{\mathbb{L}} f^*\mathcal{V}$$

and we then obtain a map

$$p_{\mathcal{V}} : R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{O}/\varpi^m) \otimes^{\mathbb{L}} \mathcal{V} \to R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{V})$$

by adjunction. The fact that this is an isomorphism follows from the case where $\mathcal{V}$ is a $\mathcal{O}/\varpi^m$-flat module. More precisely, we can replace $\mathcal{V}$ by a bounded above complex $\mathcal{F}^\bullet$ of $\mathcal{O}/\varpi^m$-flat objects in $\mathrm{Mod}_{\mathrm{sm}}(H, \mathcal{O}/\varpi^m)$, and replace $\mathcal{O}/\varpi^m$ by a bounded complex $\mathcal{I}^\bullet$ of Rf_*-acyclic objects in $\mathrm{Sh}_H(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{O}/\varpi^m)$. Using [KS94, Prop. 2.6.6] again, we see that each sheaf $\mathcal{I}^i \otimes \mathcal{F}^j$ is Rf_*-acyclic and the natural map

$$f_*\mathcal{I}^i \otimes \mathcal{F}^j \to f_*(\mathcal{I}^i \otimes \mathcal{F}^j)$$

is an isomorphism. The total complexes of the double complexes $f_*\mathcal{I}^\bullet \otimes \mathcal{F}^\bullet \to f_*(\mathcal{I}^\bullet \otimes \mathcal{F}^\bullet)$ respectively compute the source and target of $p_{\mathcal{V}}$, so we see that $p_{\mathcal{V}}$ is an isomorphism in $D^b_{\mathrm{sm}}(\mathrm{G}^T \times K_{\mathrm{G},T}, \mathcal{O}/\varpi^m)$. □

Remark 2.1.10. With Lemma 2.1.9 in hand, we can prove that $R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{O}/\varpi^m)$ has bounded Tor-dimension, and then extend our projection formula to handle $\mathcal{V}$ in the unbounded derived category. See for example [Fu15, Corollary 6.5.6] for the classical projection formula in l-adic cohomology.

Assume now that $R = \mathcal{O}$, and that $\mathcal{V}$ is an $\mathcal{O}[K_{\mathrm{G},S}]$-module, which is finite free as an $\mathcal{O}$-module and such that $\mathcal{V}/\varpi^m$ is a smooth $\mathcal{O}/\varpi^m[K_{\mathrm{G},S}]$-module for each $m \in \mathbb{Z}_{\geq 1}$. We then consider

$$R\Gamma(\overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}}, \mathcal{V}) := \varprojlim_m R\Gamma(\overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}}, \mathcal{V}/\varpi^m)$$

in $\mathrm{D}^+(\mathcal{O})$, where the projective limit should be understood as a homotopy limit. We also consider the analogue with coefficient system $j_!\mathcal{V}$. These limits can be endowed with an action of the Hecke algebra $\mathcal{H}(\mathrm{G}^S, K^S_{\mathrm{G}}) \otimes_{\mathbb{Z}} \mathcal{O}$[7].

Continue to assume $R = \mathcal{O}$. If $S \subseteq S_p(F)$ is a set of places of F above p, let $\mathcal{V}$ be an $\mathcal{O}[K_{S_p \setminus S}]$-module which is finite free as an $\mathcal{O}$-module and such that $\mathcal{V}/\varpi^m$ is a smooth $\mathcal{O}/\varpi^m[K_{S_p \setminus S}]$-module. We consider

$$R\Gamma(K^S_{\mathrm{G}}, R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{V})) := \varprojlim_m R\Gamma(K^S_{\mathrm{G}}, R\Gamma(\overline{\mathfrak{X}}_{\mathrm{G}}, \mathcal{V}/\varpi^m))$$

in $\mathrm{D}^+(\mathrm{G}_S, \mathcal{O})$, where again the projective limit should be understood as a homotopy limit. There is also the analogue with coefficient system $j_!\mathcal{V}$. When $T \supseteq S_p(F)$ is a finite set of finite places of F, these limits can also be endowed with an action of the Hecke algebra $\mathcal{H}(\mathrm{G}^T, K^T_{\mathrm{G}}) \otimes_{\mathbb{Z}} \mathcal{O}$.

[7]As the proof of Lemma 2.1.6 shows, we have explicit perfect complexes that compute these derived functors, so we could simply take a projective limit on the level of complexes. To endow the projective limit with a Hecke action, we can instead consider adelic complexes that compute these derived functors as in [CGJ19, §5.1]. Combining the fact that derived limits commute with cohomology [Sta13, Tag 08U1] and Lemma 2.1.5 we can show that we will obtain the same Hecke actions as in [NT16].

We now assume that G is reductive, and let $\mathrm{P} = \mathrm{MN}$ be a parabolic subgroup with Levi subgroup M. Let $K_{\mathrm{G}} \subset \mathrm{G}(\mathbb{A}_{F,f})$ be a good subgroup. In this situation, we define $K_{\mathrm{P}} = K_{\mathrm{G}} \cap \mathrm{P}(\mathbb{A}_{F,f})$, $K_{\mathrm{N}} = K_{\mathrm{G}} \cap \mathrm{N}(\mathbb{A}_{F,f})$, and define K_{M} to be the image of K_{P} in $\mathrm{M}(\mathbb{A}_{F,f})$. We say that K_{G} is *decomposed* with respect to $\mathrm{P} = \mathrm{MN}$ if we have $K_{\mathrm{P}} = K_{\mathrm{M}} \ltimes K_{\mathrm{N}}$; equivalently, if $K_{\mathrm{M}} = K_{\mathrm{G}} \cap \mathrm{M}(\mathbb{A}_{F,f})$.

Assume now that K_{G} is decomposed with respect to $\mathrm{P} = \mathrm{MN}$, and let S be a finite set of finite places of F such that for all $v \notin S$, $K_{\mathrm{G},v}$ is a hyperspecial maximal compact subgroup of $\mathrm{G}(F_v)$. In this case, we can define homomorphisms

$$r_{\mathrm{P}} : \mathcal{H}(\mathrm{G}^S, K_{\mathrm{G}}^S) \to \mathcal{H}(\mathrm{P}^S, K_{\mathrm{P}}^S) \text{ and } r_{\mathrm{M}} : \mathcal{H}(\mathrm{P}^S, K_{\mathrm{P}}^S) \to \mathcal{H}(\mathrm{M}^S, K_{\mathrm{M}}^S),$$

given respectively by "restriction to P" and "integration along N"; see [NT16, §2.2.3] and [NT16, §2.2.4] respectively for the definitions of these maps, along with the proofs that they are indeed algebra homomorphisms, and that r_{M} preserves integrality. We use $\mathcal{S} := r_{\mathrm{M}} \circ r_{\mathrm{P}}$ to denote the unnormalised Satake transform.

Finally, we remark that the above formalism also applies to the case of the Hecke algebra of a monoid, see [ACC+18, §2.1.8].

2.1.11. *The general linear group and the quasi-split unitary group.* From now on, we fix an integer $n \geq 2$ and we let F be an imaginary CM field containing the maximal totally real subfield F^+. Let $c \in \mathrm{Gal}(F/F^+)$ denote complex conjugation. We set $\overline{S}_p := S_p(F^+)$ and $S_p := S_p(F)$. We let Ψ_n be the matrix with 1's on the anti-diagonal and 0's elsewhere, and we let

$$J_n = \begin{pmatrix} 0 & \Psi_n \\ -\Psi_n & 0 \end{pmatrix}.$$

We let $\widetilde{G}/\mathcal{O}_{F^+}$ be the group scheme defined by

$$\widetilde{G}(R) = \{g \in \mathrm{GL}_{2n}(R \otimes_{\mathcal{O}_{F^+}} \mathcal{O}_F) \mid {}^t g J_n g^c = J_n\}$$

for any $\mathcal{O}_{F^+}$-algebra R. The generic fibre of $\widetilde{G}$ over F^+ is a quasi-split unitary group, which becomes isomorphic to GL_{2n}/F after base change from F^+ to F. In particular, if $\bar{v}$ is a place of F^+ that splits in F, a choice of place $v \mid \bar{v}$ of F determines a canonical isomorphism $\iota_v : G(F^+_{\bar{v}}) \xrightarrow{\sim} \mathrm{GL}_{2n}(F_v)$.

We let $P \subset \widetilde{G}$ denote the Siegel parabolic consisting of block upper-triangular matrices with blocks of size $n \times n$. We let $P = U \rtimes G$ be a Levi decomposition such that we can identify G with $\mathrm{Res}_{\mathcal{O}_F/\mathcal{O}_{F^+}} \mathrm{GL}_n$[8]. To simplify the notation, from now on we write $\widetilde{X}$ for $X^{\widetilde{G}}$ and X for X^G. We also write $\widetilde{K}$ and K for good subgroups of $\widetilde{G}(\mathbb{A}_{F^+,f})$ and of $G(\mathbb{A}_{F^+,f}) = \mathrm{GL}_n(\mathbb{A}_{F,f})$. Note that the locally symmetric spaces $\widetilde{X}_{\widetilde{K}}$ are complex manifolds of (complex) dimension $d := n^2[F^+ : \mathbb{Q}]$, whereas the locally symmetric spaces X_K are real manifolds of (real) dimension $d - 1$.

We now describe some explicit (integral and rational) coefficient systems for these symmetric spaces. These will depend on a choice of a prime p and on a choice of a dominant weight for either G or $\widetilde{G}$. We fix a coefficient field $E/\mathbb{Q}_p$ which is assumed to be sufficiently large, so that it contains the image of every embedding $\mathrm{Hom}(F, \overline{\mathbb{Q}}_p)$. Let $T \subset \widetilde{B} \subset \widetilde{G}$ be the maximal torus of diagonal matrices and the upper triangular Borel subgroup, respectively. Set $B := \widetilde{B} \cap G$, this can be identified with the upper triangular Borel subgroup in G.

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[8] We use the same identification as in [ACC+18, §2.2.1], namely $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right) \in G(R) \mapsto D \in \mathrm{GL}_n(R \otimes_{\mathcal{O}_{F^+}} \mathcal{O}_F)$ for an $\mathcal{O}_{F^+}$-algebra R.

We first treat the case of G. We identify the character group of $(\mathrm{Res}_{F^+/\mathbb{Q}}T)_E$ with $(\mathbb{Z}^n)^{\mathrm{Hom}(F,E)}$ in the usual way. A weight $(\lambda_{\tau,i}) \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,E)}$ with $\tau \in \mathrm{Hom}(F,E)$ and $i \in 1,\dots,n$ is dominant for $(\mathrm{Res}_{F^+/\mathbb{Q}}B)_E$ if it satisfies

$$\lambda_{\tau,1} \geq \lambda_{\tau,2} \geq \cdots \geq \lambda_{\tau,n}$$

for each $\tau \in \mathrm{Hom}(F,E)$. We denote by $(\mathbb{Z}^n_+)^{\mathrm{Hom}(F,E)}$ the subset of dominant weights. The expression 'λ is a dominant weight for G' will indicate that a weight $\lambda \in X^*((\mathrm{Res}_{F^+/\mathbb{Q}}T)_E)$ is dominant for $(\mathrm{Res}_{F^+/\mathbb{Q}}B)_E$.

Assume now that $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F,E)}$. We define the $G(\mathcal{O}_{F^+,p}) = \prod_{v\in S_p} G(\mathcal{O}_{F_v})$-representation $\mathcal{V}_\lambda$ to be the integral dual Weyl module of highest weight λ with coefficients in $\mathcal{O}$, obtained from the Borel–Weil construction. More precisely, if we let $\mathrm{B}_n \subset \mathrm{GL}_n$ denote the standard Borel consisting of upper-triangular matrices and $w_{0,n}$ denote the longest element in the Weyl group of GL_n, we consider the algebraic induction

$$(\mathrm{Ind}_{\mathrm{B}_n}^{\mathrm{GL}_n} w_{0,n}\lambda_\tau)_{/\mathcal{O}} := \{f \in \mathcal{O}[\mathrm{GL}_n] \mid f(bg) = (w_{0,n}\lambda_\tau)(b)f(g),$$
$$\forall\ \mathcal{O} \to R, b \in \mathrm{B}_n(R), g \in \mathrm{GL}_n(R)\},$$

and we set $\mathcal{V}_{\lambda_\tau}$ to be the finite free $\mathcal{O}$-module obtained by evaluating this on $\mathcal{O}$ and $V_{\lambda_\tau} := \mathcal{V}_{\lambda_\tau} \otimes_{\mathcal{O}} E$. When τ induces the place v of F, these modules come with an action of $\mathrm{GL}_n(\mathcal{O}_{F_v})$ and $\mathrm{GL}_n(F_v)$ respectively.

Finally, we set $\mathcal{V}_\lambda := \otimes_{\tau,\mathcal{O}}\mathcal{V}_{\lambda_\tau}$ and $V_\lambda := \mathcal{V}_\lambda \otimes_{\mathcal{O}} E$. Then V_λ is the absolutely irreducible algebraic representation of $(\mathrm{Res}_{F^+/\mathbb{Q}}G)_E$ of highest weight λ and it is finite-dimensional over E; the lattice $\mathcal{V}_\lambda \subset V_\lambda$ is $G(\mathcal{O}_{F^+,p})$-stable. For every $m \in \mathbb{Z}_{\geq 1}$, $\mathcal{V}_\lambda/\varpi^m$, is a smooth $\mathcal{O}/\varpi^m[G(\mathcal{O}_{F^+,p})]$-module that is finite free as an $\mathcal{O}/\varpi^m$-module. Therefore, the formalism of the previous section applies to $\mathcal{V}_\lambda$.

We now treat the case of $\widetilde{G}$. Assume that each place in $\overline{S}_p$ splits from F^+ to F and that we have a partition of the form $S_p = \widetilde{S}_p \sqcup \widetilde{S}_p^c$, with $\tilde{v} \in \widetilde{S}_p$ the place lying above a place $\bar{v} \in \overline{S}_p$. This induces a partition on $\mathrm{Hom}(F,E)$, by choosing the embedding $\tilde{\tau} : F \hookrightarrow E$ above a given embedding $\tau : F^+ \hookrightarrow E$ that induces a place in $\widetilde{S}_p$. In turn, this induces an identification

$$(\mathrm{Res}_{F^+/\mathbb{Q}}\widetilde{G})_E = \prod_{\mathrm{Hom}(F^+,E)} \mathrm{GL}_{2n,E}$$

and therefore an identification of the character group of $(\mathrm{Res}_{F^+/\mathbb{Q}}T)_E$ with $(\mathbb{Z}^{2n})^{\mathrm{Hom}(F^+,E)}$. More precisely, this identifies a weight $\lambda = (\lambda_{\tilde{\tau},i})$ with a weight $\tilde{\lambda} = (\tilde{\lambda}_{\tau,i})$ where

$$\tilde{\lambda}_\tau = (-\lambda_{\tilde{\tau}c,n},\dots,,-\lambda_{\tilde{\tau}c,1},\lambda_{\tilde{\tau},1},\dots,\lambda_{\tilde{\tau},n}). \tag{2.1.4}$$

The set of weights that are dominant for $(\mathrm{Res}_{F^+/\mathbb{Q}}\widetilde{B})_E$ are the ones in the subset $(\mathbb{Z}^{2n}_+)^{\mathrm{Hom}(F^+,E)}$. For such weights, we can therefore define the integral dual Weyl module of highest weight $\tilde{\lambda}$, $\mathcal{V}_{\tilde{\lambda}} \subset V_{\tilde{\lambda}}$, a $\widetilde{G}(\mathcal{O}_{F^+,p})$-stable $\mathcal{O}$-lattice in the highest weight $\tilde{\lambda}$ representation of $(\mathrm{Res}_{F^+/\mathbb{Q}}\widetilde{G})_E$. For every $m \in \mathbb{Z}_{\geq 1}$, $\mathcal{V}_{\tilde{\lambda}}/\varpi^m$, is a smooth $\mathcal{O}/\varpi^m[\prod_{\bar{v}\in\overline{S}_p}\widetilde{G}(\mathcal{O}_{F^+_{\bar{v}}})]$-module that is finite free as an $\mathcal{O}/\varpi^m$-module. Therefore, the formalism of the previous section also applies to $\mathcal{V}_{\tilde{\lambda}}$. We say '$\tilde{\lambda}$ is a dominant weight for $\widetilde{G}$' to indicate that a weight $\tilde{\lambda} \in X^*((\mathrm{Res}_{F^+/\mathbb{Q}}T)_E)$ is dominant for $(\mathrm{Res}_{F^+/\mathbb{Q}}\widetilde{B})_E$.

We now define appropriate quotients of the Hecke algebras acting on the cohomology groups with these coefficient systems. Again, we treat G first. Let $S \supseteq S_p$ be a finite set of finite places of F and let $K \subset \mathrm{GL}_n(\mathbb{A}_{F,f})$ be a good subgroup such that $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ for $v \notin S$ and $K_v \subseteq \mathrm{GL}_n(\mathcal{O}_{F_v})$ for $v \in S_p$. For any $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F,E)}$, the complex $R\Gamma(X_K, \mathcal{V}_\lambda)$ is well-defined as an object of $D^+(\mathcal{O})$ (up to unique isomorphism) and equipped with a Hecke action. We set $\mathbb{T}^S := \mathcal{H}(G^S, K^S) \otimes_{\mathbb{Z}} \mathcal{O}$ and

$$\mathbb{T}^S(K,\lambda) := \mathrm{Im}\left(\mathbb{T}^S \to \mathrm{End}_{D^+(\mathcal{O})}(R\Gamma(X_K, \mathcal{V}_\lambda))\right).$$

In the case of $\widetilde{G}$, let $S \supseteq S_p$ be a finite set of finite places of F satisfying $S = S^c$. Let $\overline{S}$ denote the set of finite places of F^+ below S. Let $\widetilde{K} \subset \widetilde{G}(\mathbb{A}_{F^+,f})$ be a good compact open subgroup such that $\widetilde{K}_{\bar{v}} = \widetilde{G}(\mathcal{O}_{F^+_{\bar{v}}})$ for $\bar{v} \notin \overline{S}$ and $\widetilde{K}_{\bar{v}} \subseteq \widetilde{G}(\mathcal{O}_{F^+_{\bar{v}}})$ for $\bar{v} \in \overline{S}_p$. To simplify notation, we write $\widetilde{G}^S = \widetilde{G}^{\overline{S}}$ etc. For any $\tilde{\lambda} \in (\mathbb{Z}^{2n}_+)^{\mathrm{Hom}(F^+,E)}$, the complex $R\Gamma(X_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}})$ is well-defined as an object of $D^+(\mathcal{O})$ (up to unique isomorphism) and equipped with a Hecke action. We consider the abstract Hecke $\mathcal{O}$-algebra $\widetilde{\mathbb{T}}^S := \mathcal{H}(\widetilde{G}^S, \widetilde{K}^S) \otimes_{\mathbb{Z}} \mathcal{O}$ and its quotient $\mathbb{T}^S(\widetilde{K}, \tilde{\lambda})$ acting faithfully on $R\Gamma(X_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}})$.

As a consequence of Lemma 2.1.6, we see that both $\mathbb{T}^S(K,\lambda)$ and $\widetilde{\mathbb{T}}^S(\widetilde{K}, \tilde{\lambda})$ are finite $\mathcal{O}$-modules. There are obvious versions of all of this with $\mathcal{O}/\varpi^m$-coefficients and for compactly supported cohomology and for the cohomology of the boundary $\partial X_{\widetilde{K}}$ of the Borel–Serre compactification of $X_{\widetilde{K}}$.

We will make use of particular elements of some Weyl groups, besides the longest element $w_{0,n}$ in the Weyl group of GL_n which we have already mentioned. For $\mathrm{G} = G$ or $\widetilde{G}$, we will write w_0^{G} for the longest element in the Weyl group $W((\mathrm{Res}_{F^+/\mathbb{Q}}\mathrm{G})_E, (\mathrm{Res}_{F^+/\mathbb{Q}}T)_E)$. We set $w_0^P = w_0^G w_0^{\widetilde{G}}$. It is the longest element in the set W^P of minimal length coset representatives for

$$W((\mathrm{Res}_{F^+/\mathbb{Q}}\widetilde{G})_E, (\mathrm{Res}_{F^+/\mathbb{Q}}T)_E)/W((\mathrm{Res}_{F^+/\mathbb{Q}}G)_E, (\mathrm{Res}_{F^+/\mathbb{Q}}T)_E).$$

In our development of P-ordinary Hida theory, it will be important to compare coefficient systems for $\widetilde{G}$ and G.

Let $\mathrm{P}_{n,n} \subset \mathrm{GL}_{2n}$ be the parabolic subgroup of block-upper triangular matrices with Levi quotient $\mathrm{GL}_n \times \mathrm{GL}_n$. By the transitivity of algebraic induction, $\mathcal{V}_{\tilde{\lambda}_\tau}$ and $V_{\tilde{\lambda}_\tau}$ can be identified with the evaluation on $\mathcal{O}$ and E respectively of the algebraic induction

$$\left(\mathrm{Ind}_{\mathrm{P}_{n,n}}^{\mathrm{GL}_{2n}} \mathcal{V}_{\lambda_{\tilde{\tau}}} \otimes \mathcal{V}_{-w_{0,n}\lambda_{\tilde{\tau}c}}\right)_{/\mathcal{O}}.$$

Lemma 2.1.12. *The natural* $\mathrm{P}_{n,n}(\mathcal{O})$*-equivariant morphism*

$$\mathcal{V}_{\tilde{\lambda}_\tau} \to \mathcal{V}_{\lambda_{\tilde{\tau}}} \otimes \mathcal{V}_{-w_{0,n}\lambda_{\tilde{\tau}c}}$$

given by evaluation of functions at the identity is surjective.

Proof. By transitivity of parabolic induction, we can identify $\mathcal{V}_{\tilde{\lambda}_\tau}$ with the evaluation on $\mathcal{O}$ of

$$\left(\mathrm{Ind}_{\mathrm{B}_{2n}}^{\mathrm{GL}_{2n}} w_{0,2n}\tilde{\lambda}_\tau\right)_{/\mathcal{O}} \xrightarrow{\sim} \left(\mathrm{Ind}_{\mathrm{P}_{n,n}}^{\mathrm{GL}_{2n}} \circ \mathrm{Ind}_{\mathrm{B}_n \times B_n}^{\mathrm{GL}_n \times \mathrm{GL}_n} w_{0,2n}\tilde{\lambda}_\tau\right)_{/\mathcal{O}},$$

where, by [Jan03, §I.3.5], the map is given by $f \mapsto \tilde{f}(g)(h) = f(hg)$ for all $\mathcal{O}$-algebras R, $h \in \mathrm{GL}_n(R) \times \mathrm{GL}_n(R)$, and $g \in \mathrm{GL}_{2n}(R)$. By Nakayama, it is enough to check surjectivity after base change to $\overline{\mathbb{F}}_p$, in which case the evaluation at identity map can be rewritten in geometric terms as the restriction map

$$H^0(X, \mathcal{L}) \to H^0(X', \mathcal{L}),$$

where $X = \mathrm{B}_{2n} \backslash \mathrm{GL}_{2n}$ is the full flag variety for GL_{2n}, $X' \subset X$ is the Schubert variety for the longest Weyl group element in $\mathrm{GL}_n \times \mathrm{GL}_n$ and $\mathcal{L}$ is the line bundle on X determined by $w_{0,2n}\tilde{\lambda}$. The result now follows from the main theorem of [And85] applied to $\mathrm{SL}_{2n}/\overline{\mathbb{F}}_p$. □

2.1.13. *Explicit Hecke operators.* Fix once and for all a choice $\varpi_{\bar{v}}$ of uniformiser of $F^+_{\bar{v}}$ for every finite place $\bar{v}$ of F^+. When $\bar{v}$ is unramified in F we set $\varpi_v = \varpi_{\bar{v}}$ for $v|\bar{v}$.

We define some explicit Hecke operators at unramified primes first. If v is a finite place of F and $1 \le i \le n$ is an integer then we write $T_{v,i} \in \mathcal{H}(\mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_{F_v}))$ for the double coset operator

$$T_{v,i} = [\mathrm{GL}_n(\mathcal{O}_{F_v})\mathrm{diag}(\varpi_v, \dots, \varpi_v, 1, \dots, 1)\,\mathrm{GL}_n(\mathcal{O}_{F_v})],$$

where ϖ_v appears i times on the diagonal. This is the same as the operator denoted by $T_{M,v,i}$ in [NT16, Prop.-Def. 5.3]. We define a polynomial

$$\begin{aligned} P_v(X) = X^n - T_{v,1}X^{n-1} + \dots + (-1)^i q_v^{i(i-1)/2} T_{v,i} X^{n-i} + \dots \\ + q_v^{n(n-1)/2} T_{v,n} \in \mathcal{H}(\mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_{F_v}))[X]. \end{aligned} \tag{2.1.5}$$

It corresponds to the characteristic polynomial of a Frobenius element on $\mathrm{rec}^T_{F_v}(\pi_v)$, where π_v is an unramified representation of $\mathrm{GL}_n(F_v)$.

If $\overline{v}$ is a place of F^+ unramified in F, and v is a place of F above $\overline{v}$, and $1 \le i \le 2n$ is an integer, then we write $\widetilde{T}_{v,i} \in \mathcal{H}(\widetilde{G}(F^+_{\overline{v}}), \widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}})) \otimes_{\mathbb{Z}} \mathbb{Z}[q_{\overline{v}}^{-1}]$ for the operator denoted $T_{G,v,i}$ in [NT16, Prop.-Def. 5.2]. We define a polynomial

$$\begin{aligned} \widetilde{P}_v(X) = X^{2n} - \widetilde{T}_{v,1}X^{2n-1} + \dots + (-1)^j q_v^{j(j-1)/2} \widetilde{T}_{v,j} + \dots \\ + q_v^{n(2n-1)} \widetilde{T}_{v,2n} \in \mathcal{H}(\widetilde{G}(F^+_{\overline{v}}), \widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}})) \otimes_{\mathbb{Z}} \mathbb{Z}[q_{\overline{v}}^{-1}][X]. \end{aligned} \tag{2.1.6}$$

It corresponds to the characteristic polynomial of a Frobenius element on $\mathrm{rec}^T_{F_v}(\pi_v)$, where π_v is the base change of an unramified representation $\sigma_{\overline{v}}$ of the group $\widetilde{G}(F^+_{\overline{v}})$.

We now describe the behaviour of these Hecke operators under the unnormalised Satake transform with respect to the Siegel parabolic. We use the following convention: if $f(X)$ is a polynomial of degree d, with constant term a unit a_0, we set $f^\vee(X) := a_0^{-1} X^d f(X^{-1})$.

Proposition 2.1.14. *Let v be a place of F, unramified over the place $\overline{v}$ of F^+. Let*

$$\mathcal{S} : \mathcal{H}(\widetilde{G}(F^+_{\overline{v}}), \widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}})) \to \mathcal{H}(G(F^+_{\overline{v}}), G(\mathcal{O}_{F^+_{\overline{v}}}))$$

denote the homomorphism defined at the end of §2.1.1. Then we have

$$\mathcal{S}(\widetilde{P}_v(X)) = P_v(X) q_v^{n(2n-1)} P^\vee_{v^c}(q_v^{1-2n}X).$$

Proof. See [NT16, §5.1]. □

We now discuss some Hecke operators at (possibly ramified) places in $\overline{S}_p$. Assume that each prime $\bar{v}$ of F^+ above p splits in F. Let $\bar{v} \in \overline{S}_p$, and recall that $\tilde{v}$ is a chosen prime of F above it. For integers $c \ge b \ge 0$, we define subgroups

$$\mathcal{P}_{\bar{v}}(b, c) \subset \widetilde{G}(\mathcal{O}_{F^+_{\bar{v}}}) = \mathrm{GL}_{2n}(\mathcal{O}_{F_{\tilde{v}}})$$

which reduce to block upper-triangular matrices (with two $n \times n$ blocks) modulo $\varpi_{\tilde{v}}^c$ and to block unipotent matrices modulo $\varpi_{\tilde{v}}^b$. We set $\mathcal{P}_{\bar{v}} = \mathcal{P}_{\bar{v}}(0,1)$, which is identified with the standard parahoric subgroup $\mathcal{P}_{n,n}$ of GL_{2n}. For each parabolic subgroup $Q_{\bar{v}}$ of $P_{\bar{v}}$ which contains $\widetilde{B}_{\bar{v}}$ we have an associated parahoric subgroup $\mathcal{Q}_{\bar{v}} \subset \mathcal{P}_{\bar{v}}$. We note that these subgroups all admit an Iwahori decomposition with respect to $P_{\bar{v}}$, and therefore the formalism of [ACC+18, §2.1.9] applies when we consider the Hecke algebras of monoids.

Write $\tilde{u}_{\tilde{v},n} := \mathrm{diag}(\varpi_{\tilde{v}}, \ldots, \varpi_{\tilde{v}}, 1, \ldots, 1) \in \mathrm{GL}_{2n}(F_{\tilde{v}})$, where $\varpi_{\tilde{v}}$ appears exactly n times on the diagonal. If $c \geq 1$, we write $\widetilde{U}_{\tilde{v},n} \in \mathcal{H}\left(\widetilde{G}(\mathcal{O}_{F^+_{\bar{v}}}), \mathcal{P}_{\bar{v}}(b,c)\right)$ for the double coset operator $\widetilde{U}_{\tilde{v},n} = [\mathcal{P}_{\bar{v}}(b,c)\iota_{\tilde{v}}^{-1}\tilde{u}_{\tilde{v},n}\mathcal{P}_{\bar{v}}(b,c)]$. Also write $\tilde{u}_{\tilde{v},2n} := \mathrm{diag}(\varpi_{\tilde{v}}, \ldots, \varpi_{\tilde{v}}) \in \mathrm{GL}_{2n}(F_{\tilde{v}})$ and denote by $\widetilde{U}_{\tilde{v},2n} \in \mathcal{H}\left(\widetilde{G}(\mathcal{O}_{F^+_{\bar{v}}}), \mathcal{P}_{\bar{v}}(b,c)\right)$ the corresponding double coset operator. Note that these depend on both the choice of uniformiser $\varpi_{\tilde{v}}$ and on the chosen level. We write $\widetilde{\Delta}_{\bar{v}} \subset \widetilde{G}(F^+_{\bar{v}})$ for the subset

$$\widetilde{\Delta}_{\bar{v}} := \iota_{\tilde{v}}^{-1}\left(\sqcup_{\mu_1 \in \mathbb{Z}_+} \sqcup_{\mu_2 \in \mathbb{Z}} \mathcal{P}_{n,n}(\tilde{u}_{\tilde{v},n})^{\mu_1}(\tilde{u}_{\tilde{v},2n})^{\mu_2}\mathcal{P}_{n,n}\right),$$

which is independent of the choice of $\tilde{v} \mid \bar{v}$.

Considering cohomology at level $\mathcal{P}_{\bar{v}}$ and the ordinary subspace for the Hecke operator $\widetilde{U}_{\tilde{v},n}$ will be most important for us. However, we will work a little more generally to allow us to keep track of additional Hecke operators at $\bar{v}$ and prove a local–global compatibility result for ordinary as well as crystalline representations.

So, more generally, we suppose we have a parabolic subgroup $\widetilde{B}_{\bar{v}} \subset Q_{\bar{v}} \subset P_{\bar{v}}$ corresponding to a subset $I \subset \Delta$ of the simple roots, and with Levi decomposition $Q_{\bar{v}} = M_{Q_{\bar{v}}}N_{Q_{\bar{v}}}$ compatible with the decomposition $P_{\bar{v}} = G_{\bar{v}}U_{\bar{v}}$. We consider the monoid of cocharacters

$$X_{Q_{\bar{v}}} := \{\nu \in X_*(Z(M_{Q_{\bar{v}}})) : \langle \nu, \delta \rangle \geq 0 \text{ for all } \delta \in \Delta - I\}.$$

In fact, this is simply the subset of $\widetilde{B}_{\bar{v}}$-dominant cocharacters in $X_*(Z(M_{Q_{\bar{v}}}))$. We then define a subset $\widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}} \subset \widetilde{G}(F^+_{\bar{v}})$ containing the parahoric subgroup $\mathcal{Q}_{\bar{v}} \subset \mathcal{P}_{\bar{v}}$ by

$$\widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}} := \coprod_{\nu \in X_{Q_{\bar{v}}}} \mathcal{Q}_{\bar{v}}\nu(\varpi_{\bar{v}})\mathcal{Q}_{\bar{v}}.$$

We have $\widetilde{\Delta}_{\bar{v}}^{\mathcal{P}_{\bar{v}}} = \widetilde{\Delta}_{\bar{v}}$.

We set $\Delta_{\bar{v}}^{\mathcal{Q}_{\bar{v}},+} := \widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}} \cap G_{F^+_{\bar{v}}}$ and $\Delta_{\bar{v}}^{\mathcal{Q}_{\bar{v}}} := \Delta_{\bar{v}}^{\mathcal{Q}_{\bar{v}},+}[\iota_{\tilde{v}}^{-1}(\tilde{u}_{\tilde{v},n}^{-1})]$ (the submonoid of $G(F^+_{\bar{v}})$ generated by $\Delta_{\bar{v}}^{\mathcal{Q}_{\bar{v}},+}$ and its central element $\iota_{\tilde{v}}^{-1}(\tilde{u}_{\tilde{v},n}^{-1})$).

Lemma 2.1.15. (1) *For $\nu \in X_{Q_{\bar{v}}}$, the element $\nu(\varpi_{\bar{v}})$ is $\mathcal{Q}_{\bar{v}}$-*positive*; i.e. we have*

$$\nu(\varpi_{\bar{v}})\left(N_{Q_{\bar{v}}} \cap \mathcal{Q}_{\bar{v}}\right)\nu(\varpi_{\bar{v}})^{-1} \subset N_{Q_{\bar{v}}} \cap \mathcal{Q}_{\bar{v}}$$
$$\text{and } \nu(\varpi_{\bar{v}})^{-1}\left(\overline{N}_{Q_{\bar{v}}} \cap \mathcal{Q}_{\bar{v}}\right)\nu(\varpi_{\bar{v}}) \subset \overline{N}_{Q_{\bar{v}}} \cap \mathcal{Q}_{\bar{v}}.$$

(2) *$\widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}$ is a monoid under multiplication.*
(3) *The map $[(M_{Q_{\bar{v}}} \cap \mathcal{Q}_{\bar{v}})\nu(\varpi_{\bar{v}})(M_{Q_{\bar{v}}} \cap \mathcal{Q}_{\bar{v}})] \mapsto [\mathcal{Q}_{\bar{v}}\nu(\varpi_{\bar{v}})\mathcal{Q}_{\bar{v}}]$ defines a ring isomorphism of Hecke algebras*

$$\mathcal{H}(M_{Q_{\bar{v}}} \cap \widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}, M_{Q_{\bar{v}}} \cap \mathcal{Q}_{\bar{v}}) \xrightarrow{\sim} \mathcal{H}(\widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}, \mathcal{Q}_{\bar{v}})$$

which also factors through an isomorphism to $\mathcal{H}(\Delta_{\bar{v}}^{\mathcal{Q}_{\bar{v}},+}, G(F^+_{\bar{v}}) \cap \mathcal{Q}_{\bar{v}})$.

Proof. The first part can be checked directly, or using root groups. The second part follows from the first, using the Iwahori decomposition of $\mathcal{Q}_{\bar{v}}$. The third part is [BK98, Corollary 6.12]. $\square$

Remark 2.1.16. Our monoids are usually strictly contained in those defined in [ACC+18, §2.1.9]. We only need to consider Hecke operators supported on double cosets of central elements in the Levi subgroup, which in particular implies (as shown in the preceding lemma) that the Hecke algebra $\mathcal{H}(\widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}, \mathcal{Q}_{\bar{v}})$ is commutative.

Fix $\bar{v} \in \bar{S}$ and $\tau \in \mathrm{Hom}(F_{\bar{v}}^+, E)$. Let $\tilde{\lambda} \in \left(\mathbb{Z}_+^{2n}\right)^{\mathrm{Hom}(F^+,E)}$. We define a character $\tilde{\alpha}_{\tilde{\lambda}_\tau}^{\mathcal{Q}_{\bar{v}}} : \widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}} \to E^\times$ by setting

$$\tilde{\alpha}_{\tilde{\lambda}_\tau}^{\mathcal{Q}_{\bar{v}}}(\nu(\varpi_{\bar{v}})) = \tau(\varpi_{\bar{v}})^{\langle \nu, w_0^{\widetilde{G}} \tilde{\lambda}_\tau \rangle}$$

and setting the character to be trivial on $\mathcal{Q}_{\bar{v}}$.

Lemma 2.1.17. *Fix $\bar{v} \in \bar{S}$ and $\tau \in \mathrm{Hom}(F_{\bar{v}}^+, E)$. Define an action of $\mathcal{O}[\widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}]$ on $V_{\tilde{\lambda}_\tau}$ by*

$$g \cdot_{\tilde{\lambda}_\tau}^{\mathcal{Q}_{\bar{v}}} x := \tilde{\alpha}_{\tilde{\lambda}_\tau}^{\mathcal{Q}_{\bar{v}}}(g)^{-1} g \cdot x, \tag{2.1.7}$$

where $g \cdot x$ is the usual action of $g \in \widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}} \subset \widetilde{G}(F_{\bar{v}}^+)$ on $x \in V_{\tilde{\lambda}_\tau}$. The lattice $\mathcal{V}_{\tilde{\lambda}_\tau}$ is stable under the $\cdot_{\tilde{\lambda}_\tau}^{\mathcal{Q}_{\bar{v}}}$-action of $\mathcal{O}[\widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}]$.

Proof. This follows from the fact that the re-scaled action of $\widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}$ stabilizes each weight space in $\mathcal{V}_{\tilde{\lambda}_\tau}$, which has lowest weight $w_{0,2n}\tilde{\lambda}_\tau$. Cf. [Ger19, Definition 2.8]. $\square$

Suppose we have a subset $\bar{S} \subseteq \bar{S}_p$ and standard parabolic subgroups $Q_{\bar{v}} \subset P_{\bar{v}}$ for each $\bar{v} \in \bar{S}$. Then we set $\widetilde{\Delta}_{\bar{S}}^{\mathcal{Q}_{\bar{S}}} := \prod_{\bar{v} \in \bar{S}} \widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}$, with similar notation for Δ. Let $\tilde{\lambda} \in \left(\mathbb{Z}_+^{2n}\right)^{\mathrm{Hom}(F^+,E)}$. If we omit the superscript $\mathcal{Q}_{\bar{S}}$, we take $Q_{\bar{v}} = P_{\bar{v}}$ for all $\bar{v} \in \bar{S}$.

As a consequence of Lemma 2.1.17, we have constructed a twisted action of $\mathcal{O}[\widetilde{\Delta}_{\bar{S}}^{\mathcal{Q}_{\bar{S}}}]$ on $V_{\tilde{\lambda}}$ which stabilizes the lattice $\mathcal{V}_{\tilde{\lambda}}$. This action is obtained from the usual action by rescaling with the inverse of the character

$$\tilde{\alpha}_{\tilde{\lambda}}^{\mathcal{Q}_{\bar{S}}} := \prod_{\bar{v} \in \bar{S}} \prod_{\tau \in \mathrm{Hom}(F_{\bar{v}}^+, E)} \tilde{\alpha}_{\tilde{\lambda}_\tau}^{\mathcal{Q}_{\bar{v}}}.$$

We construct a similar rescaled action for the Levi subgroup G. Suppose $\lambda \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,E)}$ is a dominant weight for G. Recall that we have identified λ with a (not necessarily dominant) weight $\tilde{\lambda}$ of $\widetilde{G}$. We define a character $\alpha_{\lambda}^{\mathcal{Q}_{\bar{v}}} : \Delta_{\bar{v}}^{\mathcal{Q}_{\bar{v}}} \to E^\times$ using the formula

$$\alpha_{\lambda}^{\mathcal{Q}_{\bar{v}}}(\nu(\varpi_{\bar{v}})) = \prod_{\tau \in \mathrm{Hom}(F_{\bar{v}}^+, E)} \tau(\varpi_{\bar{v}})^{\langle \nu, w_0^G \tilde{\lambda}_\tau \rangle}$$

and a rescaled action of $\Delta^{\mathcal{Q}_{\bar{v}}}$ on $\mathcal{V}_\lambda$ by $g \cdot_\lambda x = \alpha_{\lambda}^{\mathcal{Q}_{\bar{v}}}(x)^{-1} g \cdot x$. Note that the rescaling means that $\iota_{\tilde{v}}^{-1}(\tilde{u}_{\tilde{v},n}) \cdot_\lambda x = x$.

Let $T \supseteq S_p$ be a finite set of finite places of F that satisfies $T = T^c$. The formalism of [ACC+18, §2.1.8] implies then that, for each good subgroup $\widetilde{K} \subset$

$\widetilde{G}(\mathbb{A}_{F^+,f})$ such that $\widetilde{K}_{\bar{v}} = \mathcal{P}_{\bar{v}}(b,c)$ with $c \geq 1$ for each $\bar{v} \in \bar{S} \subseteq \bar{S}_p$, there is a canonical homomorphism

$$\mathcal{H}(\widetilde{G}^T, \widetilde{K}^T) \otimes_{\mathbb{Z}} \mathcal{H}(\widetilde{\Delta}_{\bar{S}}, \widetilde{K}_{\bar{S}}) \to \mathrm{End}_{D^+(\mathcal{O})}\left(R\Gamma\left(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}}\right)\right) \tag{2.1.8}$$

and in particular all the Hecke operators $\widetilde{U}_{\tilde{v},n}$ and $\widetilde{U}_{\tilde{v},2n}$ for $\bar{v} \in \bar{S}$ act as endomorphisms of $R\Gamma\left(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}}\right)$. In fact, the Hecke operators $\widetilde{U}_{\tilde{v},2n}$ for $\bar{v} \in \bar{S}$ act as automorphisms of $R\Gamma\left(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}}\right)$ because the elements $\tilde{u}_{\tilde{v},2n}$ are central.

We will also consider $\widetilde{K}$ with $\widetilde{K}_{\bar{v}} = \mathcal{Q}_{\bar{v}}$ for each $\bar{v} \in \overline{S}$, and then we have a Hecke action

$$\mathcal{H}(\widetilde{G}^T, \widetilde{K}^T) \otimes_{\mathbb{Z}} \mathcal{H}(\widetilde{\Delta}_{\bar{S}}^{\mathcal{Q}_{\bar{S}}}, \widetilde{K}_{\bar{S}}) \to \mathrm{End}_{D^+(\mathcal{O})}\left(R\Gamma\left(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}}\right)\right). \tag{2.1.9}$$

Similarly, for the Levi subgroup G, let $K \subset G(\mathbb{A}_{F^+,f})$ be a good subgroup with $K_{\bar{v}} = \mathcal{Q}_{\bar{v}} \cap G(F^+_{\bar{v}})$ for $\bar{v} \in S$. Let $\{\lambda_\tau\}_{\tau \in \mathrm{Hom}(F^+_{\bar{v}}, E)}$ be sets of dominant weights for G at primes $\bar{v} \in \bar{S}$, giving rise to an $\mathcal{O}[K_{\bar{S}}]$-module $\mathcal{V}_{\lambda_{\bar{S}}}$. Let $\mathcal{V}$ be an $\mathcal{O}[K_{\bar{S}_p \setminus \bar{S}}]$-module, which is finite free as an $\mathcal{O}$-module and such that $\mathcal{V}/\varpi^m$ is a smooth $\mathcal{O}/\varpi^m[K_{\bar{S}_p \setminus \bar{S}}]$-module for every $m \in \mathbb{Z}_{\geq 1}$. We get a Hecke action

$$\mathcal{H}(G^T, K^T) \otimes_{\mathbb{Z}} \mathcal{H}(\Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}}}, K_{\bar{S}}) \to \mathrm{End}_{D^+(\mathcal{O})}\left(R\Gamma\left(X_K, \mathcal{V} \otimes_{\mathcal{O}} \mathcal{V}_{\lambda_{\bar{S}}}\right)\right). \tag{2.1.10}$$

This Hecke action generalizes in the natural way to the case when $\mathcal{V}$ is a complex of $\mathcal{O}[K_{\bar{S}_p \setminus \bar{S}}]$-modules as above. For $v|\bar{v}$ we will be interested in the (invertible) Hecke operator U_v corresponding to the central element $\iota_v^{-1}(\tilde{u}_{v,n}^{-1}\tilde{u}_{v,2n})$. Under our identification of G with $\mathrm{Res}_{F/F^+} \mathrm{GL}_n$, this element is $\mathrm{diag}(\varpi_v, \ldots, \varpi_v)$.

2.1.18. *Automorphic Galois representations and middle degree cohomology.* We start by recalling some well-known results about Galois representations associated to automorphic representations, and more generally to systems of Hecke eigenvalues occurring in the cohomology of locally symmetric spaces with integral coefficients.

Theorem 2.1.19. *Assume that F contains an imaginary quadratic field and that π is a cuspidal automorphic representation of $\widetilde{G}(\mathbb{A}_{F^+})$ that is ξ-cohomological*[9] *for some irreducible algebraic representation ξ of $(\mathrm{Res}_{F^+/\mathbb{Q}}\widetilde{G})_{\mathbb{C}}$. For any isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$, there exists a continuous, semisimple Galois representation*

$$r_\iota(\pi) : G_F \to \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_p)$$

satisfying the following conditions:

1. *For each prime $\ell \neq p$ which is unramified in F and above which π is unramified, and for each prime $v \mid \ell$ of F, $r_\iota(\pi)|_{G_{F_v}}$ is unramified and the characteristic polynomial of $r_\iota(\pi)(\mathrm{Frob}_v)$ is equal to the image of $\widetilde{P}_v(X)$ in $\overline{\mathbb{Q}}_p[X]$ corresponding to the base change of $\iota^{-1}(\pi_v)$.*
2. *For each prime $v \mid p$ of F, $r_\iota(\pi)$ is de Rham, and for each $\tau : F \hookrightarrow \overline{\mathbb{Q}}_p$, we have*

$$\mathrm{HT}_\tau(r_\iota(\pi)) = \{\tilde{\lambda}_{\tau,1} + 2n - 1, \tilde{\lambda}_{\tau,2} + 2n - 2, \ldots, \tilde{\lambda}_{\tau,2n}\},$$

[9] As in [Shi14], ξ-cohomological means that $\pi_\infty \otimes \xi$ has non-zero $(\mathfrak{g}, K_\infty)$-cohomology.

where $\tilde{\lambda} \in (\mathbb{Z}^{2n}_+)^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_p)}$ *is the highest weight of the representation* $\iota^{-1}(\xi \otimes \xi)^\vee$ *of* $(\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_{2n})_{\overline{\mathbb{Q}}_p}$.[10]

(3) *If* $F_0 \subset F$ *is an imaginary quadratic field and* ℓ *is a prime which splits in* F_0 *(including possibly* $\ell = p$*), then for each prime* $v \mid \ell$ *of* F *lying above a prime* $\bar{v}$ *of* F^+*, there is an isomorphism*

$$\mathrm{WD}\left(r_\iota(\pi)|_{G_{F_v}}\right)^{F-ss} \simeq \mathrm{rec}^T_{F_v}(\pi_{\bar{v}} \circ \iota_v).$$

Proof. This is [ACC⁺18, Theorem 2.3.3]. We mention that it relies on the base change result of [Shi14] and on the existence and properties of the Galois representations associated to regular algebraic, conjugate self-dual cuspidal automorphic representations of GL_m. □

Theorem 2.1.20. *Let* $\mathfrak{m} \subset \mathbb{T}^T(K, \lambda)$ *be a maximal ideal. Suppose* F *contains an imaginary quadratic field, the finite set of finite places* T *of* F *is stable under complex conjugation, and the following condition is satisfied:*

- *Let* $v \notin T$ *be a finite place, with residue characteristic* ℓ*. Then either* T *contains no* ℓ*-adic places and* ℓ *is unramified in* F*, or there exists an imaginary quadratic subfield of* F *in which* ℓ *splits.*[11]*.*

Then there exists a continuous, semi-simple Galois representation

$$\bar{\rho}_{\mathfrak{m}} : G_{F,T} \to \mathrm{GL}_n\left(\mathbb{T}^T(K, \lambda)/\mathfrak{m}\right)$$

such that, for each finite place $v \notin T$ *of* F*, the characteristic polynomial of* $\bar{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ *is equal to the image of* $P_v(X)$ *in* $\left(\mathbb{T}^T(K, \lambda)/\mathfrak{m}\right)[X]$*.*

Proof. This is [ACC⁺18, Theorem 2.3.5]: it essentially follows from [Sch15, Corollary 5.4.3]. □

Lemma 2.1.21. *Let* $\mathfrak{m} \subset \mathbb{T}^T(K, \lambda)$ *be a maximal ideal as in Theorem 2.1.20. Suppose* $k = k(\mathfrak{m})$*. Let* v *be a* p*-adic place of* F*. The Hecke operator* U_v *has a unique eigenvalue on* $H^*(X_K, \mathcal{V}_\lambda/\varpi)_{\mathfrak{m}}$*, equal to* $\bar{\epsilon}_p^{\frac{n(n-1)}{2}}(\mathrm{Art}_{F_v}(\varpi_v)) \cdot \det \bar{\rho}_{\mathfrak{m}}(\mathrm{Art}_{F_v}(\varpi_v))$*.*

Proof. Our proof of this will be global, computing the action of U_v in terms of a central Hecke operator at a suitable unramified prime. We write Z_n for the centre of $\mathrm{GL}_{n,F}$. We have a right action of $Z_n(\mathbb{A}_F)$ on X_K (by right multiplication of the finite adelic part on $\mathrm{GL}_n(\mathbb{A}_{F,f})$ and of the archimedean part on $X^{\mathrm{GL}_{n,F}}$). The rescaled action of u_v on $\mathcal{V}_\lambda$ allows us to define an action of $Z_n(\mathbb{A}_F)$ on $\mathcal{V}_\lambda$ which factors through the p-adic part and is compatible with its existing K action. We obtain an action of $Z_n(\mathbb{A}_F)$ on $H^*(X_K, \mathcal{V}_\lambda/\varpi)_{\mathfrak{m}}$ which factors through the quotient $Z_n(\mathbb{A}_F)/F_\infty^\times(Z_n(\mathbb{A}_{F,f}) \cap K)$ for continuity reasons[12].

We also have a continuous character $\psi_{\mathfrak{m}} : \mathbb{A}_F^\times = Z_n(\mathbb{A}_F) \to k^\times$ determined by

$$\psi_{\mathfrak{m}} = \bar{\epsilon}_p^{\frac{n(n-1)}{2}} \det \bar{\rho}_{\mathfrak{m}} \circ \mathrm{Art}_F.$$

---

[10]For each $\overline{\tau} : F^+ \to \mathbb{C}$, ξ gives a representation of $\widetilde{G}_{\overline{\tau}}$ and hence for $\tau, \tau c$ extending $\overline{\tau}$ to F we have a representention $\xi \otimes \xi$ of $\widetilde{G}_{\overline{\tau}} \times \widetilde{G}_{\overline{\tau}} = (\mathrm{GL}_{2n,F})_\tau \times (\mathrm{GL}_{2n,F})_{\tau c}$. Note that $r_\iota^{\vee,c}(1-2n) \cong r_\iota(\pi)$ so $\mathrm{HT}_\tau(r_\iota(\pi))$ and $\mathrm{HT}_{\tau c}(r_\iota(\pi))$ can be read off from each other.

[11]This condition can always be realised after enlarging T and is used to ensure that the results of [Sch15] that we appeal to are unconditional.

[12]In fact we do not even need continuity of the $F_\infty^\times$ action. The cohomology groups are finite, so the action of $F_\infty^\times$ gives a homomorphism from a product of copies of $\mathbb{C}^\times$ to a finite group. This is necessarily trivial, since $\mathbb{C}^\times$ has no finite index subgroups.

Since $F_\infty^\times$ is connected, and $\bar{\rho}_{\mathfrak{m}}$ is unramified away from T, $\psi_{\mathfrak{m}}$ factors through the quotient $F^\times\backslash Z_n(\mathbb{A}_F)/F_\infty^\times K_Z$ for a compact open subgroup $K_Z = \prod_w K_{Z,w}$ of $Z_n(\mathbb{A}_{F,f})$ with $K_{Z,w} = \mathcal{O}_{F_w}^\times$ for $w \notin T$. Shrinking K_Z if necessary, we assume that $K_Z \subset Z_n(\mathbb{A}_{F,f}) \cap K$. Note that for a finite place $w \notin T$, $\psi_{\mathfrak{m}}(\mathrm{Frob}_w)$ is equal to $T_{w,n} \mod \mathfrak{m}$.

By Chebotarev density, we can find a place $w \notin T$ such that the uniformiser ϖ_w and ϖ_v map to the same element in the ray class group $F^\times\backslash Z_n(\mathbb{A}_F)/F_\infty^\times K_Z$.

The action of $Z_n(\mathbb{A}_F)$ on X_K factors through the quotient $Z_n(\mathbb{A}_F)/(F_\infty^\times \cap K_\infty \mathbb{R}^\times)(Z_n(\mathbb{A}_{F,f}) \cap K)$. We can choose $z_\infty \in F_\infty^\times$ such that $z_\infty \varpi_w$ and ϖ_v map to the same element in this quotient. Now we can compare the action of the two elements $z_\infty \varpi_w$ and ϖ_v on cohomology. By construction, they act in the same way on X_K. They both act trivially on $\mathcal{V}_\lambda$. So they act the same on $H^*(X_K, \mathcal{V}_\lambda/\varpi)_{\mathfrak{m}}$. Since the action of $F_\infty^\times$ on cohomology is trivial, we deduce that ϖ_w and ϖ_v have the same action on $H^*(X_K, \mathcal{V}_\lambda/\varpi)_{\mathfrak{m}}$. The unique eigenvalue of ϖ_w (i.e. of the Hecke operator $T_{w,n}$) on $H^*(X_K, \mathcal{V}_\lambda/\varpi)_{\mathfrak{m}}$ is $\psi_{\mathfrak{m}}(\varpi_w)$. Our choice of w means this is equal to $\psi_{\mathfrak{m}}(\varpi_v)$, and we are done. □

Remark 2.1.22. A more conceptual proof for Lemma 2.1.21 can be given by arguing with (mod ϖ) completed cohomology at level K^p, localized at $\mathfrak{m}$. We can then assume λ is trivial, in which case completed cohomology is equipped with a continuous action of $F^\times\backslash \mathbb{A}_F^\times/F_\infty^\times(\mathbb{A}_{F,f} \cap K^p)$ with unique system of eigenvalues corresponding to the character $\bar{\epsilon}_p^{\frac{n(n-1)}{2}} \cdot \det \bar{\rho}_{\mathfrak{m}}$.

Definition 2.1.23. *We say that a maximal ideal $\mathfrak{m} \subset \mathbb{T}^T(K, \lambda)$ is* non-Eisenstein *if $\bar{\rho}_{\mathfrak{m}}$ is absolutely irreducible.*

Our convention is that when we ask for a maximal ideal $\mathfrak{m}$ to be non-Eisenstein, we are implicitly imposing the assumptions of Theorem 2.1.20.

Theorem 2.1.24. *Let $\mathfrak{m} \subset \mathbb{T}^T(K, \lambda)$ be a non-Eisenstein maximal ideal. There exist an integer $N \geq 1$, which depends only on n and $[F : \mathbb{Q}]$, an ideal $I \subset \mathbb{T}^T(K, \lambda)$ satisfying $I^N = 0$, and a continuous homomorphism*

$$\rho_{\mathfrak{m}} : G_{F,T} \to \mathrm{GL}_n\left(\mathbb{T}^T(K, \lambda)/I\right)$$

such that, for each finite place $v \notin T$ of F, the characteristic polynomial of $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ is equal to the image of $P_v(X)$ in $\left(\mathbb{T}^T(K, \lambda)/I\right)[X]$.

Proof. This is [Sch15, Corollary 5.4.4]. □

Let $\bar{S} \subseteq \bar{S}_p$ and let $\widetilde{K} \subset \widetilde{G}(\mathbb{A}_{F^+,f})$ be a compact open subgroup, which is decomposed with respect to the Levi decomposition $P = G \ltimes U$, and which satisfies $\widetilde{K}_{\bar{v}} = \mathcal{P}_{\bar{v}}(b, c)$ or $\mathcal{Q}_{\bar{v}}$ for all $\bar{v} \in \bar{S}$. We set $K := \widetilde{K} \cap G(\mathbb{A}_{F^+,f})$ and $K_U := \widetilde{K} \cap U(\mathbb{A}_{F^+,f})$. Let $\mathfrak{m} \subset \mathbb{T}^T(K, \lambda)$ be a non-Eisenstein maximal ideal and let $\widetilde{\mathfrak{m}} \subset \widetilde{\mathbb{T}}^T$ denote its pullback under the unnormalised Satake transform $\widetilde{\mathbb{T}}^T \to \mathbb{T}^T$. Recall that the boundary $\partial \widetilde{X}_{\widetilde{K}}$ of the Borel–Serre compactification of $\widetilde{X}_{\widetilde{K}}$ has a $\widetilde{G}(\mathbb{A}_{F^+,f})$-equivariant stratification indexed by the rational parabolic subgroups of $\widetilde{G}$ which contain $\widetilde{B}$. See [NT16, §3.1.2], especially [NT16, Lemma 3.10] for more details. For such a standard parabolic subgroup Q, we denote by $\widetilde{X}_{\widetilde{K}}^Q$ the stratum labeled by Q. This stratum can be written as a double quotient:

$$\widetilde{X}_{\widetilde{K}}^Q = Q(F^+)\backslash X^Q \times \widetilde{G}(\mathbb{A}_{F^+,f})/K$$

By applying the formalism in § 2.1.2, there is, for any $\tilde{\lambda} \in (\mathbb{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$ a homomorphism

$$\widetilde{\mathbb{T}}^T \to \mathrm{End}_{D^+(\mathcal{O})}\left(R\Gamma\left(\widetilde{X}^Q_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}}\right)\right).$$

Therefore, we can define the localisation $R\Gamma\left(\widetilde{X}^Q_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}}\right)_{\widetilde{\mathfrak{m}}}$.

Theorem 2.1.25. *Let $\mathfrak{m} \subset \mathbb{T}^T(K,\lambda)$ be a non-Eisenstein maximal ideal and let $\widetilde{\mathfrak{m}} := \mathcal{S}^*(\mathfrak{m}) \subset \widetilde{\mathbb{T}}^T$. Let $\tilde{\lambda} \in (\mathbb{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$. Pullback along the natural inclusion induces a $\widetilde{\mathbb{T}}^T$-equivariant isomorphism in $D^+(\mathcal{O})$:*

$$R\Gamma\left(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}}\right)_{\widetilde{\mathfrak{m}}} \xrightarrow{\sim} R\Gamma\left(\widetilde{X}^P_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}}\right)_{\widetilde{\mathfrak{m}}}.$$

Proof. This is [ACC+18, Thm. 2.4.2]. □

Theorem 2.1.26. *Let $\widetilde{\mathfrak{m}} \subset \widetilde{\mathbb{T}}^T(\widetilde{K},\tilde{\lambda})$ be a maximal ideal. Suppose F contains an imaginary quadratic field, the finite set of finite places T of F is stable under complex conjugation, and the following condition is satisfied:*

- *Let $v \notin T$ be a finite place, with residue characteristic ℓ. Then either T contains no ℓ-adic places and ℓ is unramified in F, or there exists an imaginary quadratic subfield of F in which ℓ splits.*

Then there exists a continuous, semi-simple Galois representation

$$\bar{\rho}_{\widetilde{\mathfrak{m}}} : G_{F,T} \to \mathrm{GL}_{2n}\left(\widetilde{\mathbb{T}}^T(\widetilde{K},\tilde{\lambda})/\widetilde{\mathfrak{m}}\right)$$

such that, for each finite place $v \notin T$ of F, the characteristic polynomial of $\bar{\rho}_{\widetilde{\mathfrak{m}}}(\mathrm{Frob}_v)$ is equal to the image of $\widetilde{P}_v(X)$ in $\left(\widetilde{\mathbb{T}}^T(\widetilde{K},\tilde{\lambda})/\widetilde{\mathfrak{m}}\right)[X]$.

Proof. We will use the Hecke algebra $\mathbb{T}^T_{cl}$ defined in [CGH+20, §6.5]. By reducing to $\mathbb{F}$-coefficients and increasing the level to sufficiently small compact opens $\widetilde{K}'_{\bar{v}}$ at all primes $\bar{v} \in \bar{S}_p$, we can trivialise the local system $\mathcal{V}_{\tilde{\lambda}} \otimes_{\mathcal{O}} \mathbb{F}$. The proof of [CGH+20, Theorem 6.5.3] shows that the map

$$\mathbb{T}^T_{cl} \to \mathrm{End}_{D^+\left(\prod_{\bar{v}\in\bar{S}_p} \widetilde{G}(\mathcal{O}_{F^+_{\bar{v}}})/\widetilde{K}'_{\bar{v}},\mathbb{F}\right)}\left(R\Gamma(\widetilde{X}_{\widetilde{K}'},\mathbb{F})_{\widetilde{\mathfrak{m}}} \otimes_{\mathbb{F}} \left(\mathcal{V}_{\tilde{\lambda}} \otimes_{\mathcal{O}} \mathbb{F}\right)\right)$$

is continuous for the discrete topology on the target. By the Hochschild–Serre spectral sequence, this implies that the map

$$\mathbb{T}^T_{cl} \to \mathrm{End}_{D^+(\mathbb{F})}\left(R\Gamma\left(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}} \otimes_{\mathcal{O}} \mathbb{F}\right)_{\widetilde{\mathfrak{m}}}\right)$$

is also continuous for the discrete topology on the target. The existence of a determinant valued in $\widetilde{\mathbb{T}}^T(\widetilde{K},\tilde{\lambda})/\widetilde{\mathfrak{m}}$ now follows from [CGH+20, Lemma 6.5.2], which is the version of [Sch15, Corollary 5.1.11] for usual cohomology and which can be made unconditional by using Theorem 2.1.19 as an input. This determinant corresponds to a semi-simple $\overline{\mathbb{F}}_p$-valued representation by [Che14, Theorem A]. This representation can be realised over $\widetilde{\mathbb{T}}^T(\widetilde{K},\tilde{\lambda})/\widetilde{\mathfrak{m}}$ by the same argument as in the proof of [ACC+18, Theorem 2.3.5]. □

We now introduce the key technical condition that our residual representations must satisfy in order to appeal to the main result of [CS19].

Definition 2.1.27. *A continuous representation* $\bar{\rho} : G_F \to \mathrm{GL}_m(\mathbb{F})$ *is* decomposed generic[13] *if there exists a prime* $\ell \neq p$ *such that the following are satisfied:*

(1) *the prime* ℓ *splits completely in* F*;*
(2) *for every prime* $v \mid \ell$ *of* F*, the representation* $\bar{\rho}|_{G_{F_v}}$ *is unramified and the eigenvalues* $\alpha_1, \ldots, \alpha_m$ *of* $\bar{\rho}(\mathrm{Frob}_v)$ *satisfy* $\alpha_i/\alpha_j \neq \ell$ *for* $i \neq j$*.*

We note that if a representation $\bar{\rho}$ is decomposed generic, then by the Chebotarev density theorem there exist infinitely many primes $\ell \neq p$ as in Definition 2.1.27, cf. [ACC+18, Lemma 4.3.2].

Theorem 2.1.28. *Keep the same assumptions on* F *as in Theorem 2.1.26. Let* $\widetilde{\mathfrak{m}} \subset \widetilde{\mathbb{T}}^T(\widetilde{K}, \tilde{\lambda})$ *be a maximal ideal such that the associated Galois representation* $\bar{\rho}_{\widetilde{\mathfrak{m}}}$ *constructed in Theorem 2.1.26 is decomposed generic, in the sense of Definition 2.1.27. Recall that* $d = \dim_{\mathbb{C}} \widetilde{X}_{\widetilde{K}}$*. Then we have* $\widetilde{\mathbb{T}}^T$*-equivariant morphisms*

$$H^d\left(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}}[1/p]\right)_{\widetilde{\mathfrak{m}}} \hookleftarrow H^d\left(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}}\right)_{\widetilde{\mathfrak{m}}} \twoheadrightarrow H^d\left(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}}\right)_{\widetilde{\mathfrak{m}}}.$$

Proof. This follows from [CS19, Thm. 1.1] as in the proof of [ACC+18, Thm. 4.3.3]. Moreover, we can remove the technical hypotheses that $[F^+ : \mathbb{Q}] > 1$ and $\bar{\rho}_{\widetilde{\mathfrak{m}}}$ has length at most two by appealing to Koshikawa's work [Kos21, Theorem 1.4]. □

2.2. **P-ordinary Hida theory.** In this section, we develop a P-ordinary version of Hida theory for group $\widetilde{G}$ and the Betti cohomology of the locally symmetric spaces $\widetilde{X}_{\widetilde{K}}$, by extending the theory developed in [TU99] for GSp_4 and the Betti cohomology of Siegel modular threefolds. We relate this construction to the P-ordinary part of completed cohomology.

For this entire section, fix a subset $\bar{S} \subset \bar{S}_p$, where we will take the P-ordinary (or, slightly more generally, Q-ordinary) part of the cohomology of the $\widetilde{X}_{\widetilde{K}}$.

2.2.1. *P-ordinary Hida theory at finite level.* In this section, we will only consider good subgroups $\widetilde{K} \subset \widetilde{G}(\mathbb{A}_{F,f})$ such that the tame level $\widetilde{K}^p$ is fixed and such that, for all $\bar{v} \in \bar{S}$, $\widetilde{K}_{\bar{v}} = \mathcal{P}_{\bar{v}}(b, c)$ for some integers $c \geq b \geq 0$. We denote such a good subgroup by $\widetilde{K}(b, c)$ and assume from now on that $c \geq 1$. Also set $\mathcal{P}_{\bar{S}}(b, c) := \prod_{\bar{v} \in \bar{S}} \mathcal{P}_{\bar{v}}(b, c)$.

Recall from (2.1.8) that we have well-defined actions of the Hecke algebras $\mathcal{H}(\widetilde{\Delta}_{\bar{S}}, \widetilde{K}_{\bar{S}})$ on the complexes $R\Gamma(\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\tilde{\lambda}})$. These actions are compatible with the natural pullback maps as b, c vary. We define the P-ordinary part $R\Gamma(\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\tilde{\lambda}})^{\mathrm{ord}}$ of the complex $R\Gamma(\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\tilde{\lambda}})$ to be the maximal direct summand on which all the $\widetilde{U}_{\tilde{v},n}$ act invertibly. This is a well-defined object of $D^+\left(\mathcal{P}_{\bar{S}}(0, c)/\mathcal{P}_{\bar{S}}(b, c), \mathcal{O}\right)$ by Lemma 2.1.6 and by the theory of ordinary parts, cf. [KT17, §2.4]. Moreover, it inherits an action of the abstract Hecke algebra $\widetilde{\mathbb{T}}^T \otimes_{\mathbb{Z}} (\bigotimes_{\bar{v} \in \bar{S}} \mathcal{H}(\widetilde{\Delta}_{\bar{v}}, \widetilde{K}_{\bar{v}})[\widetilde{U}_{\tilde{v},n}^{-1}])$.

We similarly have well-defined actions of the Hecke algebras $\mathcal{H}(\widetilde{\Delta}_{\bar{S}}, \widetilde{K}_{\bar{S}})$ on the complexes $R\Gamma(\partial\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\tilde{\lambda}})$. Therefore, we can also define the P-ordinary part $R\Gamma(\partial\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\tilde{\lambda}})^{\mathrm{ord}}$ of the complex $R\Gamma(\partial\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\tilde{\lambda}})$, which is an object of $D^+\left(\mathcal{P}_{\bar{S}}(0, c)/\mathcal{P}_{\bar{S}}(b, c), \mathcal{O}\right)$ equipped with an action of $\widetilde{\mathbb{T}}^T \otimes_{\mathbb{Z}} (\bigotimes_{\bar{v} \in \bar{S}} \mathcal{H}(\widetilde{\Delta}_{\bar{v}}, \widetilde{K}_{\bar{v}})[\widetilde{U}_{\tilde{v},n}^{-1}])$.

[13] This is slightly weaker than the condition called *decomposed generic* in [CS17]. See [CS19, Remark 1.4] and [CS19, Corollary 5.1.3] for an explanation.

2.2.2. *The P-ordinary part of a smooth representation.* In this section, we will define various functors that will allow us to study P-ordinary Hida theory using completed cohomology. These will be variants of the functors considered in [ACC+18, §5.2.1], with essentially the same properties, and we will appeal to the basic results in *loc. cit.* throughout. We will introduce a variant with more general parahoric level in §2.2.11 — we find it clearer to introduce the simplest version of the theory first, which is already sufficient for our results on local-global compatibility in the crystalline case.

For an integer $b \geq 0$, we set

$$K_{\bar{v}}(b) := \ker\left(G(\mathcal{O}_{F^+_{\bar{v}}}) \to G(\mathcal{O}_{F^+_{\bar{v}}}/\varpi^b_{\bar{v}})\right), K_{\bar{S}}(b) := \prod_{\bar{v}} K_{\bar{v}}(b), K_{\bar{S}} := K_{\bar{S}}(0).$$

We also set $U^0_{\bar{S}} := \prod_{\bar{v}\in\bar{S}} U(\mathcal{O}_{F^+_{\bar{v}}})$. Let $\Delta^+_{\bar{S}} \subset G_{\bar{S}}$ denote the monoid generated by $K_{\bar{S}}$ and by $\{\tilde{u}_{\bar{v},n} \mid \bar{v} \in \bar{S}\}$ and $\Delta_{\bar{S}} \subset G_{\bar{S}}$ the subgroup obtained by adjoining the inverses of the elements $\tilde{u}_{\bar{v},n}$. We set $\widetilde{\Delta}_{\bar{S},P} := \widetilde{\Delta}_{\bar{S}} \cap \prod_{\bar{v}\in\bar{S}} P(F^+_{\bar{v}})$ and note that we have $\widetilde{\Delta}_{\bar{S},P} = \Delta^+_{\bar{S}} \ltimes U^0_{\bar{S}}$.

We let

$$\Gamma(U^0_{\bar{S}},\ ) : \mathrm{Mod}_{\mathrm{sm}}(\widetilde{\Delta}_{\bar{S}}, \mathcal{O}/\varpi^m) \to \mathrm{Mod}_{\mathrm{sm}}(\Delta^+_{\bar{S}}, \mathcal{O}/\varpi^m).$$

denote the functor of $U^0_{\bar{S}}$-invariants.

For $V \in \mathrm{Mod}_{\mathrm{sm}}(\widetilde{\Delta}_{\bar{S}}, \mathcal{O}/\varpi^m)$, we define the action of an element $g \in \Delta^+_{\bar{S}}$ on $v \in \Gamma(U^0_{\bar{S}}, V)$ by the formula

$$v \mapsto g \cdot v := \sum_{n\in U^0_{\bar{S}}/gU^0_{\bar{S}}g^{-1}} ngv, \tag{2.2.1}$$

cf. [Eme10a, §3]. We obtain a derived functor

$$R\Gamma(U^0_{\bar{S}},\ ) : D^+_{\mathrm{sm}}(\widetilde{\Delta}_{\bar{S}}, \mathcal{O}/\varpi^m) \to D^+_{\mathrm{sm}}(\Delta^+_{\bar{S}}, \mathcal{O}/\varpi^m).$$

Since $U^0_{\bar{S}}$ is compact, an injective object in $\mathrm{Mod}_{\mathrm{sm}}(\widetilde{\Delta}_{\bar{S}}, \mathcal{O}/\varpi^m)$ remains $\Gamma(U^0_{\bar{S}},\ )$-acyclic on restriction to $\widetilde{\Delta}_{\bar{S},P}$. So this derived functor factors through the restriction functor to $D^+_{\mathrm{sm}}(\widetilde{\Delta}_{\bar{S},P}, \mathcal{O}/\varpi^m)$.

We also define a functor

$$\mathrm{ord} : \mathrm{Mod}_{\mathrm{sm}}(\Delta^+_{\bar{S}}, \mathcal{O}/\varpi^m) \to \mathrm{Mod}_{\mathrm{sm}}(\Delta_{\bar{S}}, \mathcal{O}/\varpi^m)$$

that is the composition of the localisation functors $\otimes_{\mathcal{O}/\varpi^m[\tilde{u}_{\bar{v},n}]} \mathcal{O}/\varpi^m[(\tilde{u}_{\bar{v},n})^{\pm 1}]$ for all $\bar{v} \in \bar{S}$. Note that ord is an exact functor because localisation is an exact functor, and it preserves injectives by the same argument as in [ACC+18, Lemma 5.2.7].

Definition 2.2.3. *We have a functor of* P-ordinary parts

$$D^+_{\mathrm{sm}}(\widetilde{\Delta}_{\bar{S}}, \mathcal{O}/\varpi^m) \to D^+_{\mathrm{sm}}(\Delta_{\bar{S}}, \mathcal{O}/\varpi^m), \pi \mapsto R\Gamma(U^0_{\bar{S}}, \pi)^{\mathrm{ord}}$$

obtained by composing the functor $R\Gamma(U^0_{\bar{S}},\ )$ *with the functor* ord.

For $b \geq 0$, we also have a functor

$$\Gamma(U^0_{\bar{S}} \rtimes K_{\bar{S}}(b),\ ) : \mathrm{Mod}_{\mathrm{sm}}(\widetilde{\Delta}_{\bar{S}}, \mathcal{O}/\varpi^m) \to \mathrm{Mod}(\Delta^+_{\bar{S}}/K_{\bar{S}}(b), \mathcal{O}/\varpi^m),$$

where the action of $g \in \Delta^+_{\bar{S}}/K_{\bar{S}}(b)$ is given by the same formula (2.2.1). We denote the corresponding derived functor by $R\Gamma(U^0_{\bar{S}} \rtimes K_{\bar{S}}(b),\ )$. We also define the functor

$$\mathrm{ord}_b : \mathrm{Mod}(\Delta^+_{\bar{S}}/K_{\bar{S}}(b), \mathcal{O}/\varpi^m) \to \mathrm{Mod}(\Delta_{\bar{S}}/K_{\bar{S}}(b), \mathcal{O}/\varpi^m)$$

by localisation. Note that ord_b is also an exact functor that preserves injectives. Finally, for $c \geq b \geq 0$ and $c \geq 1$, we also have a functor

$$\Gamma(\mathcal{P}_{\bar{S}}(b,c),\ ) : \mathrm{Mod}_{\mathrm{sm}}(\widetilde{\Delta}_{\bar{S}}, \mathcal{O}/\varpi^m) \to \mathrm{Mod}(\Delta_{\bar{S}}^+/K_{\bar{S}}(b), \mathcal{O}/\varpi^m),$$

where the action of $g \in \Delta_{\bar{S}}^+/K_{\bar{S}}(b)$ is given by the same formula (2.2.1). Note here that on the right hand side we are considering the natural action of the Hecke algebra

$$\mathcal{H}(\mathcal{P}_{\bar{S}}(0,c)\Delta_{\bar{S}}^+\mathcal{P}_{\bar{S}}(0,c), \mathcal{P}_{\bar{S}}(b,c)) = \mathcal{O}[\Delta_{\bar{S}}^+/K_{\bar{S}}(b)].$$

Lemma 2.2.4. *There is a natural isomorphism*

$$\mathrm{ord}_b \circ \Gamma(K_{\bar{S}}(b),\ ) \simeq \Gamma(K_{\bar{S}}(b),\ ) \circ \mathrm{ord}$$

of functors $\mathrm{Mod}_{\mathrm{sm}}(\Delta_{\bar{S}}^+, \mathcal{O}/\varpi^m) \to \mathrm{Mod}(\Delta_{\bar{S}}/K_{\bar{S}}(b), \mathcal{O}/\varpi^m)$*, which extends to an isomorphism of derived functors*

$$\mathrm{ord}_b \circ R\Gamma(K_{\bar{S}}(b),\ ) \simeq R\Gamma(K_{\bar{S}}(b),\ ) \circ \mathrm{ord}\,.$$

Proof. The same argument as for [ACC+18, Lemma 5.2.6] works for the un-derived statement. Since ord is exact and preserves injectives and ord_b is exact, the statement for derived functors follows. □

Lemma 2.2.5. [14] *For all $c \geq b \geq 0$ with $c \geq 1$, there is a natural isomorphism*

$$\mathrm{ord}_b \circ \Gamma(U_{\bar{S}}^0 \rtimes K_{\bar{S}}(b),\ ) \simeq \mathrm{ord}_b \circ \Gamma(\mathcal{P}_{\bar{S}}(b,c),\ )$$

of functors

$$\mathrm{Mod}_{\mathrm{sm}}(\widetilde{\Delta}_{\bar{S}}, \mathcal{O}/\varpi^m) \to \mathrm{Mod}(\Delta_{\bar{S}}/K_{\bar{S}}(b), \mathcal{O}/\varpi^m).$$

Proof. Let $V \in \mathrm{Mod}_{\mathrm{sm}}(\widetilde{\Delta}_{\bar{S}}), \mathcal{O}/\varpi^m)$. We first claim that the natural inclusion $\Gamma(\mathcal{P}_{\bar{S}}(b,c), V) \hookrightarrow \Gamma(U_{\bar{S}}^0 \rtimes K_{\bar{S}}(b), V)$ is a morphism of $\mathcal{O}/\varpi^m[\Delta_{\bar{S}}^+]$-modules. Indeed, by the formula (2.2.1), it is enough to check that, for all $g \in \Delta_{\bar{S}}^+/K_{\bar{S}}(b)$, the map

$$U_{\bar{S}}^0/gU_{\bar{S}}^0g^{-1} \to \mathcal{P}_{\bar{S}}(b,c)/g\mathcal{P}_{\bar{S}}(b,c)g^{-1}$$

is bijective, which holds by the Iwahori decomposition of $\mathcal{P}_{\bar{S}}(b,c)$ with respect to P. By the exactness of ord_b, we obtain an injection $\mathrm{ord}_b \Gamma(\mathcal{P}_{\bar{S}}(b,c), V) \hookrightarrow \mathrm{ord}_b \Gamma(U_{\bar{S}}^0 \rtimes K_{\bar{S}}(b), V)$.

We are left to show that this injection is an equality. Let $\tilde{u}_{\bar{S}} := \prod_{\bar{v}\in\bar{S}} \tilde{u}_{\bar{v},n}$ and $\widetilde{U}_{\bar{S}} := \prod_{\bar{v}\in\bar{S}} \widetilde{U}_{\bar{v},n}$. The result will follow if we show that, for any $v \in \Gamma(U_{\bar{S}}^0 \rtimes K_{\bar{S}}(b), V)$, there exists $N \geq 0$ such that $(\tilde{u}_{\bar{S}})^N v \in V^{\mathcal{P}_{\bar{S}}(b,c)}$. Since V is smooth, there exists $c' \geq c$ such that $v \in V^{\mathcal{P}_{\bar{S}}(b,c')}$. However, if $c' \geq 2$, then $\widetilde{U}_{\bar{S}} v \in V^{\mathcal{P}_{\bar{S}}(b,c'-1)}$ by [Eme10a, Lemma 3.3.2]. We conclude by induction. □

Lemma 2.2.6. *Let $\pi \in D_{\mathrm{sm}}^+(\widetilde{\Delta}_{\bar{S}}, \mathcal{O}/\varpi^m)$. Then for any $c \geq b \geq 0$ with $c \geq 1$ there is a natural isomorphism*

$$R\Gamma(K_{\bar{S}}(b), \mathrm{ord}\, R\Gamma(U_{\bar{S}}^0, \pi)) \xrightarrow{\sim} \mathrm{ord}_b\, R\Gamma(\mathcal{P}_{\bar{S}}(b,c), \pi)$$

in $D^+(\Delta_{\bar{S}}/K_{\bar{S}}(b), \mathcal{O}/\varpi^m)$.

Proof. This is proved in the same way as [ACC+18, Lemma 5.2.9], except we appeal to Lemma 2.2.5 above instead of Lemma 5.2.8 of *op. cit.* □

[14] Compare with [ACC+18, Lemma 5.2.8]

2.2.7. *Independence of level.* Recall that the finite free $\mathcal{O}$-module $\mathcal{V}_{\tilde{\lambda}}$ is equipped with an action of $\widetilde{\Delta}_{\bar{S}}$. We consider the completed cohomology at $\bar{S}$ with coefficients in $\mathcal{V}_{\tilde{\lambda}}/\varpi^m$:

$$\pi(\widetilde{K}^{\bar{S}}, \tilde{\lambda}, m) := R\Gamma\left(\widetilde{K}^{\bar{S}}, R\Gamma\left(\overline{\mathfrak{X}}_{\widetilde{G}}, \mathcal{V}_{\tilde{\lambda}}/\varpi^m\right)\right).$$

We are using the discussion following Lemma 2.1.8 to regard $R\Gamma\left(\overline{\mathfrak{X}}_{\widetilde{G}}, \mathcal{V}_{\tilde{\lambda}}/\varpi^m\right)$ as an object of $D^+_{\mathrm{sm}}(\widetilde{G}^T \times \widetilde{K}_{T\setminus\bar{S}} \times \widetilde{\Delta}_{\bar{S}}, \mathcal{O}/\varpi^m)$. Then $\pi(\widetilde{K}^{\bar{S}}, \tilde{\lambda}, m)$ is an object of $D^+_{\mathrm{sm}}(\widetilde{\Delta}_{\bar{S}}, \mathcal{O}/\varpi^m)$ equipped with an action of $\widetilde{\mathbb{T}}^T$. We note that, for any $c \geq b \geq 0$, $\mathcal{P}_{\bar{S}}(b,c) \subset \widetilde{\Delta}_{\bar{S}}$ and we have a canonical $\widetilde{\mathbb{T}}^T$-equivariant isomorphism

$$R\Gamma\left(\mathcal{P}_{\bar{S}}(b,c), \pi(\widetilde{K}^{\bar{S}}, \tilde{\lambda}, m)\right) \xrightarrow{\sim} R\Gamma\left(\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\tilde{\lambda}}/\varpi^m\right)$$

in $D^+\left(\mathcal{P}_{\bar{S}}(0,c)/\mathcal{P}_{\bar{S}}(b,c), \mathcal{O}/\varpi^m\right)$.

We define $\pi^{\mathrm{ord}}(\widetilde{K}^{\bar{S}}, \tilde{\lambda}, m)$ to be the P-ordinary part of $\pi(\widetilde{K}^{\bar{S}}, \tilde{\lambda}, m)$ as in Definition 2.2.3. This is an object in $D^+_{\mathrm{sm}}(\Delta_{\bar{S}}, \mathcal{O}/\varpi^m)$ equipped with an action of $\widetilde{\mathbb{T}}^T$.

Proposition 2.2.8. *For any integers $m \geq 1$ and $c \geq b \geq 0$ with $c \geq 1$, there is a natural $\widetilde{\mathbb{T}}^T$-equivariant isomorphism*

$$R\Gamma\left(K_{\bar{S}}(b), \pi^{\mathrm{ord}}(\widetilde{K}^{\bar{S}}, \tilde{\lambda}, m)\right) \simeq R\Gamma\left(\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\tilde{\lambda}}/\varpi^m\right)^{\mathrm{ord}}$$

in $D^+(K_{\bar{S}}/K_{\bar{S}}(b), \mathcal{O}/\varpi^m)$.

Proof. This is proved in the same way as [ACC+18, Prop. 5.2.15], given Lemma 2.2.6 as an input. Again, the key point is that the two definitions of *P-ordinary parts* (via Hida's idempotent as in [KT17, §2.4] or via localisation as in § 2.2.2) agree on finite $\mathcal{O}/\varpi^m$-modules, and that the cohomology groups of $R\Gamma\left(\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\tilde{\lambda}}/\varpi^m\right)$ are finite $\mathcal{O}/\varpi^m$-modules by Lemma 2.1.6. □

Corollary 2.2.9 (Independence of level[15])**.** *For any integers $m \geq 1$ and $c \geq b \geq 0$ with $c \geq 1$, the natural $\widetilde{\mathbb{T}}^T$-equivariant morphism*

$$R\Gamma\left(\widetilde{X}_{\widetilde{K}(b,\max\{1,b\})}, \mathcal{V}_{\tilde{\lambda}}/\varpi^m\right)^{\mathrm{ord}} \to R\Gamma\left(\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\tilde{\lambda}}/\varpi^m\right)^{\mathrm{ord}}$$

is an isomorphism in $D^+(K_{\bar{S}}/K_{\bar{S}}(b), \mathcal{O}/\varpi^m)$.

Proof. This follows immediately from Proposition 2.2.8, upon noting that the LHS of the main isomorphism is independent of $c \geq \max\{1, b\}$. □

The same results hold for the cohomology of the Borel–Serre boundary, with the same proof.

Proposition 2.2.10. *For any integers $m \geq 1$ and $c \geq b \geq 0$ with $c \geq 1$, there is a natural $\widetilde{\mathbb{T}}^T$-equivariant isomorphism*

$$R\Gamma\left(K_{\bar{S}}(b), \pi^{\mathrm{ord}}_{\partial}(\widetilde{K}^{\bar{S}}, \tilde{\lambda}, m)\right) \simeq R\Gamma\left(\partial\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\tilde{\lambda}}/\varpi^m\right)^{\mathrm{ord}}$$

in $D^+(K_{\bar{S}}/K_{\bar{S}}(b), \mathcal{O}/\varpi^m)$.

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[15]Compare with [ACC+18, Cor. 5.2.16]

2.2.11. *A variant at parahoric level.* We now suppose we have standard parabolic subgroups $Q_{\bar{v}} \subset P_{\bar{v}}$ for each $\bar{v} \in \bar{S}$, with corresponding parahoric subgroup $\mathcal{Q}_{\bar{v}} \subset \widetilde{G}(\mathcal{O}_{F^+_{\bar{v}}})$. We will still be interested in P-ordinary parts — in other words, we only invert the Hecke operator for the element $\tilde{u}_{\overline{S}}$, as in the previous subsection. Our degree shifting arguments will apply to this P-ordinary part. Later on, we will take Q-ordinary parts after specialising to finite level $\mathcal{Q}_{\overline{S}}$, but this means that we need to keep track of additional Hecke operators on our cohomology groups.

Recall that we have defined monoids $\widetilde{\Delta}_{\bar{S}}^{\mathcal{Q}_{\bar{S}}}, \Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}},+}, \Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}}}$. We will also make use of $\widetilde{\Delta}_{\bar{S},P}^{\mathcal{Q}_{\bar{S}}} := \widetilde{\Delta}_{\bar{S}}^{\mathcal{Q}_{\bar{S}}} \cap \prod_{\bar{v}\in\bar{S}} P(F^+_{\bar{v}})$. Note that $\widetilde{\Delta}_{\bar{S},P}^{\mathcal{Q}_{\bar{S}}} = \Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}},+} \ltimes U^0_{\bar{S}}$. We set $K_{\bar{v}} := \mathcal{Q}_{\bar{v}} \cap G(F^+_{\bar{v}})$ for each $\bar{v} \in \bar{S}$.

Using the formula (2.2.1), we define a functor

$$R\Gamma(U^0_{\bar{S}}, -) : D^+_{\mathrm{sm}}(\widetilde{\Delta}_{\bar{S}}^{\mathcal{Q}_{\bar{S}}}, \mathcal{O}/\varpi^m) \to D^+_{\mathrm{sm}}(\Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}},+}, \mathcal{O}/\varpi^m)$$

which factors through $D^+_{\mathrm{sm}}(\widetilde{\Delta}_{\bar{S},P}^{\mathcal{Q}_{\bar{S}}}, \mathcal{O}/\varpi^m)$, and the localisation, inverting $\tilde{u}_{\tilde{v},n}$ for all $\bar{v} \in \bar{S}$,

$$\mathrm{ord} : D^+_{\mathrm{sm}}(\Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}},+}, \mathcal{O}/\varpi^m) \to D^+_{\mathrm{sm}}(\Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}}}, \mathcal{O}/\varpi^m)$$

with composition denoted by $\pi \mapsto R\Gamma(U^0_{\bar{S}}, \pi)^{\mathrm{ord}}$.

Note that $K_{\bar{S}}$ is not in general normal in $\Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}},+}$, but it is in the submonoid $K_{\bar{S}}[\tilde{u}_{\tilde{v},n}|\bar{v} \in \bar{S}]$. This means we can define a functor

$$\mathrm{ord}_0\, R\Gamma\,(K_{\bar{S}}, -) : D^+_{\mathrm{sm}}(\Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}},+}, \mathcal{O}/\varpi^m) \to D^+_{\mathrm{sm}}(K_{\bar{S}}[\tilde{u}^{\pm 1}_{\tilde{v},n}|\bar{v} \in \bar{S}]/K_{\bar{S}}, \mathcal{O}/\varpi^m)$$

as above, and equip an object in the image of this functor with an action of the Hecke algebra $\mathcal{H}(\Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}}}, K_{\bar{S}})$. We can also compute this functor as $R\Gamma(K_{\bar{S}}, -) \circ \mathrm{ord}$, as in Lemma 2.2.4. We have an analogue of Lemma 2.2.6:

Lemma 2.2.12. *Let $\pi \in D^+_{\mathrm{sm}}(\widetilde{\Delta}_{\bar{S}}^{\mathcal{Q}_{\bar{S}}}, \mathcal{O}/\varpi^m)$. There is a natural isomorphism*

$$R\Gamma(K_{\bar{S}}, \mathrm{ord}\, R\Gamma(U^0_{\bar{S}}, \pi)) \xrightarrow{\sim} \mathrm{ord}_0\, R\Gamma\,(\mathcal{Q}_{\bar{S}}, \pi)$$

in $D^+_{\mathrm{sm}}(K_{\bar{S}}[\tilde{u}^{\pm 1}_{\tilde{v},n}|\bar{v} \in \bar{S}]/K_{\bar{S}}, \mathcal{O}/\varpi^m)$ under which the action of $[K_{\bar{v}}\nu(\varpi_{\bar{v}})K_{\bar{v}}] \in \mathcal{H}(\Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}}}, K_{\bar{S}})$ on the left hand side matches with the action of $[\mathcal{Q}_{\bar{v}}\nu(\varpi_{\bar{v}})\mathcal{Q}_{\bar{v}}]$ on the right hand side.

Proof. The proof is essentially the same as Lemma 2.2.6. The description of the Hecke action follows from the decomposition $\widetilde{\Delta}_{\bar{S},P}^{\mathcal{Q}_{\bar{S}}} = \Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}},+} \ltimes U^0_{\bar{S}}$ and the fact that the formula defining the double coset operator $[\mathcal{Q}_{\bar{v}} \cap P(F^+_{\bar{v}})\nu(\varpi_{\bar{v}})\mathcal{Q}_{\bar{v}} \cap P(F^+_{\bar{v}})]$ also defines $[\mathcal{Q}_{\bar{v}}\nu(\varpi_{\bar{v}})\mathcal{Q}_{\bar{v}}]$.

More precisely, if $V \in \mathrm{Mod}_{\mathrm{sm}}(\widetilde{\Delta}_{\bar{S}}^{\mathcal{Q}_{\bar{S}}}, \mathcal{O}/\varpi^m)$, we consider the natural inclusion

$$\Gamma(\mathcal{Q}_{\bar{S}}, V) \hookrightarrow \Gamma(U^0_{\bar{S}} \ltimes K_{\bar{S}}, V).$$

For $\delta \in \Delta_{\bar{v}}^{\mathcal{Q}_{\bar{v}},+}$, the action of $[\mathcal{Q}_{\bar{v}}\delta\mathcal{Q}_{\bar{v}}]$ on the left hand side is given by $\sum_{\gamma\in\mathcal{Q}_{\bar{v}}/(\mathcal{Q}_{\bar{v}}\cap\delta\mathcal{Q}_{\bar{v}}\delta^{-1})} \gamma\delta$. The action on the right hand side is given by $\sum_{k\in K_{\bar{v}}/(K_{\bar{v}}\cap\delta K_{\bar{v}}\delta^{-1})} \sum_{n\in U^0_{\bar{v}}/k\delta U^0_{\bar{v}}(k\delta)^{-1}} nk\delta$. The Iwahori decomposition of $\mathcal{Q}_{\bar{v}}$ with respect to P shows that the map $(k, n) \mapsto nk$ gives a well-defined bijection

$$\{(k, n) : k \in K_{\bar{v}}/(K_{\bar{v}} \cap \delta K_{\bar{v}}\delta^{-1}), n \in U^0_{\bar{v}}/k\delta U^0_{\bar{v}}(k\delta)^{-1}\} \xrightarrow{\sim} \mathcal{Q}_{\bar{v}}/(\mathcal{Q}_{\bar{v}} \cap \delta\mathcal{Q}_{\bar{v}}\delta^{-1}).$$

It remains to show that the inclusion

$$\mathrm{ord}_0\, \Gamma(\mathcal{Q}_{\bar{S}}, V) \hookrightarrow \mathrm{ord}_0\, \Gamma(U^0_{\bar{S}} \ltimes K_{\bar{S}}, V)$$

is bijective. As in the proof of Lemma 2.2.5, if v is in the right hand side, there exists $c \geq 1$ such that $v \in V^{\mathcal{Q}_{\bar{S}} \cap \mathcal{P}_{\bar{S}}(0,c)}$. We can apply [Eme10a, Lemma 3.3.2] again to show that $\widetilde{U}_{\bar{S}}^{N} v \in V^{\mathcal{Q}_{\bar{S}}}$ for some $N \geq 0$. □

Using this lemma, we get natural analogues of our independence of level statements. In this context, we will have $\pi^{\mathrm{ord}}(\widetilde{K}^{\bar{S}}, \tilde{\lambda}, m) \in D^{+}_{\mathrm{sm}}(\widetilde{\Delta}_{\bar{S}}^{\mathcal{Q}_{\bar{S}}}, \mathcal{O}/\varpi^{m})$ and the same for boundary cohomology.

2.2.13. *Independence of weight.* We retain our current set-up, with parabolic subgroups $Q_{\bar{v}}$ for $\bar{v} \in \bar{S}$ and a parahoric level subgroup $K_{\bar{S}}$. Assume that we have a dominant weight λ for G. For a subset $\bar{S} \subseteq \bar{S}_p$, set

$$\mathcal{V}_{\lambda_{\bar{S}}} := \bigotimes_{v \in S} \bigotimes_{\tau \in \mathrm{Hom}(F_v, E), \mathcal{O}} \mathcal{V}_{\lambda_\tau}.$$

This is, a priori, a finite free $\mathcal{O}$-module with an action of $K_{\bar{S}}$. We have explained how to extend the inflate the $K_{\bar{S}}$-action to an action of $\Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}}}$ (in particular, $\tilde{u}_{\bar{v},n}$ acts trivially for each $\bar{v} \in \bar{S}$).

Lemma 2.2.14. *Using our identification of $K_{\bar{S}}$ with the block diagonal Levi subgroup of $\prod_{\bar{v} \in \bar{S}} \mathrm{P}_{n,n}(\mathcal{O}_{F_{\tilde{v}}})$, we can identify*

$$\mathcal{V}_{\lambda_{\bar{S}}} = \bigotimes_{\bar{v} \in \bar{S}} \bigotimes_{\tau \in \mathrm{Hom}(F^{+}_{\bar{v}}, E), \mathcal{O}} \mathcal{V}_{-w_{0,n}\lambda_{\tilde{\tau}c}} \otimes \mathcal{V}_{\lambda_{\tilde{\tau}}}$$

with the action on both factors $\mathcal{V}_{-w_{0,n}\lambda_{\tilde{\tau}c}} \otimes \mathcal{V}_{\lambda_{\tilde{\tau}}}$ defined using the embedding $\tilde{\tau}$.

Proof. Recall that for each $\bar{v} \in \overline{S}$ and place $\tilde{v}|\bar{v}$ of F we have an isomorphism $\iota_{\tilde{v}}$: $\widetilde{G}(F^{+}_{\bar{v}}) \cong \mathrm{GL}_{2n}(F_{\tilde{v}})$ identifying the Levi subgroup $G(F^{+}_{\bar{v}}) = \mathrm{GL}_n(F_{\tilde{v}}) \times \mathrm{GL}_n(F_{\tilde{v}^c})$ with block diagonal matrices in $\mathrm{P}_{n,n}(F_{\tilde{v}})$ via $(A_{\tilde{v}}, A_{\tilde{v}^c}) \mapsto \begin{pmatrix} (\Psi_n\ {}^tA_{\tilde{v}^c}^{-1}\Psi_n)^c & 0 \\ 0 & A_{\tilde{v}} \end{pmatrix}$. We write θ for the isomorphism $\mathrm{GL}_n(F_{\tilde{v}^c}) \cong \mathrm{GL}_n(F_{\tilde{v}})$ defined by

$$\theta(A) = (\Psi_n\ {}^tA^{-1}\Psi_n)^c,$$

which preserves our chosen Borel subgroup B.

For each $\tau \in \mathrm{Hom}(F^{+}_{\bar{v}}, E)$, $K_{\bar{v}} = \mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}}}) \times \mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}^c}})$ acts 'factor-by-factor' on $\mathcal{V}_{\lambda_{\tilde{\tau}}} \otimes \mathcal{V}_{\lambda_{\tilde{\tau}c}}$. Describing this representation in terms of block diagonal matrices, we get $\theta^{-1}\mathcal{V}_{\lambda_{\tilde{\tau}c}} \otimes \mathcal{V}_{\lambda_{\tilde{\tau}}}$, where $\theta^{-1}\mathcal{V}_{\lambda_{\tilde{\tau}c}}$ denotes the representation of $\mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}}})$ given by pulling back the representation $\mathcal{V}_{\lambda_{\tilde{\tau}c}}$ by θ^{-1}. To finish the proof, we need to explain why $\theta^{-1}\mathcal{V}_{\lambda_{\tilde{\tau}c}} \cong \mathcal{V}_{-w_{0,n}\lambda_{\tilde{\tau}c}}$. To see this, consider the map

$$\begin{aligned}(\mathrm{Ind}_{\mathrm{B}_n}^{\mathrm{GL}_n} w_{0,n}\lambda_{\tilde{\tau}c})_{/\mathcal{O}} &\to (\mathrm{Ind}_{\mathrm{B}_n}^{\mathrm{GL}_n} -\lambda_{\tilde{\tau}c})_{/\mathcal{O}} \\ f &\mapsto \big(g \mapsto f(\Psi_n\ {}^tg^{-1}\Psi_n)\big)\end{aligned}$$

and note that it gives the desired isomorphism. □

Proposition 2.2.15. *Given a dominant weight $\tilde{\lambda}$ for $\widetilde{G}$ and a subset $\bar{S} \subseteq \bar{S}_p$, let $\tilde{\lambda}^{\bar{S}}$ be defined as follows:*

- *if $\bar{v} \in \bar{S}$ and $\tau \in \mathrm{Hom}(F^{+}_{\bar{v}}, E)$, then $\tilde{\lambda}^{\bar{S}}_{\tau} := (0, \ldots, 0)$.*
- *if $\bar{v} \in \bar{S}_p \setminus \bar{S}$ and $\tau \in \mathrm{Hom}(F^{+}_{\bar{v}}, E)$, then $\tilde{\lambda}^{\bar{S}}_{\tau} := \tilde{\lambda}_{\tau}$.*

Identify $\tilde{\lambda}$ with a dominant weight λ for G as in 2.1.4. For any integer $m \geq 1$ there is a natural $\widetilde{\mathbb{T}}^T$-equivariant isomorphism

$$\pi^{\mathrm{ord}}(\widetilde{K}^{\bar{S}}, \tilde{\lambda}, m) \xrightarrow{\sim} \pi^{\mathrm{ord}}(\widetilde{K}^{\bar{S}}, \tilde{\lambda}^{\bar{S}}, m) \otimes \mathcal{V}_{w_0^P \lambda_{\bar{S}}}/\varpi^m$$

in $D^+_{\mathrm{sm}}(\Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}}}, \mathcal{O}/\varpi^m)$.

Proof. Note first that $\pi^{\mathrm{ord}}(\widetilde{K}^{\bar{S}}, \tilde{\lambda}, m)$ only depends on $\mathcal{V}_{\tilde{\lambda}_{\bar{S}}}/\varpi^m$ as an object in $D^+_{\mathrm{sm}}(U^0_{\bar{S}} \rtimes \Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}},+}, \mathcal{O}/\varpi^m)$. As in Lemma 2.1.12, there is a $U^0_{\bar{S}} \rtimes \Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}},+}$-equivariant morphism of finite free $\mathcal{O}$-modules

$$\mathrm{ev} : \mathcal{V}_{\tilde{\lambda}_{\bar{S}}} \to \mathcal{V}_{w_0^P \lambda_{\bar{S}}}$$

given by evaluation of functions at the identity, where the action of $U^0_{\bar{S}} \rtimes \Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}},+} \subset \widetilde{\Delta}_{\bar{S}}^{\mathcal{Q}_{\bar{S}}}$ on the LHS is as in Lemma 2.1.17 and the action on the RHS factors through the action of $\Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}},+}$. By Lemma 2.1.12, this morphism is surjective. Let $\mathcal{K}_{\tilde{\lambda}_{\bar{S}}} := \ker(\mathrm{ev})$, a finite free $\mathcal{O}$-module with an action of $U^0_{\bar{S}} \rtimes \Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}},+}$.

For any $m \geq 1$, we have a short exact sequence of $\mathcal{O}/\varpi^m[U^0_{\bar{S}} \rtimes \Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}},+}]$-modules

$$0 \to \mathcal{K}_{\tilde{\lambda}_{\bar{S}}}/\varpi^m \to \mathcal{V}_{\tilde{\lambda}_{\bar{S}}}/\varpi^m \to \mathcal{V}_{w_0^P \lambda_{\bar{S}}}/\varpi^m \to 0.$$

We first claim that ev $\pmod{\varpi^m}$ induces an isomorphism between

$$\pi^{\mathrm{ord}}(\widetilde{K}^{\bar{S}}, \tilde{\lambda}, m) \stackrel{\mathrm{def}}{=} \mathrm{ord}\, R\Gamma\left(U^0_{\bar{S}}, R\Gamma\left(\widetilde{K}^{\bar{S}}, R\Gamma(\overline{\mathfrak{X}}_{\widetilde{G}}, \mathcal{V}_{\tilde{\lambda}}/\varpi^m)\right)\right)$$

and

$$\mathrm{ord}\, R\Gamma\left(U^0_{\bar{S}}, R\Gamma\left(\widetilde{K}^{\bar{S}}, R\Gamma(\overline{\mathfrak{X}}_{\widetilde{G}}, \mathcal{V}_{\tilde{\lambda}^{\bar{S}}}/\varpi^m) \otimes \mathcal{V}_{w_0^P \lambda_{\bar{S}}}/\varpi^m\right)\right). \tag{2.2.2}$$

By definition (cf. Lemma 2.1.8), we have

$$R\Gamma(\overline{\mathfrak{X}}_{\widetilde{G}}, \mathcal{V}_{\tilde{\lambda}}/\varpi^m) = R\Gamma(\overline{\mathfrak{X}}_{\widetilde{G}}, \mathcal{V}_{\tilde{\lambda}^{\bar{S}}}/\varpi^m) \otimes \mathcal{V}_{\tilde{\lambda}_{\bar{S}}}/\varpi^m.$$

To prove the claim, it is therefore enough to show that

$$\mathrm{ord}\, R\Gamma\left(U^0_{\bar{S}}, R\Gamma\left(\widetilde{K}^{\bar{S}}, R\Gamma(\overline{\mathfrak{X}}_{\widetilde{G}}, \mathcal{V}_{\tilde{\lambda}^{\bar{S}}}/\varpi^m) \otimes \mathcal{K}_{\tilde{\lambda}_{\bar{S}}}/\varpi^m\right)\right)$$

is trivial. This follows from Lemma 2.2.16 below.

Finally, we observe that there is a natural isomorphism in $D^+_{\mathrm{sm}}(\Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}}}, \mathcal{O}/\varpi^m)$ between (2.2.2) and $\pi^{\mathrm{ord}}(\widetilde{K}^{\bar{S}}, \tilde{\lambda}^{\bar{S}}, m) \otimes \mathcal{V}_{w_0^P \lambda_{\bar{S}}}/\varpi^m$. Indeed, $\tilde{u}_{\tilde{v},n}$, $\widetilde{K}^{\bar{S}}$ and $U^0_{\bar{S}}$ all act trivially on the finite free $\mathcal{O}/\varpi^m$-module $\mathcal{V}_{w_0^P \lambda_{\bar{S}}}/\varpi^m$, so we can pull this factor outside all the functors being applied in (2.2.2). □

Lemma 2.2.16. *Let $\tau \in \mathrm{Hom}(F^+_{\bar{v}}, E)$ and let*

$$\mathcal{K}_{\tilde{\lambda}_\tau} := \ker\left(\mathcal{V}_{\tilde{\lambda}_\tau} \to \mathcal{V}_{\lambda_{\tilde{\tau}}} \otimes \mathcal{V}_{\lambda_{-w_{0,n}\lambda_{\tilde{\tau}c}}}\right)$$

be the kernel of the evaluation at identity map. For any $m \geq 1$, we have $(\tilde{u}_{\tilde{v},n})^m(\mathcal{K}_{\tilde{\lambda}_\tau}/\varpi^m) = 0$.

Proof. Since $\mathcal{K}_{\tilde{\lambda}_\tau}$ is the evaluation at $\mathcal{O}$ of an algebraic representation of $\mathrm{GL}_n/\mathcal{O}$, it has a decomposition into weight spaces for the diagonal torus $\mathrm{T}_n/\mathcal{O}$. The parabolic subgroup $\mathrm{P}_{n,n}$ corresponds to the set of simple roots $I = \Delta \backslash (e_n - e_{n+1})$. It follows from [Cab84, Proposition 4.1] that the weights which show up in $\mathcal{K}_{\tilde{\lambda}_\tau}$ are the weights μ of $V_{\tilde{\lambda}_\tau}$ such that $\mu - w_{0,2n}\tilde{\lambda}_\tau$ contains a positive multiple of $(e_n - e_{n+1})$

in its decomposition into simple roots. This condition corresponds to the rescaled action of $\tilde{u}_{\tilde{v},n}$ on the μ-weight space acting as multiplication by a positive power of $\tilde{\tau}(\varpi_{\tilde{v}})$. □

The same result holds for the cohomology of the Borel–Serre boundary, with the same proof.

Proposition 2.2.17. *Given a weight $\tilde{\lambda}$ and a subset $\bar{S} \subseteq \bar{S}_p$, let $\tilde{\lambda}^{\bar{S}}$ be defined as in Proposition 2.2.15. For any integer $m \geq 1$ there is a natural $\widetilde{\mathbb{T}}^T$-equivariant isomorphism*

$$\pi_{\partial}^{\mathrm{ord}}(\widetilde{K}^{\bar{S}}, \tilde{\lambda}, m) \xrightarrow{\sim} \pi_{\partial}^{\mathrm{ord}}(\widetilde{K}^{\bar{S}}, \tilde{\lambda}^{\bar{S}}, m) \otimes \mathcal{V}_{w_0^P \lambda_{\bar{S}}}/\varpi^m$$

in $D^+_{\mathrm{sm}}(\Delta_{\bar{S}}^{\mathcal{Q}_{\bar{S}}}, \mathcal{O}/\varpi^m)$.

2.2.18. *A variant with dual cofficients.* We will also make use of a variant of P-ordinary Hida theory which can be applied with dual coefficient systems. It is formulated in terms of the ordinary parts $\mathrm{ord}^\vee$, $\mathrm{ord}_0^\vee$ defined using the Hecke action of $\tilde{u}_{\bar{v},n}^{-1}$ on invariants under $\overline{U}_{\bar{v}}^1$ and $\mathcal{Q}_{\bar{v}}$ respectively, where $\overline{U}_{\bar{v}}^1$ is the block-strictly-*lower*-triangular part of the parahoric $\mathcal{P}_{\bar{v}}$. We will fix $\bar{v} \in \bar{S}$ and start with a representation π of the inverse monoid $(\widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}})^{-1} = \coprod_{\nu \in X_{Q_{\bar{v}}}} \mathcal{Q}_{\bar{v}} \nu(\varpi_{\bar{v}})^{-1} \mathcal{Q}_{\bar{v}}$. Set $K_{\bar{v}} := \mathcal{Q}_{\bar{v}} \cap G(F_{\bar{v}}^+)$.

Lemma 2.2.19. *Let $\pi \in D^+_{\mathrm{sm}}((\widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}})^{-1}, \mathcal{O}/\varpi^m)$. There is a natural isomorphism*

$$R\Gamma(K_{\bar{v}}, \mathrm{ord}^\vee R\Gamma(\overline{U}_{\bar{v}}^1, \pi)) \xrightarrow{\sim} \mathrm{ord}_0^\vee R\Gamma(\mathcal{Q}_{\bar{v}}, \pi)$$

in $D^+_{\mathrm{sm}}(K_{\bar{v}}[\tilde{u}_{\bar{v},n}^{\pm 1}]/K_{\bar{v}}, \mathcal{O}/\varpi^m)$ under which the action of $[K_{\bar{v}}\nu(\varpi_{\bar{v}})^{-1}K_{\bar{v}}] \in \mathcal{H}((\Delta_{\bar{v}}^{\mathcal{Q}_{\bar{v}}})^{-1}, K_{\bar{v}})$ on the left hand side matches with the action of $[\mathcal{Q}_{\bar{v}}\nu(\varpi_{\bar{v}})^{-1}\mathcal{Q}_{\bar{v}}]$ on the right hand side.

Proof. Conjugation by $\tilde{u}_{\bar{v},n}^{-1} w_0^P$ sends $\overline{U}_{\bar{v}}^1$ to $U_{\bar{v}}^0$ and $\mathcal{Q}_{\bar{v}}$ to the parahoric $\overline{\mathcal{Q}}_{\bar{v}}^{w_0^P}$ corresponding to the standard parabolic with Levi subgroup $Q_{\bar{v}}^{w_0^P} \cap G(F_{\bar{v}}^+)$. It moreover sends $(\widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}})^{-1}$ to $\widetilde{\Delta}_{\bar{v}}^{\overline{\mathcal{Q}}_{\bar{v}}^{w_0^P}}$.

Conjugation by $\tilde{u}_{\bar{v},n}^{-1} w_0^P$ then identifies $R\Gamma(K_{\bar{v}}, \mathrm{ord}^\vee R\Gamma(\overline{U}_{\bar{v}}^1, \pi))$ and $\mathrm{ord}_0^\vee R\Gamma(\mathcal{Q}_{\bar{v}}, \pi)$ with $R\Gamma(K_{\bar{v}}^{w_0^P}, \mathrm{ord}\, R\Gamma(U_{\bar{v}}^0, \pi^{\tilde{u}_{\bar{v},n}^{-1} w_0^P}))$ and $\mathrm{ord}_0 R\Gamma\left(\overline{\mathcal{Q}}_{\bar{v}}^{w_0^P}, \pi^{\tilde{u}_{\bar{v},n}^{-1} w_0^P}\right)$ respectively, where $\pi^{\tilde{u}_{\bar{v},n}^{-1} w_0^P} \in D^+_{\mathrm{sm}}(\widetilde{\Delta}^{\overline{\mathcal{Q}}_{\bar{v}}^{w_0^P}}, \mathcal{O}/\varpi^m)$ denotes π with the action of $x \in \widetilde{\Delta}_{\bar{v}}^{\overline{\mathcal{Q}}_{\bar{v}}^{w_0^P}}$ given by the action of $(w_0^P)^{-1} \tilde{u}_{\bar{v},n} x \tilde{u}_{\bar{v},n}^{-1} w_0^P \in (\widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}})^{-1}$. Applying Lemma 2.2.12 now gives the desired result. □

An independence of weight statement for dual coefficients can be proved by following the proof of Proposition 2.2.15, using the short exact sequence

$$0 \to \mathcal{V}_{w_0^P \lambda_{\bar{S}}}^\vee \xrightarrow{\mathrm{ev}^\vee} \mathcal{V}_{\tilde{\lambda}_{\bar{S}}}^\vee \to \mathcal{K}_{\tilde{\lambda}_{\bar{S}}}^\vee \to 0$$

and the topological nilpotence of $\tilde{u}_{\bar{v},n}^{-1}$ on $\mathcal{K}_{\tilde{\lambda}_{\bar{S}}}^\vee$.

2.3. **New ingredients for degree shifting.**

2.3.1. *A computation of P-ordinary parts.* In this section, we compute the P-ordinary part of a parabolic induction from G to $\widetilde{G}$, in the same spirit as the computation of ordinary parts in [ACC+18, §5.3]. Our calculations here are purely local. The global application is to the boundary of the Borel–Serre compactification of the locally symmetric spaces for $\widetilde{G}$ and it is carried out in § 4.

Fix a prime $\bar{v}$ of F^+ dividing p and let $L := F^+_{\bar{v}}$, a p-adic field (with ring of integers $\mathcal{O}_L$ and uniformiser ϖ_L). In this section, we let $\mathrm{G}/\mathcal{O}_L$ be a split connected reductive group with split maximal torus $\mathrm{T} \subset \mathrm{G}$. Write $\mathrm{W} := W(\mathrm{G}, \mathrm{T})$ for the Weyl group, and fix a Borel subgroup B containing T and a parabolic subgroup $\mathrm{B} \subset \mathrm{P} \subset \mathrm{G}$ with Levi decomposition $\mathrm{P} = \mathrm{M} \ltimes \mathrm{U}$. The Weyl group $\mathrm{W_P}$ of M can be identified with a subgroup of W. We denote by $\mathrm{W^P} \subset \mathrm{W}$ the subset of minimal length representatives of $\mathrm{W}_P \backslash \mathrm{W}$. We denote the length of an element $w \in \mathrm{W}$ by $\ell(w) \in \mathbb{Z}_{\geq 0}$.

Recall from [BT65, Cor. 5.20] the (generalised) Bruhat decomposition

$$\mathrm{G}(L) = \bigsqcup_{w \in \mathrm{W^P}} \mathrm{P}(L) w \mathrm{B}(L).$$

Denote by ${}^{\mathrm{P}}\mathrm{W^P}$ the intersection $\mathrm{W^P} \cap (\mathrm{W^P})^{-1}$. This is a set of minimal length representatives for the double cosets $\mathrm{W_P} \backslash \mathrm{W} / \mathrm{W_P}$, cf. [DM91, Lemma 3.2.2].

Lemma 2.3.2. *We have a set-theoretic decomposition*

$$\mathrm{G}(L) = \coprod_{w \in {}^{\mathrm{P}}\mathrm{W^P}} \mathrm{P}(L) w \mathrm{P}(L).$$

The closure relations (for the p-adic topology) are given by the Bruhat ordering

$$\overline{\mathrm{P}(L) w \mathrm{P}(L)} = \coprod_{w' \leq w \in {}^{\mathrm{P}}\mathrm{W^P}} \mathrm{P}(L) w' \mathrm{P}(L).$$

Moreover, if $\Omega \subset {}^{\mathrm{P}}\mathrm{W^P}$ is an upper subset[16]*, then $\mathrm{P}(L)\Omega\mathrm{P}(L)$ is open in $\mathrm{G}(L)$.*

Proof. See [Hau18, Lemma 2.1.2]. □

We are interested in the parabolic induction functor

$$\mathrm{Ind}_{\mathrm{P}(L)}^{\mathrm{G}(L)} : D^+_{\mathrm{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m) \to D^+_{\mathrm{sm}}(\mathrm{G}(L), \mathcal{O}/\varpi^m).$$

This functor is exact and preserves injectives. We define several functors related to it.

For $w \in {}^{\mathrm{P}}\mathrm{W^P}$, define $S_w := \mathrm{P}(L) w \mathrm{P}(L)$ and $S_w^\circ := \mathrm{P}(L) w \mathrm{M}(L) \mathrm{U}^0$, where $\mathrm{U}^0 := \mathrm{U}(\mathcal{O}_L)$. The subset $S_w^\circ \subset \mathrm{G}(L)$ is invariant under left multiplication by $\mathrm{P}(L)$ and right multiplication by inverses of elements in $\mathrm{M}(L)^+ \ltimes \mathrm{U}^0$, where $\mathrm{M}(L)^+ = \{m \in \mathrm{M}(L) : m\mathrm{U}^0 m^{-1} \subset \mathrm{U}^0\}$ (this means that functions with support in S_w° are stable under right translation by $\mathrm{M}(L)^+ \ltimes \mathrm{U}^0$).

For any $i \in \mathbb{Z}_{\geq 0}$, we define

$$\mathrm{G}_{\geq i} := \bigsqcup_{\ell(w) \geq i} S_w,$$

which is an open subset of $\mathrm{G}(L)$ by Lemma 2.3.2, and which is invariant under left and right multiplication by $\mathrm{P}(L)$. For any $i \in \mathbb{Z}_{\geq 0}$, we define a functor

$$I_{\geq i} : \mathrm{Mod_{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m) \to \mathrm{Mod_{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m)$$

[16] This means that, if $w \in \Omega$ and $w' \in {}^{\mathrm{P}}\mathrm{W^P}$ satisfies $w' \geq w$, then $w' \in \Omega$.

by sending $\pi \in \mathrm{Mod}_{\mathrm{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m)$ to

$$I_{\geq i}(\pi) = \{f : \mathrm{G}_{\geq i} \to \pi \mid f \text{ locally constant, of compact support modulo } \mathrm{P}(L), \\ \forall p \in \mathrm{P}(L), g \in \mathrm{G}_{\geq i}, f(pg) = pf(g)\},$$

where $\mathrm{P}(L)$ acts by right translation. For $w \in {}^{\mathrm{P}}\mathrm{W}^{\mathrm{P}}$, we define a functor

$$I_w : \mathrm{Mod}_{\mathrm{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m) \to \mathrm{Mod}_{\mathrm{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m)$$

by sending $\pi \in \mathrm{Mod}_{\mathrm{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m)$ to

$$I_w(\pi) = \{f : S_w \to \pi \mid f \text{ locally constant, of compact support modulo } \mathrm{P}(L), \\ \forall p \in \mathrm{P}(L), g \in S_w, f(pg) = pf(g)\},$$

where again $\mathrm{P}(L)$ acts by right translation. Finally, for $w \in {}^{\mathrm{P}}\mathrm{W}^{\mathrm{P}}$, we also define a functor

$$I_w^\circ : \mathrm{Mod}_{\mathrm{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m) \to \mathrm{Mod}_{\mathrm{sm}}(\mathrm{M}(L)^+ \ltimes U^0, \mathcal{O}/\varpi^m)$$

by defining $I_w^\circ(\pi) \subset I_w(\pi)$ to be the subset of functions with support in S_w°.

Proposition 2.3.3.

(1) *We have* $I_{\geq 0} = \mathrm{Res}_{\mathrm{P}(L)}^{\mathrm{G}(L)} \circ \mathrm{Ind}_{\mathrm{P}(L)}^{\mathrm{G}(L)}$.
(2) *Each functor* $I_{\geq i}$, I_w *and* I_w° *is exact.*
(3) *For each* $i \in \mathbb{Z}_{\geq 0}$ *and* $\pi \in \mathrm{Mod}_{\mathrm{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m)$, *there is a functorial short exact sequence*

$$0 \to I_{\geq i+1}(\pi) \to I_{\geq i}(\pi) \to \bigoplus_{\ell(w)=i} I_w(\pi) \to 0.$$

Proof. The first part follows from the definition of parabolic induction and from the fact that $\mathrm{P}(L)\backslash \mathrm{G}(L)$ is compact. The second part follows from the fact that the natural map $\mathrm{G}(L) \twoheadrightarrow \mathrm{P}(L)\backslash\mathrm{G}(L)$ admits a continuous section, which can be deduced from [Jan03, Part II, §1.10] and [Hau16, Lemma 2.1.1]. The third part follows from [Hau18, Lemma 2.2.1], noting that the length function ℓ is strictly monotonic for the Bruhat order. □

We deduce that, for any $\pi \in D^+_{\mathrm{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m)$, there is a functorial distinguished triangle

$$I_{\geq i+1}(\pi) \to I_{\geq i}(\pi) \to \bigoplus_{\ell(w)=i} I_w(\pi) \to I_{\geq i+1}(\pi)[1] \tag{2.3.1}$$

in $D^+_{\mathrm{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m)$.

Proposition 2.3.4. *Let* $\pi \in D^+_{\mathrm{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m)$ *and let* $\mathcal{V}$ *be a finite free* $\mathcal{O}/\varpi^m$-*module equipped with a smooth representation of an open submonoid* $\Delta^+ \subset \mathrm{M}(L)$ *containing an open subgroup* $\mathrm{K} \subset \mathrm{M}(\mathcal{O}_L)$. *For any* $i \in \mathbb{Z}_{\geq 0}$ *and* $j \in \mathbb{Z}$, *the sequence*

$$0 \to R^j\Gamma\left(\mathrm{K} \ltimes \mathrm{U}^0, \mathcal{V} \otimes_{\mathcal{O}/\varpi^m} I_{\geq i+1}(\pi)\right) \to R^j\Gamma\left(\mathrm{K} \ltimes \mathrm{U}^0, \mathcal{V} \otimes_{\mathcal{O}/\varpi^m} I_{\geq i}(\pi)\right)$$

$$\to \bigoplus_{\ell(w)=i} R^j\Gamma\left(\mathrm{K} \ltimes \mathrm{U}^0, \mathcal{V} \otimes_{\mathcal{O}/\varpi^m} I_w(\pi)\right) \to 0$$

associated to (2.3.1) *is an exact sequence of* $\mathcal{H}(\Delta^+, \mathrm{K})$-*modules.*

Proof. The distinguished triangle (2.3.1) gives a distinguished triangle

$$\mathcal{V}\otimes_{\mathcal{O}/\varpi^m} I_{\geq i+1}(\pi) \to \mathcal{V}\otimes_{\mathcal{O}/\varpi^m} I_{\geq i}(\pi) \to \bigoplus_{\ell(w)=i} \mathcal{V}\otimes_{\mathcal{O}/\varpi^m} I_w(\pi) \to \mathcal{V}\otimes_{\mathcal{O}/\varpi^m} I_{\geq i+1}(\pi)[1]$$

in $D^+_{\mathrm{sm}}(\Delta^+ \ltimes \mathrm{U}^0, \mathcal{O}/\varpi^m)$, and taking cohomology gives us the desired sequence of $\mathcal{H}(\Delta^+, \mathrm{K})$-modules. It remains to check exactness, for which we can forget the Hecke action and just consider $\mathcal{V}$ as a representation of K. We consider decompositions $\mathrm{G}_{\geq i} = U_1 \sqcup U_2$ into open and closed subsets that are $\mathrm{P}(L)$-invariant on the left and $\mathrm{P}(\mathcal{O}_L)$-invariant on the right, and such that $U_1 \subset \mathrm{G}_{\geq i+1}$. Any such decomposition induces a functorial decomposition $I_{\geq i}(\pi) = I_{U_1}(\pi) \oplus I_{U_2}(\pi)$ in the category $\mathrm{Mod}_{\mathrm{sm}}(\mathrm{K} \ltimes \mathrm{U}^0, \mathcal{O}/\varpi^m)$, where I_{U_1} denotes functions with support in U_1, and similarly for U_2 (we can also tensor this decomposition with $\mathcal{V}$). In particular, for any $\pi \in D^+_{\mathrm{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m)$, the associated morphism

$$R^j\Gamma\left(\mathrm{K} \ltimes \mathrm{U}^0, \mathcal{V} \otimes_{\mathcal{O}/\varpi^m} I_{U_1}(\pi)\right) \to R^j\Gamma\left(\mathrm{K} \ltimes \mathrm{U}^0, \mathcal{V} \otimes_{\mathcal{O}/\varpi^m} I_{\geq i}(\pi)\right)$$

of $\mathcal{O}/\varpi^m$-modules is injective. Lemma 2.3.5 below implies that $I_{\geq i+1}$ can be written as a filtered direct limit of of functors of the form I_{U_1}. Since the tensor product $\mathcal{V}\otimes_{\mathcal{O}/\varpi^m}$ and the functor $R^j\Gamma(\mathrm{K} \ltimes \mathrm{U}^0, \ )$ commute with filtered direct limits[17], it follows that the morphism

$$R^j\Gamma\left(\mathrm{K} \ltimes \mathrm{U}^0, \mathcal{V} \otimes_{\mathcal{O}/\varpi^m} I_{\geq i+1}(\pi)\right) \to R^j\Gamma\left(\mathrm{K} \ltimes \mathrm{U}^0, \mathcal{V} \otimes_{\mathcal{O}/\varpi^m} I_{\geq i}(\pi)\right)$$

is injective. Since the injectivity applies for any $j \in \mathbb{Z}$, the long exact sequence of cohomology groups attached to the distinguished triangle (2.3.1) (tensored with $\mathcal{V}$) gives the statement of the proposition. □

Lemma 2.3.5. *For any $i \in \mathbb{Z}_{\geq 0}$, there exist decompositions $\mathrm{G}_{\geq i} = U_1^m \sqcup U_2^m$ into open and closed subsets, indexed by $m \in \mathbb{Z}_{\geq 1}$, that are $\mathrm{P}(L)$-invariant on the left and $\mathrm{P}(\mathcal{O}_L)$-invariant on the right, such that*

$$\mathrm{G}_{\geq i+1} = \bigcup_{m \geq 1} U_1^m$$

(in particular, each U_1^m is a subset of $\mathrm{G}_{\geq i+1}$).

Proof. Let $\overline{S}_w := \overline{\mathrm{P}(L) w \mathrm{P}(L)}$, a closed subset of $\mathrm{G}(L)$. We claim that it is enough to find, for each $w \in {}^{\mathrm{P}}\mathrm{W}^{\mathrm{P}}$ with $\ell(w) = i$, decompositions $\mathrm{G}(L) = U^m_{w,1} \sqcup U^m_{w,2}$ into open and closed subsets, indexed by $m \in \mathbb{Z}_{\geq 1}$, that are $\mathrm{P}(L)$-invariant on the left and $\mathrm{P}(\mathcal{O}_L)$-invariant on the right, such that

$$\overline{S}_w = \bigcap_{m \geq 1} U^m_{w,2}.$$

Indeed, once we have found such decompositions, we set

$$U_2^m := \left(\bigcup_{\ell(w)=i} U^m_{w,2} \right) \cap \mathrm{G}_{\geq i},$$

which is open and closed in $\mathrm{G}_{\geq i}$ because we have taken a finite union. The U_2^m and their complements induce decompositions of $\mathrm{G}_{\geq i}$ with the desired properties.

We now describe how to find the decompositions for each w. Set $\mathscr{F}\ell := \mathrm{P}\backslash\mathrm{G}$; this is a projective variety defined over $\mathcal{O}_L$. The closed subset $\mathrm{P}(L)\backslash\overline{S}_w \subset \mathrm{P}(L)\backslash\mathrm{G}(L)$

[17]Since $\mathrm{K} \ltimes \mathrm{U}^0$ is compact, a filtered direct limit of injective objects in $\mathrm{Mod}_{\mathrm{sm}}(\mathrm{K} \ltimes \mathrm{U}^0, \mathcal{O}/\varpi^m)$ is again an injective object in $\mathrm{Mod}_{\mathrm{sm}}(\mathrm{K} \ltimes \mathrm{U}^0, \mathcal{O}/\varpi^m)$, cf. [Eme10b, Prop. 2.1.3].

can be identified with the $\mathcal{O}_L$-points of a closed Schubert subvariety $\overline{\mathcal{S}}_w$ of $\mathscr{F}\ell$. Let ϖ_L be a uniformiser of $\mathcal{O}_L$. For each $m \geq 1$, we consider the subset $\overline{V}_m \subset \mathscr{F}\ell(\mathcal{O}_L/\varpi_L^m)$ consisting of the points that satisfy modulo ϖ_L^m the equations for $\overline{\mathcal{S}}_w$. The preimage $V_m \subset \mathrm{P}(L)\backslash\mathrm{G}(L)$ of $\overline{V}_m$ is an open and closed subset that contains $\mathrm{P}(L)\backslash\overline{S}_w$, and is invariant under multiplication on the right by the finite index normal subgroup of $\mathrm{P}(\mathcal{O}_L)$ of elements that reduce to the identity modulo ϖ_L^m. We intersect the translates of V_m by finitely many coset representatives in $\mathrm{P}(\mathcal{O}_L)$ to obtain an open and closed subset $W_m \subset \mathrm{P}(L)\backslash\mathrm{G}(L)$ that contains $\mathrm{P}(L)\backslash\overline{S}_w$, and is invariant under multiplication on the right by $\mathrm{P}(\mathcal{O}_L)$. Finally, we define $U_{w,2}^m$ to be the preimage of W_m in $\mathrm{G}(L)$. □

Assume now that $\mathrm{G} = \mathrm{GL}_{2n}/L$ and that P is the standard upper-triangular parabolic with Levi $\mathrm{GL}_n \times \mathrm{GL}_n$. Write $\tilde{u}_L := \mathrm{diag}(\varpi_L, \ldots, \varpi_L, 1, \ldots, 1)$, where the uniformiser ϖ_L of $\mathcal{O}_L$ occurs n times on the diagonal. Then $\mathrm{M}(L)^+$ contains the monoid generated by $\mathrm{M}(\mathcal{O}_L)$ and $\tilde{u}_L$. This means that if $\pi \in D^+_{\mathrm{sm}}(\mathrm{M}(L)^+ \ltimes \mathrm{U}^0, \mathcal{O}/\varpi^m)$, we can define $R\Gamma\left(\mathrm{U}^0, \pi\right) \in D^+_{\mathrm{sm}}(\mathrm{M}(L)^+, \mathcal{O}/\varpi^m)$ and the localisation (inverting $\tilde{u}_L$) $\mathrm{ord}\, R\Gamma\left(\mathrm{U}^0, \pi\right) \in D^+_{\mathrm{sm}}(\mathrm{M}(L)^+, \mathcal{O}/\varpi^m)$.

We now compute the complexes $\mathrm{ord}\, R\Gamma\left(\mathrm{U}^0, I_w(\pi)\right)$ in two special cases: for w equal to either the longest element of ${}^{\mathrm{P}}\mathrm{W}^{\mathrm{P}}$ or to the identity element.

Lemma 2.3.6. *Let w_0^{P} be the longest element of ${}^{\mathrm{P}}\mathrm{W}^{\mathrm{P}}$. Then:*

(1) *$I^\circ_{w_0^{\mathrm{P}}}$ takes injectives to $\Gamma(\mathrm{U}^0,\ )$-acyclics.*
(2) *Let $\pi \in D^+_{\mathrm{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m)$. Then there is a natural isomorphism*

$$\mathrm{ord}\, R\Gamma(\mathrm{U}^0, I^\circ_{w_0^{\mathrm{P}}}(\pi)) \xrightarrow{\sim} \mathrm{ord}\, R\Gamma(\mathrm{U}^0, I_{w_0^{\mathrm{P}}}(\pi)).$$

Proof. Since w_0^{P} is the longest element in ${}^{\mathrm{P}}\mathrm{W}^{\mathrm{P}}$, it normalises $\mathrm{M}(L)$ and therefore we have $S^\circ_{w_0^{\mathrm{P}}} = \mathrm{P}(L)w_0^{\mathrm{P}}\mathrm{U}^0$. The proof of [ACC+18, Lemma 5.3.4] applies verbatim; for convenience, we reproduce it here.

For the first part, let $\pi \in \mathrm{Mod}_{\mathrm{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m)$ and fix an $\mathcal{O}/\varpi^m$-linear embedding $\pi \hookrightarrow I$, where I is an injective $\mathcal{O}/\varpi^m$-module. This gives rise to an embedding $\pi \hookrightarrow \mathrm{Ind}_1^{\mathrm{P}(L)} I$ of smooth $\mathcal{O}/\varpi^m[\mathrm{P}(L)]$-modules. By [Eme10b, Lemma 2.1.10], it suffices to show that $I^\circ_{w_0^{\mathrm{P}}}\left(\mathrm{Ind}_1^{\mathrm{P}(L)} I\right)$ is an injective smooth $\mathcal{O}/\varpi^m[\mathrm{U}^0]$-module. There is a natural U^0-equivariant isomorphism

$$I^\circ_{w_0^{\mathrm{P}}}\left(\mathrm{Ind}_1^{\mathrm{P}(L)} I\right) \xrightarrow{\sim} \mathcal{C}^\infty\left(\mathrm{P}(L)w_0^{\mathrm{P}}\mathrm{U}^0, I\right),$$

where $\mathcal{C}^\infty\left(\mathrm{P}(L)w_0^{\mathrm{P}}\mathrm{U}^0, I\right)$ denotes the set of locally constant I-valued functions on $\mathrm{P}(L)w_0^{\mathrm{P}}\mathrm{U}^0$ (with the action of U^0 by right translation). The isomorphism sends a function $f \in I^\circ_{w_0^{\mathrm{P}}}\left(\mathrm{Ind}_1^{\mathrm{P}(L)} I\right)$ to $F \in \mathcal{C}^\infty\left(\mathrm{P}(L)w_0^{\mathrm{P}}\mathrm{U}^0, I\right)$ given by $F(x) := f(x)(1)$. Since $\mathcal{C}^\infty\left(\mathrm{P}(L)w_0^{\mathrm{P}}\mathrm{U}^0, I\right)$ is an injective smooth $\mathcal{O}/\varpi^m[\mathrm{U}^0]$-module, we conclude.

For the second part, we first define an exact functor

$$J_{w_0^{\mathrm{P}}}(\pi) : \mathrm{Mod}_{\mathrm{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m) \to \mathrm{Mod}_{\mathrm{sm}}(\mathrm{M}(L)^+ \ltimes U^0, \mathcal{O}/\varpi^m)$$

by the formula $J_{w_0^{\mathrm{P}}}(\pi) := I_{w_0^{\mathrm{P}}}(\pi)/I^\circ_{w_0^{\mathrm{P}}}(\pi)$. For each $\pi \in D^+_{\mathrm{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m)$, we have a distinguished triangle

$$\mathrm{ord}\, R\Gamma(\mathrm{U}^0, I^\circ_{w_0^{\mathrm{P}}}(\pi)) \to \mathrm{ord}\, R\Gamma(\mathrm{U}^0, I_{w_0^{\mathrm{P}}}(\pi)) \to \mathrm{ord}\, R\Gamma(\mathrm{U}^0, J_{w_0^{\mathrm{P}}}(\pi)) \to \mathrm{ord}\, R\Gamma(\mathrm{U}^0, I^\circ_{w_0^{\mathrm{P}}}(\pi))[1].$$

It is enough to show that, for any $j \in \mathbb{Z}$, we have

$$\operatorname{ord} H^j(\mathrm{U}^0, J_{w_0^{\mathrm{P}}}(\pi)) = 0.$$

By direct computation, we see that $\tilde{u}_L$ acts locally nilpotently on $J_{w_0^{\mathrm{P}}}(\pi)$: it shrinks the support of a function in $I_{w_0^{\mathrm{P}}}(\pi)$ towards $S^\circ_{w_0^{\mathrm{P}}}$. We conclude using the same argument as in [Hau16, Lemma 3.3.1]. □

Lemma 2.3.7. *Let w_0^{P} be the longest element of ${}^{\mathrm{P}}\mathrm{W}^{\mathrm{P}}$ and let $\pi \in D^+_{\mathrm{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m)$. Then there is a natural isomorphism*

$$R\Gamma(\mathrm{U}^0, I^\circ_{w_0^{\mathrm{P}}}(\pi)) \xrightarrow{\sim} \pi^{w_0^{\mathrm{P}}}$$

in $D^+_{\mathrm{sm}}(\mathrm{M}(L)^+, \mathcal{O}/\varpi^m)$, where $m \in \mathrm{M}(L)^+$ acts on $\pi^{w_0^{\mathrm{P}}}$ via the action of $w_0^{\mathrm{P}} m (w_0^{\mathrm{P}})^{-1}$ on π.

Proof. By the first part of Lemma 2.3.6, it is enough to show that there is a natural isomorphism

$$\Gamma(\mathrm{U}^0, I^\circ_{w_0^{\mathrm{P}}}(\pi)) \xrightarrow{\sim} \pi^{w_0^{\mathrm{P}}} \tag{2.3.2}$$

of underived functors. The map sends an U^0-invariant function $f : \mathrm{P}(L) w_0^{\mathrm{P}} \mathrm{U}^0 \to \pi$ to the value $f(w_0^{\mathrm{P}}) \in \pi$. This is an isomorphism of $\mathcal{O}/\varpi^m$-modules; it remains to check that (2.3.2) is $\mathrm{M}(L)^+$-equivariant, for the Hecke action of $\mathrm{M}(L)^+$ on the LHS and the action twisted by w_0^{P} on the RHS. In other words, we have to check that, for any U^0-invariant function $f : \mathrm{P}(L) w_0^{\mathrm{P}} \mathrm{U}^0 \to \pi$ and any $m \in \mathrm{M}(L)^+$, we have the equality

$$\sum_{\bar{u} \in \mathrm{U}^0 / m \mathrm{U}^0 m^{-1}} f(w_0^{\mathrm{P}} u m) = w_0^{\mathrm{P}} m (w_0^{\mathrm{P}})^{-1} f(w_0^{\mathrm{P}}). \tag{2.3.3}$$

This will hold if and only if the only $\bar{u} \in \mathrm{U}^0 / m\mathrm{U}^0 m^{-1}$ that contributes to the LHS of (2.3.3) is the identity element. We claim that $f(w_0^{\mathrm{P}} um) = 0$ unless $u \in m\mathrm{U}^0 m^{-1}$. Assume $f(w_0^{\mathrm{P}} um) \neq 0$; then $w_0^{\mathrm{P}} um \in \mathrm{P}(L) w_0^{\mathrm{P}} \mathrm{U}^0$. We write $w_0^{\mathrm{P}} um = q w_0^{\mathrm{P}} u'$ with $q \in \mathrm{P}(L)$ and $u' \in \mathrm{U}^0$ and we obtain

$$u = (w_0^{\mathrm{P}})^{-1} q w_0^{\mathrm{P}} u' m^{-1} = ((w_0^{\mathrm{P}})^{-1} q m' w_0^{\mathrm{P}})(m u' m^{-1}),$$

with $m' := w_0^{\mathrm{P}} m^{-1} (w_0^{\mathrm{P}})^{-1} \in \mathrm{M}(L)$. But then $((w_0^{\mathrm{P}})^{-1} q m' w_0^{\mathrm{P}}) \in (w_0^{\mathrm{P}})^{-1} \mathrm{P}(L) w_0^{\mathrm{P}} \cap \mathrm{U}^0 = \{\mathrm{id}\}$ and we deduce $u \in m\mathrm{U}^0 m^{-1}$. □

Let $\chi : \mathrm{M}(L) \to \mathcal{O}^\times$ be the character defined by the formula

$$\chi(m) = \frac{\mathrm{Nm}_{L/\mathbb{Q}_p} \det_L \left(\mathrm{Ad}(m)|_{\mathrm{Lie}\ U(L)} \right)^{-1}}{|\mathrm{Nm}_{L/\mathbb{Q}_p} \det_L \left(\mathrm{Ad}(m)|_{\mathrm{Lie}\ U(L)} \right)|_p}.$$

Lemma 2.3.8. *Let $\pi \in D^+_{\mathrm{sm}}(\mathrm{M}(L), \mathcal{O}/\varpi^m)$. Then there is a natural isomorphism*

$$\operatorname{ord} R\Gamma \left(\mathrm{U}^0, \mathrm{Inf}_{\mathrm{M}(L)^+}^{\mathrm{M}(L)^+ \ltimes \mathrm{U}^0} \pi \right) \xrightarrow{\sim} \mathcal{O}/\varpi^m(\chi) \otimes_{\mathcal{O}/\varpi^m} \pi[-\mathrm{rk}_{\mathbb{Z}_p} \mathrm{U}^0]$$

in $D^+_{\mathrm{sm}}(\mathrm{M}(L)^+, \mathcal{O}/\varpi^m)$.

Proof. Since U^0 acts trivially on π, we have

$$\operatorname{ord} R\Gamma \left(\mathrm{U}^0, \mathrm{Inf}_{\mathrm{M}(L)^+}^{\mathrm{M}(L)^+ \ltimes \mathrm{U}^0} \pi \right) \xrightarrow{\sim} \pi \otimes_{\mathcal{O}/\varpi^m} \operatorname{ord} R\Gamma \left(\mathrm{U}^0, \mathcal{O}/\varpi^m \right).$$

By the proof of [ACC+18, Lemma 5.3.7], the continuous cohomology groups of U^0 vanish below the top degree $\ell := \mathrm{rk}_{\mathbb{Z}_p}\mathrm{U}^0$ after applying ordinary parts. It remains to show that

$$H^\ell(\mathrm{U}^0, \mathcal{O}/\varpi^m) \simeq \mathcal{O}/\varpi^m(\chi)$$

as a representation of $\mathrm{M}(L)^+$. This follows, just like [Hau16, Prop. 3.1.8], from the natural isomorphism in [Eme10b, Prop. 3.5.6] and from the explicit description of the corestriction map in [Eme10b, Prop. 3.5.10]. □

Remark 2.3.9. The functor I_{id} is the identity on $D^+_{\mathrm{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m)$ and the functor I°_{id} is the natural restriction $D^+_{\mathrm{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m) \to D^+_{\mathrm{sm}}(\mathrm{M}(L)^+ \ltimes \mathrm{U}^0, \mathcal{O}/\varpi^m)$. Note also that in this case we have $S^\circ_{\mathrm{id}} = S_{\mathrm{id}}$, so that we have a natural identification

$$R\Gamma(\mathrm{U}^0, I^\circ_{\mathrm{id}}(\ )) \xrightarrow{\sim} R\Gamma(\mathrm{U}^0, I_{\mathrm{id}}(\ ))$$

in $D^+_{\mathrm{sm}}(\mathrm{M}(L)^+, \mathcal{O}/\varpi^m)$. Furthermore, if we start with $\pi \in D^+_{\mathrm{sm}}(\mathrm{M}(L), \mathcal{O}/\varpi^m)$, we can identify $\mathrm{Inf}^{\mathrm{M}(L)^+\ltimes \mathrm{U}^0}_{\mathrm{M}(L)^+}\pi$ with the functor I°_{id} applied to $\mathrm{Inf}^{\mathrm{P}(L)}_{\mathrm{M}(L)}\pi$.

Corollary 2.3.10. *Let $\pi \in D^+_{\mathrm{sm}}(\mathrm{M}(L), \mathcal{O}/\varpi^m)$. Then there is a natural isomorphism*

$$\mathrm{ord}\, R\Gamma\left(\mathrm{U}^0, I_{\mathrm{id}}\left(\mathrm{Inf}^{\mathrm{P}(L)}_{\mathrm{M}(L)}\pi\right)\right) \xrightarrow{\sim} \mathcal{O}/\varpi^m(\chi) \otimes_{\mathcal{O}/\varpi^m} \pi[-\mathrm{rk}_{\mathbb{Z}_p}\mathrm{U}^0]$$

in $D^+_{\mathrm{sm}}(\mathrm{M}(L)^+, \mathcal{O}/\varpi^m)$.

We now apply our results in the setting of Section 2.2, so we adopt the notation of that section. We fix a place $\bar{v} \in \bar{S}$ and a standard parabolic $Q_{\bar{v}} \subset P_{\bar{v}}$. We apply the results of this section with $\mathrm{G} = \widetilde{G}_{\mathcal{O}_{F^+_{\bar{v}}}}$, $\mathrm{P} = P_{\bar{v}}$, $\mathrm{M} = G_{\mathcal{O}_{F^+_{\bar{v}}}}$ and $\mathrm{U} = U_{\bar{v}}$. Set $K_{\bar{v}} = \mathcal{Q}_{\bar{v}} \cap G(F^+_{\bar{v}})$. Note that $\Delta^{\mathcal{Q}_{\bar{v}},+}_{\bar{v}}$ is a submonoid of $G(F^+_{\bar{v}})^+$. The following proposition summarises the key result of this subsection.

Proposition 2.3.11. *Let $\pi \in D^+_{\mathrm{sm}}(G(F^+_{\bar{v}}), \mathcal{O}/\varpi^m)$ and let $\mathcal{V}$ be a finite free $\mathcal{O}/\varpi^m$-module equipped with a smooth representation of $\Delta^{\mathcal{Q}_{\bar{v}},+}_{\bar{v}}$ with $\tilde{u}_{\bar{v},n}$ acting trivially, which we inflate to an action of $\widetilde{\Delta}^{\mathcal{Q}_{\bar{v}}}_{\bar{v},P} = \Delta^{\mathcal{Q}_{\bar{v}},+}_{\bar{v}} \ltimes U^0_{\bar{v}}$. Then*

$$\mathrm{ord}_0\, R^j\Gamma\left(K_{\bar{v}} \ltimes U^0_{\bar{v}}, \mathrm{Ind}^{\widetilde{G}(F^+_{\bar{v}})}_{P(F^+_{\bar{v}})}\pi \otimes \mathcal{V}\right)$$

admits as $\mathcal{H}(\Delta^{\mathcal{Q}_{\bar{v}}}_{\bar{v}}, K_{\bar{v}})$-module subquotients

$$R^j\Gamma(K_{\bar{v}}, \pi^{w_0^P} \otimes \mathcal{V}) \text{ and } R^{j-\mathrm{rk}_{\mathbb{Z}_p}U^0_{\bar{v}}}\Gamma(K_{\bar{v}}, \pi \otimes \mathcal{O}/\varpi^m(\chi) \otimes \mathcal{V}).$$

Proof. By applying the exact functor ord_0 to the short exact sequences in Proposition 2.3.4, we see that $\mathrm{ord}_0\, R^j\Gamma\left(K_{\bar{v}} \ltimes U^0_{\bar{v}}, \mathrm{Ind}^{\widetilde{G}(F^+_{\bar{v}})}_{P(F^+_{\bar{v}})}\pi \otimes \mathcal{V}\right)$ admits as subquotients $\mathrm{ord}_0 R^j\Gamma(K_{\bar{v}} \ltimes U^0_{\bar{v}}, I_w(\pi) \otimes \mathcal{V})$ for $w = w_0^P$ and for $w = \mathrm{id}$. We have an isomorphism of functors

$$\mathrm{ord}_0 \circ R\Gamma(K_{\bar{v}},\ ) \simeq R\Gamma(K_{\bar{v}},\ ) \circ \mathrm{ord},$$

as in §2.2.11. By Lemma 2.3.6 and Lemma 2.3.7, we have an isomorphism

$$\mathrm{ord}\, R\Gamma(U^0_{\bar{v}}, I_{w_0^P}(\pi) \otimes \mathcal{V}) \simeq \mathrm{ord}\, R\Gamma(U^0_{\bar{v}}, I_{w_0^P}(\pi)) \otimes \mathcal{V} \simeq \pi^{w_0^P} \otimes \mathcal{V}.$$

By Corollary 2.3.10, we have an isomorphim

$$\mathrm{ord}\, R\Gamma(U^0_{\bar{v}}, I_{\mathrm{id}}(\pi) \otimes \mathcal{V}) \simeq \mathrm{ord}\, R\Gamma(U^0_{\bar{v}}, I_{\mathrm{id}}(\pi)) \otimes \mathcal{V} \simeq \pi \otimes \mathcal{O}/\varpi^m(\chi) \otimes \mathcal{V}[-\mathrm{rk}_{\mathbb{Z}_p}U^0].$$

□

We have a corollary which will be applied with 'dual' coefficients (cf. §2.2.18). This can be viewed as a computation of the ordinary part with respect to the opposite parabolic $\overline{P}$ applied to a parabolic induction from P.

Corollary 2.3.12. *Let $\pi \in D^+_{\mathrm{sm}}(G(F^+_{\bar{v}}), \mathcal{O}/\varpi^m)$ and let $\mathcal{V}$ be a finite free $\mathcal{O}/\varpi^m$-module equipped with a smooth representation of $\left(\Delta_{\bar{v}}^{\mathcal{Q}_{\bar{v}},+}\right)^{-1}$ with $\tilde{u}_{\bar{v},n}^{-1}$ acting trivially, which we inflate to an action of $\left(\widetilde{\Delta}_{\bar{v},P}^{\mathcal{Q}_{\bar{v}}}\right)^{-1}$. Then*

$$\mathrm{ord}_0^\vee \, R^j\Gamma\left(K_{\bar{v}} \ltimes \overline{U}^1_{\bar{v}}, \mathrm{Ind}_{P(F^+_{\bar{v}})}^{\widetilde{G}(F^+_{\bar{v}})} \pi \otimes \mathcal{V}\right)$$

admits $R^j\Gamma(K_{\bar{v}}, \pi \otimes \mathcal{V})$ as a $\mathcal{H}(\left(\Delta_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}\right)^{-1}, K_{\bar{v}})$-module subquotient.

Proof. We deduce the corollary from Proposition 2.3.11 by twisting. Indeed, multiplication by $\tilde{u}_{\bar{v},n}^{-1} w_0^P$ induces an isomorphism

$$R^j\Gamma\left(\mathrm{K} \ltimes \overline{U}^1_{\bar{v}}, \mathrm{Ind}_{\mathrm{P}(L)}^{\mathrm{G}(L)} \pi \otimes \mathcal{V}\right) \xrightarrow{\sim} R^j\Gamma\left(\mathrm{K}^{w_0^P} \ltimes U^0_{\bar{v}}, \mathrm{Ind}_{\mathrm{P}(L)}^{\mathrm{G}(L)} \pi \otimes \mathcal{V}^{w_0^P}\right)$$

where the action of $\mathcal{H}(\left(\Delta_{\bar{v}}^{\mathcal{Q}_{\bar{v}},+}\right)^{-1}, K_{\bar{v}})$ on the left is identified with the action of $\mathcal{H}(\Delta_{\bar{v}}^{\overline{\mathcal{Q}}_{\bar{v}}^{w_0^P},+}, K_{\bar{v}})$ on the right by sending $[K_{\bar{v}}\nu(\varpi_{\bar{v}})^{-1}K_{\bar{v}}]$ to $[K_{\bar{v}}^{w_0^P}(-w_0^P\nu)(\varpi_{\bar{v}})K_{\bar{v}}^{w_0^P}]$ on the right. We recall from the proof of Lemma 2.2.19 that the $\overline{Q}_{\bar{v}}^{w_0^P}$ is the standard parabolic with Levi subgroup $Q_{\bar{v}}^{w_0^P} \cap G(F^+_{\bar{v}})$. Now Proposition 2.3.11 tells us that we have a subquotient $R^j\Gamma(K_{\bar{v}}^{w_0^P}, \pi^{w_0^P} \otimes \mathcal{V}^{w_0^P})$, which can be identified with $R^j\Gamma(K_{\bar{v}}, \pi \otimes \mathcal{V})$ as a $\mathcal{H}(\left(\Delta_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}\right)^{-1}, K_{\bar{v}})$-module. □

2.3.13. *Parabolic induction and cohomology.* We return to the general situation of §2.3.1, for a split reductive group G defined over (the ring of integers in) a local field L. In addition, we consider a compact Hausdorff space X, equipped with a continuous action of the locally profinite group $\mathrm{P}(L)$. Set $\mathrm{K}_\mathrm{P} := \mathrm{K} \cap \mathrm{P}(L)$. We assume that X is a free K_P-space, in the sense of [NT16, Def. 2.23].

Denote by $X \times_\mathrm{P} \mathrm{G}$ the quotient of $X \times \mathrm{G}(L)$ by the right action $(x, g) \cdot p = (xp, p^{-1}g)$ of $\mathrm{P}(L)$. Right multiplication by $\mathrm{G}(L)$ on itself gives $X \times_\mathrm{P} \mathrm{G}$ a right action of $\mathrm{G}(L)$.

The Iwasawa decomposition $\mathrm{G}(L) = \mathrm{KP}(L)$ implies that there exists a K-equivariant homeomorphism

$$X \times_{\mathrm{K}_\mathrm{P}} \mathrm{K} \xrightarrow{\sim} X \times_\mathrm{P} \mathrm{G}.$$

The LHS is visibly a compact Hausdorff space. Therefore, it makes sense to consider $R\Gamma\left(\mathrm{X} \times_\mathrm{P} \mathrm{G}, \mathcal{O}/\varpi^m\right)$ as an element in $D^+_{\mathrm{sm}}(\mathrm{G}(L), \mathcal{O}/\varpi^m)$.

Lemma 2.3.14. *We have a natural isomorphism in $D^+_{\mathrm{sm}}(\mathrm{G}(L), \mathcal{O}/\varpi^m)$*

$$R\Gamma\left(\mathrm{X} \times_\mathrm{P} \mathrm{G}, \mathcal{O}/\varpi^m\right) \xrightarrow{\sim} \mathrm{Ind}_{\mathrm{P}(L)}^{\mathrm{G}(L)} \, R\Gamma(X, \mathcal{O}/\varpi^m). \tag{2.3.4}$$

Proof. First, we explain how to construct a natural map from the LHS of (2.3.4) to the RHS, then we observe that it is enough to prove the map is an isomorphism after restriction to $D^+_{\mathrm{sm}}(\mathrm{K}, \mathcal{O}/\varpi^m)$, then we prove the latter.

Consider the $\mathrm{P}(L)$-equivariant map $X \to \mathrm{X} \times_{\mathrm{P}} \mathrm{G}$ given by $x \mapsto (x, 1)$. We get an induced morphism

$$R\Gamma(X \times_{\mathrm{P}} \mathrm{G}, \mathcal{O}/\varpi^m) \to R\Gamma(X, \mathcal{O}/\varpi^m)$$

in $D^+_{\mathrm{sm}}(\mathrm{P}(L), \mathcal{O}/\varpi^m)$, which induces a morphism

$$R\Gamma(X \times_{\mathrm{P}} \mathrm{G}, \mathcal{O}/\varpi^m) \to \mathrm{Ind}_{\mathrm{P}(L)}^{\mathrm{G}(L)} \ R\Gamma(X, \mathcal{O}/\varpi^m)$$

in $D^+_{\mathrm{sm}}(\mathrm{G}(L), \mathcal{O}/\varpi^m)$, by Frobenius reciprocity for smooth representations (also using that $\mathrm{Ind}_{\mathrm{P}(L)}^{\mathrm{G}(L)}$ is an exact functor).

The same morphism can be constructed if we work with $\mathrm{K_P} \subset \mathrm{K}$ replacing $\mathrm{P} \subset \mathrm{G}$. We now observe that we have a commutative diagram in $D^+_{\mathrm{sm}}(\mathrm{K}, \mathcal{O}/\varpi^m)$

$$\begin{array}{ccc} \mathrm{Res}_{\mathrm{K}}^{\mathrm{G}(L)} \circ R\Gamma\left(X \times_{\mathrm{P}} \mathrm{G}, \mathcal{O}/\varpi^m\right) & \longrightarrow & \mathrm{Res}_{\mathrm{K}}^{\mathrm{G}(L)} \circ \mathrm{Ind}_{\mathrm{P}(L)}^{\mathrm{G}(L)} \ R\Gamma(X, \mathcal{O}/\varpi^m) \\ \downarrow \cong & & \downarrow \cong \\ R\Gamma(X \times_{\mathrm{K_P}} \mathrm{K}, \mathcal{O}/\varpi^m) & \longrightarrow & \mathrm{Ind}_{\mathrm{K_P}}^{\mathrm{K}} \ R\Gamma(X, \mathcal{O}/\varpi^m) \end{array} \tag{2.3.5}$$

whose vertical arrows are isomorphisms by the Iwasawa decomposition.

We have a diagram with Cartesian outer square

$$\begin{array}{ccccc} & & X \times \mathrm{K} & & \\ & \swarrow^{\varphi_2} & & \searrow^{\varphi_1} & \\ X \times_{\mathrm{K_P}} \mathrm{K} & & \xleftarrow{\ \phi\ } & & X \\ & \searrow_{\phi_2} & & \swarrow_{\phi_1} & \\ & & X/\mathrm{K_P} & & \end{array} .$$

The map ϕ is given by $x \mapsto (x, 1)$; the top triangle is not commutative, but the bottom one is. Since $X \to X/\mathrm{K_P}$ is a $\mathrm{K_P}$-torsor, the same holds true for $X \times \mathrm{K} \to X \times_{\mathrm{K_P}} \mathrm{K}$. This also implies that $X \times_{\mathrm{K_P}} \mathrm{K} \to X/\mathrm{K_P}$ is a K-torsor. By [NT16, Lemma 2.24], we have inverse equivalences of categories $(\varphi_2^*, \varphi_{2,*}^{\mathrm{K_P}})$ between $\mathrm{Sh}_K(X \times_{\mathrm{K_P}} \mathrm{K})$ and $\mathrm{Sh}_{\mathrm{K_P} \times \mathrm{K}}(X \times \mathrm{K})$. The analogous statements are true for φ_1, ϕ_1 and ϕ_2.

The bottom horizontal arrow in (2.3.5) is induced by ϕ^*. Let $\mathcal{O}/\varpi^m \xrightarrow{\sim} \mathcal{I}^\bullet$ be an injective resolution in $\mathrm{Sh}(X/\mathrm{K_P})$. We claim that ϕ^* induces by Frobenius reciprocity a term-wise K-equivariant isomorphism of complexes

$$\Gamma(X \times_{\mathrm{K_P}} \mathrm{K}, \phi_2^* \mathcal{I}^\bullet) \xrightarrow{\sim} \mathrm{Ind}_{\mathrm{K_P}}^{\mathrm{K}} \ \Gamma(X, \phi_1^* \mathcal{I}^\bullet). \tag{2.3.6}$$

To see the claim, note that $\phi_2^* = \varphi_{2,*}^{\mathrm{K_P}} \varphi_2^* \phi_2^* = \varphi_{2,*}^{\mathrm{K_P}} \varphi_1^* \phi_1^*$ and we can identify ϕ^* with the restriction to the identity of a function in

$$\Gamma\left(X \times_{\mathrm{K_P}} \mathrm{K}, \varphi_{2,*}^{\mathrm{K_P}} \varphi_1^* (\phi_1^* \mathcal{I}^\bullet)\right) \xrightarrow{\sim} \mathrm{Ind}_{\mathrm{K_P}}^{\mathrm{K}} \ \Gamma(X, \phi_1^* \mathcal{I}^\bullet).$$

The map in (2.3.6) is precisely the horizontal map in the bottom row of (2.3.5), which also implies that the horizontal map in the top row is an isomorphism. □

We will also make use of the following lemma about the interaction between smooth induction and restriction.

Lemma 2.3.15. *Let* $\pi_1 \in \mathrm{Rep}_{\mathrm{sm}}(\mathrm{K}_{\mathrm{P}}, \mathcal{O}/\varpi^m)$ *and* $\pi_2 \in \mathrm{Rep}_{\mathrm{sm}}(\mathrm{K}, \mathcal{O}/\varpi^m)$ *with the property that* π_2 *takes values in a finite free* $\mathcal{O}/\varpi^m$*-module. There is a natural isomorphism*

$$\left(\mathrm{Ind}_{\mathrm{K}_{\mathrm{P}}}^{\mathrm{K}} \pi_1\right) \otimes_{\mathcal{O}/\varpi^m} \pi_2 \xrightarrow{\sim} \mathrm{Ind}_{\mathrm{K}_{\mathrm{P}}}^{\mathrm{K}} \left(\pi_1 \otimes_{\mathcal{O}/\varpi^m} \mathrm{Res}_{\mathrm{K}_{\mathrm{P}}}^{\mathrm{K}} \pi_2\right)$$

in $\mathrm{Rep}_{\mathrm{sm}}(\mathrm{K}, \mathcal{O}/\varpi^m)$.

Proof. Since π_2 is a finite $\mathcal{O}/\varpi^m$-module, there exists a compact open normal subgroup $U \subset \mathrm{K}$ that acts trivially on π_2. One can define a natural map

$$\Phi : \left(\mathrm{Ind}_{\mathrm{K}_{\mathrm{P}}}^{\mathrm{K}} \pi_1\right) \otimes_{\mathcal{O}/\varpi^m} \pi_2 \to \mathrm{Ind}_{\mathrm{K}_{\mathrm{P}}}^{\mathrm{K}} \left(\pi_1 \otimes_{\mathcal{O}/\varpi^m} \mathrm{Res}_{\mathrm{K}_{\mathrm{P}}}^{\mathrm{K}} \pi_2\right).$$

by extending $\mathcal{O}/\varpi^m$-linearly from the formula

$$f \otimes v \mapsto (g \mapsto f(g) \otimes \pi_2(g)v).$$

One can check that Φ lands in $\mathrm{Ind}_{\mathrm{K}_{\mathrm{P}}}^{\mathrm{K}} \left(\pi_1 \otimes_{\mathcal{O}/\varpi^m} \mathrm{Res}_{\mathrm{K}_{\mathrm{P}}}^{\mathrm{K}} \pi_2\right)$ and is a K-equivariant homomorphism. To see that Φ does indeed take values in smooth functions, note that there is a compact open subgroup U_f that stabilises f. Then

$$f(gu) \otimes \pi_2(gu)v = f(g) \otimes \pi_2(g)v \ \forall u \in U_f \cap U \text{ and } g \in \mathrm{K},$$

which shows that each $\Phi(f \otimes v)$ is a smooth function on K.

It remains to show that Φ is an isomorphism. For $m \in \mathbb{Z}_{\geq 1}$, we can find a decreasing sequence of compact open normal subgroups U_m of K with $U_1 = U$ and that form a basis of neighbourhoods of identity in K. It is enough to prove that Φ induces an isomorphism on the level of U_m-invariants for each m. Note that, for each m, the set of double cosets $\mathrm{K}_{\mathrm{P}} \backslash \mathrm{K}/U_m$ is finite, and for any smooth representation π of K_{P}, we have an isomorphism

$$\left(\mathrm{Ind}_{\mathrm{K}_{\mathrm{P}}}^{\mathrm{K}} \pi\right)^{U_m} \xrightarrow{\sim} \bigoplus_{\gamma \in \mathrm{K}_{\mathrm{P}} \backslash \mathrm{K}/U_m} \pi^{\mathrm{K}_{\mathrm{P}} \cap \gamma U_m \gamma^{-1}}, f \mapsto (f(\gamma))_{\gamma \in \mathrm{K}_{\mathrm{P}} \backslash \mathrm{K}/U_m}.$$

Note also that each $\mathrm{K}_{\mathrm{P}} \cap \gamma U_m \gamma^{-1} = \mathrm{K}_{\mathrm{P}} \cap U_m$ acts trivially on π_2. We obtain a commutative diagram

$$\begin{array}{ccc} \left(\mathrm{Ind}_{\mathrm{K}_{\mathrm{P}}}^{\mathrm{K}} \pi_1\right)^{U_m} \otimes \pi_2 & \xrightarrow[\cong]{} & \bigoplus_{\gamma \in \mathrm{K}_{\mathrm{P}} \backslash \mathrm{K}/U_m} (\pi_1)^{\mathrm{K}_{\mathrm{P}} \cap \gamma U_m \gamma^{-1}} \otimes \pi_2 \ , \\ \downarrow & & \downarrow \\ \left(\mathrm{Ind}_{\mathrm{K}_{\mathrm{P}}}^{\mathrm{K}} \left(\pi_1 \otimes_{\mathcal{O}/\varpi^m} \mathrm{Res}_{\mathrm{K}_{\mathrm{P}}}^{\mathrm{K}} \pi_2\right)\right)^{U_m} & \xrightarrow[\cong]{} & \bigoplus_{\gamma \in \mathrm{K}_{\mathrm{P}} \backslash \mathrm{K}/U_m} (\pi_1)^{\mathrm{K}_{\mathrm{P}} \cap \gamma U_m \gamma^{-1}} \otimes \pi_2 \end{array}$$

where the left vertical map is Φ and the right vertical map is $\mathrm{id} \otimes \gamma$ in the component corresponding to γ. This is visibly an isomorphism.

□

2.3.16. *A computation of group cohomology.* Keep the notation from the previous subsection. Inside $\mathrm{K} = \mathrm{M}(\mathcal{O}_L)$, consider the congruence subgroups $\mathrm{K}_m := \{k \in \mathrm{K} \mid k \equiv \mathrm{id} \pmod{\varpi_L^m}\}$ indexed by $m \in \mathbb{Z}_{\geq 1}$. The group U^0 is equipped with the adjoint action of K, so that we can view the continuous cohomology $R\Gamma(\mathrm{U}^0, \ )$ as a functor $D^+_{\mathrm{sm}}(\mathrm{K} \ltimes \mathrm{U}^0, \mathcal{O}/\varpi^m) \to D^+_{\mathrm{sm}}(\mathrm{K}, \mathcal{O}/\varpi^m)$.

Lemma 2.3.17. *For any $m \in \mathbb{Z}_{\geq 1}$, there exists $M = M(m) \geq m$ such that*

$$R\Gamma(\mathrm{U}^0, \mathcal{O}/\varpi^m) \simeq \bigoplus_{i=0}^{\mathrm{rk}_{\mathbb{Z}_p} \mathrm{U}^0} H^i(\mathrm{U}^0, \mathcal{O}/\varpi^m)[-i].$$

as an object in $D^+_{\mathrm{sm}}(\mathrm{K}_M, \mathcal{O}/\varpi^m)$. Moreover, each $H^i(\mathrm{U}^0, \mathcal{O}/\varpi^m)$ is non-zero and equipped with the trivial action of K_M.

Proof. By [Eme10b, Prop. 2.1.11], an injective smooth representation of $\mathrm{K} \ltimes \mathrm{U}^0$ is also injective as a representation of U^0. Therefore, if we apply the forgetful functor to $R\Gamma(\mathrm{U}^0, \mathcal{O}/\varpi^m)$ in order to view it as an object of $D^+(\mathcal{O}/\varpi^m)$, we obtain the continuous group cohomology of U^0 with coefficients in $\mathcal{O}/\varpi^m$. If we forget the K-action, U^0 is a free $\mathbb{Z}_p$-module. Therefore, we can compute its continuous group cohomology via a Koszul complex, using [BMS18, Lemma 7.3, part (ii)], for example. We see that all the differentials in the Koszul complex vanish, so we obtain

$$R\Gamma(\mathrm{U}^0, \mathcal{O}/\varpi^m) \simeq \bigoplus_i H^i(\mathrm{U}^0, \mathcal{O}/\varpi^m)[-i].$$

as an object in $D_{\mathrm{perf}}(\mathcal{O}/\varpi^m)$, the full subcategory of $D^+(\mathcal{O}/\varpi^m)$ consisting of perfect complexes. Moreover, we can identify each $H^i(\mathrm{U}^0, \mathcal{O}/\varpi^m)$ with $\wedge^i_{\mathbb{Z}_p} \mathrm{U}^0$, which shows that the cohomology groups are non-zero precisely in the range $[0, \mathrm{rk}_{\mathbb{Z}_p} \mathrm{U}^0]$.

Let $D_{\mathrm{perf}}(\mathrm{K}_M, \mathcal{O}/\varpi^m)$ be the full subcategory of $D^+_{\mathrm{sm}}(\mathrm{K}_M, \mathcal{O}/\varpi^m)$ whose essential image under the forgetful functor is $D_{\mathrm{perf}}(\mathcal{O}/\varpi^m)$. It is enough to show that, for every object A of $D_{\mathrm{perf}}(\mathrm{K}, \mathcal{O}/\varpi^m)$, there exists $M \geq m$ such that the restriction A_M of A to $D_{\mathrm{perf}}(\mathrm{K}_M, \mathcal{O}/\varpi^m)$ is isomorphic to the constant object B_M of $D_{\mathrm{perf}}(\mathrm{K}_M, \mathcal{O}/\varpi^m)$ corresponding to the image B of A in $D_{\mathrm{perf}}(\mathcal{O}/\varpi^m)$. In turn, this would follow from showing that the natural map

$$\mathrm{colim}_M \left(\mathrm{Hom}_{D_{\mathrm{perf}}(\mathrm{K}_M, \mathcal{O}/\varpi^m)}(B_M, A_M)\right) \to \mathrm{Hom}_{D_{\mathrm{perf}}(\mathcal{O}/\varpi^m)}(B, A) \tag{2.3.7}$$

is a bijection. Indeed, on the RHS we have the identity morphism, which must correspond to a morphism on the LHS for some $M \geq m$. We can check that this morphism in $D_{\mathrm{perf}}(\mathrm{K}_M, \mathcal{O}/\varpi^m)$ is an isomorphism after applying the forgetful functor to $D_{\mathrm{perf}}(\mathcal{O}/\varpi^m)$.

The objects of $D_{\mathrm{perf}}(\mathcal{O}/\varpi^m)$ are precisely the dualizable objects of $D^+(\mathcal{O}/\varpi^m)$ and the dual is given by applying $\mathrm{Hom}_{\mathcal{O}/\varpi^m}(\ , \mathcal{O}/\varpi^m)$. We have the adjunction (cf. [Sta13, Lemma 07VI])

$$\mathrm{Hom}_{D_{\mathrm{perf}}(\mathcal{O}/\varpi^m)}(B, A) = \mathrm{Hom}_{D_{\mathrm{perf}}(\mathcal{O}/\varpi^m)}(\mathcal{O}/\varpi^m, B^\vee \otimes^{\mathbb{L}} A) = H^0(B^\vee \otimes^{\mathbb{L}} A).$$

Now, we have an adjunction

$$\mathrm{Hom}_{D_{\mathrm{perf}}(\mathrm{K}_M, \mathcal{O}/\varpi^m)}\left(B_M, A_M\right) = \mathrm{Hom}_{D_{\mathrm{perf}}(\mathcal{O}/\varpi^m)}\left(B, R\Gamma(\mathrm{K}_M, A_M)\right)$$

and we have identified the right hand side with $H^0(B^\vee \otimes^{\mathbb{L}} R\Gamma(\mathrm{K}_M, A_M))$. Fixing a strictly perfect complex representing B, and equipping it with the trivial K_M action to get a complex representing B_M, we identify $B^\vee \otimes^{\mathbb{L}} R\Gamma(\mathrm{K}_M, A_M) = R\Gamma(\mathrm{K}_M, B_M^\vee \otimes^{\mathbb{L}} A_M)$.

Therefore, the bijectivity in (2.3.7) reduces to the statement that $\mathrm{colim}_M H^0(\mathrm{K}_M, B_M^\vee \otimes^{\mathbb{L}} A_M)$ (degree 0 hypercohomology) is equal to the degree 0 cohomology of the underlying complex of $B^\vee \otimes^{\mathbb{L}} A$. This follows from the fact that $B^\vee \otimes^{\mathbb{L}} A$ is a complex of smooth representations of K. □

We will apply the preceding lemma when proving the crucial Proposition 4.2.6. In that proof, we will also need the following consequence of the Artin–Rees lemma:

Lemma 2.3.18. *Let N be a finite $\mathbb{Z}_p$-module and M be a subquotient of N. For any positive integer $m \in \mathbb{Z}_{\geq 1}$, there exists an integer $m' \geq m$ such that $M/p^m M$ is a subquotient of $N/p^{m'}N$.*

Proof. If M is a quotient of N, then we may take $m' = m$. Therefore, it suffices to consider the case when $M \hookrightarrow N$ is a subobject. By the Artin–Rees lemma, cf. [AM16, Cor. 10.10], there exists $k \gg 0$ such that

$$p^{m+k}N \cap M = p^m(p^k N \cap M) \subseteq p^m M.$$

Set $m' := m + k$. We then have an inclusion $M/(p^{m'}N \cap M) \hookrightarrow N/p^{m'}N$ and a surjection $M/(p^{m'}N \cap M) \twoheadrightarrow M/p^m M$. □

3. Determinants, P-ordinary representations and deformation rings.

3.1. **The P-ordinary condition on the Galois side.** We place ourselves in the setting of §2.1.11, so we have a CM field F, unitary group $\widetilde{G}$, etc. We fix a standard parabolic $Q_{\bar{v}} \subset P_{F^+_{\bar{v}}}$. After fixing a place $v|\bar{v}$, this corresponds under ι_v to a standard parabolic $P_{n_1,\dots,n_t} \subset \mathrm{GL}_{2n}$, with $(n_1, \dots, n_t)$ a partition of $2n$ refining (n, n). Recall that $Q_{\bar{v}} = M_{\bar{v}} \ltimes N_{\bar{v}}$ is a Levi decomposition for $Q_{\bar{v}}$. We use the notation of §2.1.13, so we have a parahoric subgroup $\mathcal{Q}_{\bar{v}}$ associated with $Q_{\bar{v}}$.

For $1 \leq k \leq t$, we define a cocharacter $\nu_k \in X_{Q_{\bar{v}}}$ by

$$\nu_k(\varpi_{\bar{v}}) := \iota_v^{-1}\mathrm{diag}(\varpi_v, \dots, \varpi_v, 1, \dots, 1) \in \widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}},$$

where there are $n_1 + \cdots + n_k$ entries equal to ϖ, and denote the Hecke operator $[\mathcal{Q}_{\bar{v}}\nu_k(\varpi_{\bar{v}})\mathcal{Q}_{\bar{v}}] \in \mathcal{H}(\widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}, \mathcal{Q}_{\bar{v}})$ by $\widetilde{U}_v^k$. It follows from Lemma 2.1.15 that

$$\mathcal{H}(\widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}, \mathcal{Q}_{\bar{v}}) \cong \mathbb{Z}[\widetilde{U}_v^1, \dots, \widetilde{U}_v^{t-1}, (\widetilde{U}_v^t)^{\pm 1}].$$

If $\tilde{\lambda} \in (\mathbb{Z}^{2n}_+)^{\mathrm{Hom}(F^+, \overline{\mathbb{Q}}_p)}$ is a dominant weight for $\widetilde{G}$ and σ is a smooth $\overline{\mathbb{Q}}_p$-representation of $\widetilde{G}(F^+_{\bar{v}})$, we define the $\tilde{\lambda}$-*rescaled* action of $\mathcal{H}(\widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}, \mathcal{Q}_{\bar{v}})$ on $\sigma^{\mathcal{Q}_{\bar{v}}}$ to be given by multiplying the usual double coset operator action of $[\mathcal{Q}_{\bar{v}} g \mathcal{Q}_{\bar{v}}]$ by $\tilde{\alpha}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}(g)^{-1}$ (cf. Lemma 2.1.17).

Definition 3.1.1. *Let π be a cuspidal automorphic representation of $\widetilde{G}(\mathbb{A}_{F^+})$ and fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. Let $\tilde{\lambda} \in (\mathbb{Z}^{2n}_+)^{\mathrm{Hom}(F^+, \overline{\mathbb{Q}}_p)}$ be a dominant weight for $\widetilde{G}$. We say that π is ι-$Q_{\bar{v}}$-ordinary of weight $\tilde{\lambda}$ if π is $\iota V_{\tilde{\lambda}}^\vee$-cohomological and the $\tilde{\lambda}$-rescaled Hecke operators $\{\widetilde{U}_v^k : 1 \leq k \leq t\}$ have a simultaneous eigenvector with p-adic unit eigenvalues in $\iota^{-1}\pi^{\mathcal{Q}_{\bar{v}}}$.*

*If π is ι-$Q_{\bar{v}}$-ordinary of weight $\tilde{\lambda}$, we define the $\mathcal{Q}_{\bar{v}}$-*ordinary subspace *of $\iota^{-1}\pi_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}$ to be the largest $\mathcal{H}(\widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}, \mathcal{Q}_{\bar{v}})$-submodule on which the rescaled operators $\widetilde{U}_v^k$ have only p-adic unit eigenvalues for $1 \leq k \leq t$.*

The goal of this section is to establish the following result, generalising [Ger19, Corollary 2.33], [Tho15, Theorem 2.4].

Theorem 3.1.2. *Suppose that π is a cuspidal automorphic representation of $\widetilde{G}(\mathbb{A}_{F^+})$, ι is an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ and $\bar{v}$ is a p-adic place of F^+ such that π is ι-$Q_{\bar{v}}$-ordinary of weight $\tilde{\lambda}$.*

Then we have the following conclusions:

(1) *The associated p-adic Galois representation* $r_\iota(\pi) : G_F \to \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_p)$ *satisfies*

$$r_\iota(\pi)|_{G_{F_v}} \simeq \begin{pmatrix} r_1(\pi) & * & \cdots & * \\ 0 & r_2(\pi) & \cdots & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & \cdots & r_t(\pi) \end{pmatrix}, \tag{3.1.1}$$

where $r_j(\pi) : G_{F_v} \to \mathrm{GL}_{n_j}(\overline{\mathbb{Q}}_p)$ *is a crystalline representation for each* $j = 1, 2, \ldots, t$.

(2) *The* $\mathcal{Q}_{\bar{v}}$*-ordinary subspace of* $\iota^{-1}\pi_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}$ *is one-dimensional.*

(3) *For each embedding* $\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_p$, *the* τ*-Hodge–Tate weights of the* $r_j(\pi)$ *are given by decomposing* $\tilde{\lambda}_{\tau,2n} < \tilde{\lambda}_{\tau,2n-1}+1 < \cdots < \tilde{\lambda}_{\tau,1}+2n-1$ *according to the partition* $(n_1, \ldots, n_t)$.

(4) *The determinants* $\det r_j(\pi)$ *are given by the formulas:*
- $\prod_{j=1}^k \det r_j(\pi)(\mathrm{Art}_{F_v}(u)) = \prod_{i=1}^{n_1+\cdots+n_k} \prod_{\tau: F_v \hookrightarrow \overline{\mathbb{Q}}_p} \tau(u)^{-\tilde{\lambda}_{\tau,2n-i+1}-i+1}$ *for* $u \in \mathcal{O}_{F_v}^\times$.
- $\prod_{j=1}^k \det r_j(\pi)(\mathrm{Art}_{F_v}(\varpi_v))$ *is equal to* $\epsilon_p^{\sum_{i=1}^{n_1+\cdots+n_k}(1-i)}(\mathrm{Art}_{F_v}(\varpi_v))$ *times the eigenvalue of* $\widetilde{U}_v^k$ *on the* $\mathcal{Q}_{\bar{v}}$*-ordinary subspace.*

Before giving the proof, we first establish a preliminary result.

Lemma 3.1.3. *Assume that* $r : G_{F_v} \to \mathrm{GL}_m(\overline{\mathbb{Q}}_p)$ *is a semi-stable Galois representation. Let* $v_1 \le v_2 \le \cdots \le v_m$ *denote the valuations of the eigenvalues of the geometric Frobenius acting on* $\mathrm{WD}(r)$, *and, for each* $\tau : F \hookrightarrow \overline{\mathbb{Q}}_p$, *let* $h_{\tau,1} < h_{\tau,2} < \cdots < h_{\tau,m}$ *denote the* τ*-Hodge–Tate weights of* r. *Then*

$$\sum_{i=1}^{j} v_i \ge \frac{1}{e_v}\sum_{i=1}^{j}\sum_{\tau} h_{\tau,i} \tag{3.1.2}$$

for any $0 \le j \le m$, *where* e_v *is the ramification degree of* $F/\mathbb{Q}_p$.

Furthermore, if we have an equality

$$\sum_{i=1}^{j} v_{\sigma(i)} = \frac{1}{e_v}\sum_{i=1}^{j}\sum_{\tau} h_{\tau,i}$$

for some $1 \le j \le m-1$ *and permutation* $\sigma \in S_n$, *then* $r \simeq \begin{pmatrix} r_1 & * \\ 0 & r_2 \end{pmatrix}$ *with* $r_1 : G_{F_v} \to \mathrm{GL}_j(\overline{\mathbb{Q}}_p)$ *and* $r_2 : G_{F_v} \to \mathrm{GL}_{m-j}(\overline{\mathbb{Q}}_p)$. *The representation* r_1 *has* τ*-Hodge–Tate weights equal to* $h_{\tau,1} < \cdots < h_{\tau,j}$ *for each* τ, *the eigenvalues of the geometric Frobenius acting on* $\mathrm{WD}(r_1)$ *have valuations* $v_1 \le v_2 \le \cdots \le v_j$ *and we have* $v_j < v_{j+1}$.

Proof. The first part follows from [HKV20, Lemma 6.4.1]. For the second part, if equality holds for some permutation σ, then by the first part and by the inequalities on the v_i, equality must also hold for the identity permutation. Furthermore, we have $\sum_{i=1}^j v_i = \sum_{i=1}^j v_{\sigma(i)}$. If $\{\sigma(1), \ldots, \sigma(j)\} \ne \{1, \ldots, j\}$, then $\sigma(i) \ge j+1$ for some $i \le j$, and so we must have $v_j = v_{j+1}$. We show that $v_j = v_{j+1}$ is impossible. As in *loc. cit.*, using the inequality in (3.1.2) for $j+1$ and $j-1$, we deduce that the inequality for $j-1$ is actually an equality. Moreover, this implies that $h_{\tau,j} = h_{\tau,j+1}$

for all τ, which is a contradiction. We deduce that $\{\sigma(1), \dots, \sigma(j)\} = \{1, \dots, j\}$ and that $v_j < v_{j+1}$. The last paragraph in the proof of [HKV20, Lemma 6.4.1] gives a j-dimensional sub-G_{F_v}-representation r_1 of r with τ-Hodge–Tate weights equal to $h_{\tau,1} < \cdots < h_{\tau,j}$ for each τ and such that the eigenvalues of the geometric Frobenius acting on $\mathrm{WD}(r_1)$ are $v_1 \le v_2 \le \cdots \le v_j$. □

Proof of Theorem 3.1.2. We use ι_v to identify $\widetilde{G}(F_{\bar{v}}^+)$ with $\mathrm{GL}_{2n}(F_v)$. Since $\mathcal{Q}_{\bar{v}}$ contains the Iwahori subgroup of $\widetilde{G}(F_{\bar{v}}^+)$, $\iota^{-1}\pi_{\bar{v}}$ is a subquotient of a normalised induction $\sigma = \mathrm{n-Ind}_{\mathrm{B}_{2n}(F_v)}^{\mathrm{GL}_{2n}(F_v)}(\chi_1 \otimes \chi_2 \otimes \cdots \otimes \chi_{2n})$ with each $\chi_i : F_v^\times \to \overline{\mathbb{Q}}_p^\times$ an unramified character.

We claim that, up to reordering the χ_i, for each $1 \le k \le t$ the p-adic numbers $\prod_{i=1}^{n_1+\cdots n_k} \chi_i(\varpi_v)$ and $\delta_{\mathrm{B}_{2n}}(\nu_k(\varpi_{\bar{v}}))^{1/2}\alpha_{\tilde{\lambda}}^{\mathcal{Q}_{\bar{v}}}(\nu_k(\varpi_{\bar{v}}))$ differ by a p-adic unit.

By the proof of [Tho12, Prop. 5.4], we have an isomorphism $\sigma^{\mathcal{Q}_{\bar{v}}} \cong (J_{Q_{\bar{v}}}(\sigma))^{\mathcal{Q}_{\bar{v}} \cap M_{\bar{v}}(F_{\bar{v}}^+)}$ (normalized Jacquet module) with the $\tilde{\lambda}$-rescaled action of the Hecke operator $\widetilde{U}_v^k$ on the left given by the action of $\alpha_{\tilde{\lambda}}^{\mathcal{Q}_{\bar{v}}}(\nu_k(\varpi_{\bar{v}}))^{-1}\delta_{Q_{\bar{v}}}^{-1/2}(\nu_k(\varpi_{\bar{v}}))\nu_k(\varpi_{\bar{v}})$ on the right. We can further compose with the injection $(J_{Q_{\bar{v}}}(\sigma))^{\mathcal{Q}_{\bar{v}} \cap M_{\bar{v}}(F_{\bar{v}}^+)} \hookrightarrow (J_{\mathrm{B}_{2n}}(\sigma))^{\mathcal{Q}_{\bar{v}} \cap \mathrm{T}_{2n}(F_v)}$, with $\widetilde{U}_v^k$ acting by $\alpha_{\tilde{\lambda}}^{\mathcal{Q}_{\bar{v}}}(\nu_k(\varpi_{\bar{v}}))^{-1}\delta_{\mathrm{B}_{2n}}^{-1/2}(\nu_k(\varpi_{\bar{v}}))\nu_k(\varpi_{\bar{v}})$ on the target. Note that since $\nu_k(\varpi_{\bar{v}})$ is central in $M_{\bar{v}}$, we have $\delta_{\mathrm{B}_{2n}}(\nu_k(\varpi_{\bar{v}})) = \delta_{Q_{\bar{v}}}(\nu_k(\varpi_{\bar{v}}))$. We have $J_{\mathrm{B}_{2n}}(\sigma)^{ss} = \bigoplus_{w \in W} \chi_{w(1)} \otimes \cdots \otimes \chi_{w(2n)}$. Choosing w so that the unit-eigenvalue eigenvector for the $\widetilde{U}_v^k$ contributes to the summand indexed by w gives the desired reordering of the χ_i.

By Theorem 2.1.19, since $\tilde{\lambda}$ is dominant, the τ-Hodge–Tate weights of $r_\iota(\pi)|_{G_{F_v}}$ are equal to $\tilde{\lambda}_{\tau,2n} < \tilde{\lambda}_{\tau,2n-1} + 1 < \cdots < \tilde{\lambda}_{\tau,1} + 2n - 1$. In addition, by making part (3) of Theorem 2.1.19 explicit at v, the eigenvalues of the geometric Frobenius acting on $\mathrm{WD}(r_\iota(\pi)|_{G_{F_v}})$ are $q_v^{\frac{2n-1}{2}}\chi_i(\varpi_v)$. Our claim about the p-adic valuations of the $\chi_i(\varpi_v)$ implies that the p-adic numbers

$$\prod_{i=1}^{n_1+\cdots+n_k} \chi_i(\varpi_v) \text{ and}$$

$$q_v^{-\langle \nu_k, \rho_{\mathrm{B}_{2n}} \rangle} \prod_{\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_p} \tau(\varpi_v)^{\langle \nu_k, w_0^{\widetilde{G}} \lambda_\tau \rangle}$$

differ by a p-adic unit. The latter term has the same p-adic valuation as

$$\prod_{i=1}^{n_1+\cdots+n_k} q_v^{\frac{1-2n}{2}} \prod_{\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_p} \tau(\varpi_v)^{\lambda_{\tau,2n-i+1}+i-1}.$$

In turn, this implies that the inequality in (3.1.2) is an equality for $j = n_1 + \cdots + n_k$ for each k. We conclude that $r_\iota(\pi)|_{G_{F_v}}$ has the desired shape by Lemma 3.1.3.

To show that the $\mathcal{Q}_{\bar{v}}$-ordinary subspace of $\iota^{-1}\pi_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}$ is one-dimensional, it suffices to prove the analogous statement for $\sigma^{\mathcal{Q}_{\bar{v}}} \cong (J_{Q_{\bar{v}}}(\sigma))^{\mathcal{Q}_{\bar{v}} \cap M_{\bar{v}}(F_{\bar{v}}^+)}$. By the geometric lemma [BZ77, Lemma 2.12], we have

$$J_{Q_{\bar{v}}}(\sigma)^{ss} = \bigoplus_{w \in [W/W_{Q_{\bar{v}}}]} \left(\mathrm{n-Ind}_{\mathrm{B}_{2n} \cap M_{\bar{v}}}^{M_{\bar{v}}} \chi_{w(1)} \otimes \cdots \otimes \chi_{w(2n)} \right). \tag{3.1.3}$$

The notation $w \in [W/W_{Q_{\bar{v}}}]$ means that we take the minimal length representative $w \in W$ for each coset. Explicitly, this is given by permutations $w \in S_{2n}$ which are order-preserving on each of the subsets $\{1,\ldots,n_1\},\{n_1+1,\ldots,n_1+n_2\},\ldots,\{n_1+\cdots+n_{t-1}+1,\ldots,2n\}$. Each summand in (3.1.3) has a one-dimensional space of invariants under the maximal compact subgroup $\mathcal{Q}_{\bar{v}}\cap M_{\bar{v}}(F_{\bar{v}}^+)$. On the invariants of the summand indexed by w, for each $1\leq k\leq t$ the Hecke operator $\widetilde{U}_v^k$ acts by the scalar

$$\alpha_{k,w}:=\alpha_{\tilde{\lambda}}^{\mathcal{Q}_{\bar{v}}}(\nu_k(\varpi_{\bar{v}}))^{-1}\delta_{\mathrm{B}_{2n}}^{-1/2}(\nu_k(\varpi_{\bar{v}}))\prod_{i=1}^{n_1+\cdots+n_k}\chi_{w(i)}(\varpi_v).$$

We know that $\alpha_{k,1}$ is a p-adic unit for all k. The inequality in the conclusion of Lemma 3.1.3 shows that the $v_p(\alpha_{1,w})>0$ if w does not preserve $\{1,\ldots,n_1\}$. For a fixed w, inducting on k and repeating this argument shows that $v_p(\alpha_{k,w})=0$ for all k if and only if $w=1$. This shows that the $\mathcal{Q}_{\bar{v}}$-ordinary subspace of $\iota^{-1}\pi_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}$ is one-dimensional.

The statements about the Hodge–Tate weights and determinant of the $r_i(\pi)$ follow from the Lemma and the identifications above.

It remains to see that each $r_i(\pi)$ is crystalline. We write the argument for $r_1(\pi)$, it is similar for the other factors. Let $w_i=v_p(\chi_i(\varpi_v))$ for $i=1,\ldots,2n$. The eigenvalues of a geometric Frobenius on $\mathrm{WD}(r_1(\pi))$ are $q_v^{\frac{2n-1}{2}}\chi_i(\varpi_v)$ for $i=1,\ldots,n_1$. Lemma 3.1.3 implies that, up to reordering the χ_i for $i=1,\ldots,n_1$, we may assume that $w_1\leq w_2\cdots\leq w_{n_1}<w_{n_1+1}$. If $r_1(\pi)$ is not crystalline, then by the Bernstein–Zelevinsky classification there exists some $i\in 2,\ldots n_1$ such that $\chi_{i-1}=\chi_i\cdot|\;|$ and $\pi_{\bar{v}}$ is a subquotient of the normalised parabolic induction

$$\sigma'=\mathrm{n}-\mathrm{Ind}_{Q'(F_v)}^{\mathrm{GL}_{2n}(F_v)}(\chi_1\otimes\cdots\otimes\chi_{i-2}\otimes\mathrm{Sp}_2(\chi_i)\otimes\chi_{i+1}\otimes\cdots\otimes\chi_{2n}),$$

for an appropriate standard parabolic subgroup $Q'=M'N'\subset Q_{\bar{v}}$, where $\mathrm{Sp}_2(\chi_i)$ denotes a twist of the Steinberg representation. We let $\sigma_0'=\chi_1\otimes\cdots\otimes\chi_{i-2}\otimes\mathrm{Sp}_2(\chi_i)\otimes\chi_{i+1}\otimes\cdots\otimes\chi_{2n}$.

Applying the geometric lemma again, we have

$$J_{Q_{\bar{v}}}(\sigma')^{ss}=\bigoplus_{w\in[W_{Q'}\backslash W/W_{Q_{\bar{v}}}]}\left(\mathrm{n}-\mathrm{Ind}_{w^{-1}Q'w\cap M_{\bar{v}}}^{M_{\bar{v}}}\left(J_{M'\cap wQ_{\bar{v}}w^{-1}}\sigma_0'\right)^w\right)^{ss}$$

As in the proof that the $\mathcal{Q}_{\bar{v}}$-ordinary subspace is one-dimensional, it follows from considering valuations that the $\mathcal{Q}_{\bar{v}}$-ordinary subspace of $\iota^{-1}\pi_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}\cong(J_{Q_{\bar{v}}}(\sigma'))^{\mathcal{Q}_{\bar{v}}\cap M_{\bar{v}}(F_{\bar{v}}^+)}$ can only contribute to the $w=1$ term in this decomposition. We deduce that

$$\left(\mathrm{n}-\mathrm{Ind}_{Q'\cap M_{\bar{v}}}^{M_{\bar{v}}}(\sigma_0')\right)^{\mathcal{Q}_{\bar{v}}\cap M_{\bar{v}}(F_{\bar{v}}^+)}\neq 0,$$

but this is impossible because $\mathcal{Q}_{\bar{v}}\cap M_{\bar{v}}(F_{\bar{v}}^+)$ is maximal compact and σ_0' has a Steinberg factor. □

3.2. **Determinants.** In this section we prove a key proposition which will be combined with Theorem 3.1.2 to pass information about local–global compatibility for Galois representations with coefficients in p-torsion free Hecke algebras for $\widetilde{G}$ to Galois representations with coefficients in torsion Hecke algebras for G. The initial set-up is as follows: recall our coefficient field $E\supset\mathcal{O}\twoheadrightarrow k$ and assume we have an absolutely irreducible continuous representation

$$\overline{\rho}_{\mathfrak{m}}:G_F\to\mathrm{GL}_n(k)$$

together with a continuous lift

$$\rho_{\mathfrak{m}} : G_F \to \mathrm{GL}_n(A)$$

with coefficients in $A \in \mathrm{CNL}_{\mathcal{O}}$.

We moreover have a finite flat $\mathcal{O}$-algebra $\widetilde{A} \in \mathrm{CNL}_{\mathcal{O}}$ with $\widetilde{A}[1/p] = \prod_{i=1}^r K_i$ a product of fields, equipped with a surjective map $\widetilde{A} \twoheadrightarrow A$. Extending E if necessary, we can assume that every K_i has residue field k (indeed, we can also assume that each K_i is equal to E, but we won't need to do this). We suppose we have a continuous representation

$$\widetilde{\rho}_{\mathfrak{m}} = \prod \widetilde{\rho}_{i,\mathfrak{m}} : G_F \to \prod \mathrm{GL}_{2n}(K_i) = \mathrm{GL}_{2n}(\widetilde{A}[1/p])$$

such that the associated determinant of G_F (in the sense of [Che14]) arises from a continuous $\widetilde{A}$-valued determinant $D_{\widetilde{\rho}_{\mathfrak{m}}} : \widetilde{A}[G_F] \to \widetilde{A}$. This is equivalent to the characteristic polynomials of $\widetilde{\rho}_{\mathfrak{m}}(g)$ having coefficients in $\widetilde{A}$ for all $g \in G_F$.

We recall from [Che14] the important notion of a *Cayley–Hamilton* determinant: a determinant $D : R \to A$ is Cayley–Hamilton if the characteristic polynomial $\chi(r,t) = D(t-r) \in A[t]$ vanishes when evaluated at $t = r$ for all $r \in R$.

The determinant $D_{\widetilde{\rho}_{\mathfrak{m}}}$ factors through a Cayley–Hamilton determinant (which we also denote by $D_{\widetilde{\rho}_{\mathfrak{m}}}$) of $\widetilde{B} := \widetilde{\rho}_{\mathfrak{m}}(\widetilde{A}[G_F]) \subset M_{2n}(\widetilde{A}[1/p])$. Since each element of $\widetilde{B}$ has characteristic polynomial with coefficients in $\widetilde{A}$, $\widetilde{B}$ is integral over $\widetilde{A}$. We also assume that we have a factorisation of A-valued determinants of $A[G_F]$

$$D_{\widetilde{\rho}_{\mathfrak{m}}} \otimes_{\widetilde{A}} A = D_{\rho_{\mathfrak{m}}} D_{\rho_{\mathfrak{m}}^{\vee,c}(1-2n)}.$$

Fix a place $v|p$ of F. Extending E if necessary, we may assume that every irreducible constituent of the local representation $\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}}$ is absolutely irreducible. We assume that, for each i, we have an n-dimensional G_{F_v}-sub-representation $(\widetilde{\rho}^0_{i,\mathfrak{m}}, V_i^0) \subset (\widetilde{\rho}_{i,\mathfrak{m}}, K_i^{2n})$ with quotient $(\widetilde{\rho}^1_{i,\mathfrak{m}}, V_i^1)$, Finally, we assume that:

(1) the isomorphism classes of the irreducible constituents of $\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}}$ are disjoint from those of $\overline{\rho}_{\mathfrak{m}}^{\vee,c}(1-2n)|_{G_{F_v}}$;
(2) for all i the isomorphism classes of the irreducible constituents of the residual representation $\overline{(\widetilde{\rho}^0_{i,\mathfrak{m}})}$ coincide with those of $\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}}$.

As in the proof of [Che14, Thm. 2.22], we are going to use a basic fact about idempotents, cf. [Bou98, Chapter III, §4, Exercise 5 (a)].

Lemma 3.2.1. *Let S be a Henselian local ring, R a (not necessarily commutative) S-algebra which is integral over S. Let I be a two-sided ideal of R. Every idempotent in R/I lifts to an idempotent in R.*

Proof. The statement can be reduced to the case of a finite S-algebra R generated by one element, in particular with R commutative (take the subalgebra of R generated by an arbitrary lift of the idempotent in R/I). In this case the statement is clear, since R is then a product of finitely many Henselian local rings. □

The following lemma constructs an element $\widetilde{e} \in M_{2n}(\widetilde{A}[1/p])$ which is an idempotent projection onto $\prod_i V_i^0$, lies in $\widetilde{\rho}_{\mathfrak{m}}(\widetilde{A}[G_{F_v}])$ and moroever cuts out the first factor in the product decomposition $D_{\widetilde{\rho}_{\mathfrak{m}}} \otimes_{\widetilde{A}} k = D_{\overline{\rho}_{\mathfrak{m}}} D_{\overline{\rho}_{\mathfrak{m}}^{\vee,c}(1-2n)}$ of residual determinants. Roughly speaking, this is possible because our assumptions on the irreducible constituents of $\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}}$ mean the global factors $D_{\overline{\rho}_{\mathfrak{m}}}$, $D_{\overline{\rho}_{\mathfrak{m}}^{\vee,c}(1-2n)}$ can be distinguished locally at v.

Lemma 3.2.2. (1) *The natural map* $k[G_F] \xrightarrow{\overline{\rho}_{\mathfrak{m}} \times \overline{\rho}_{\mathfrak{m}}^{\vee,c}(1-2n)} M_n(k) \times M_n(k)$ *induces an isomorphism* $\iota : k[G_F]/\ker(D_{\widetilde{\rho}_{\mathfrak{m}}} \otimes_{\widetilde{A}} k) \cong M_n(k) \times M_n(k)$.

(2) *The idempotent* $e = \iota^{-1}(1_{M_n(k)}, 0)$ *is contained in the image of* $k[G_{F_v}]$.

(3) *There is a lift* $\widetilde{e} \in \widetilde{B}$ *of* e *which is in the image* $\widetilde{B}_v \subset \widetilde{B}$ *of* $\widetilde{A}[G_{F_v}]$. *(Note that* $D_{\widetilde{\rho}_{\mathfrak{m}}} \otimes_{\widetilde{A}} k$ *factors through* $\widetilde{B} \otimes_{\widetilde{A}} k$, *so* $k[G_F]/\ker(D_{\widetilde{\rho}_{\mathfrak{m}}} \otimes_{\widetilde{A}} k)$ *is a quotient of* $\widetilde{B}$.*)*

(4) *Choose* $\widetilde{e}$ *as in the previous part. For each* $i \in \{1, \ldots, r\}$ *let* π_i *denote the projection* $\widetilde{A}[1/p]^{2n} \to V_i^1$. *Thinking of* $\widetilde{e}$ *as an element of* $\mathrm{End}(\widetilde{A}[1/p]^{2n})$, *we have* $\pi_i \circ \widetilde{e} = 0$ *and the image of* $\widetilde{e}$ *in* $\mathrm{End}(V_i^0)$ *is the identity. In other words,* $\widetilde{e}$ *is an idempotent projection onto* $\prod_i V_i^0$.

Proof. The first claim follows from our assumption that $\overline{\rho}_{\mathfrak{m}}$ and $\overline{\rho}_{\mathfrak{m}}^{\vee,c}(1-2n)$ are distinct and absolutely irreducible, by [Che14, Thm. 2.16].

For the second claim we consider the subalgebra $B = \iota(k[G_{F_v}])$ of $M_n(k) \times M_n(k)$. The semisimple quotient $B/\mathrm{Rad}(B)$ contains the idempotent $\overline{e}_v$ which acts as the identity on each irreducible constituent of $\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}}$ and as zero on each irreducible constituent of $\overline{\rho}_{\mathfrak{m}}^{\vee,c}(1-2n)|_{G_{F_v}}$; here we are using our assumption that these two collections of irreducible constituents are disjoint. Since $\mathrm{Rad}(B)$ is nilpotent, we can lift $\overline{e}_v$ to an idempotent $e_v \in B$. By considering the composition series of $\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}}$, we see that e_v maps to an idempotent unit (i.e. the identity) under the first projection to $M_n(k)$. Considering the composition series of $\overline{\rho}_{\mathfrak{m}}^{\vee,c}(1-2n)$, e_v maps to a nilpotent idempotent (i.e. zero) under the second projection, so we have $e_v = e$.

For the third claim, we know that $\widetilde{B}_v$ is integral over $\widetilde{A}$ (since it is a subalgebra of $\widetilde{B}$). We apply Prop. 3.2.1 with $S = \widetilde{A}$, which is Henselian as it is a local ring and finite over $\mathcal{O}$, and $R = \widetilde{B}_v$. We deduce that the idempotent e lifts to an idempotent $\widetilde{e}$ in $\widetilde{B}_v$.

Now we come to the fourth claim. Fix an index i. We choose a G_F-stable $\mathcal{O}_{K_i}$-lattice $T \subset K_i^{2n}$. We have a short exact sequence

$$0 \to V_i^0 \cap T \to T \to \pi_i(T) \to 0$$

of $\mathcal{O}_{K_i}[G_{F_v}]$-modules (in particular the submodule $V_i^0 \cap T$ is stable under $\widetilde{e}$). The image of $\widetilde{e}$ in $\mathrm{End}_k(\pi_i(T) \otimes_{\mathcal{O}_{K_i}} k)$ is equal to zero, since $\widetilde{e}$ lifts the idempotent which acts as zero on each irreducible constituent of the G_{F_v} representation on $\pi_i(T) \otimes_{\mathcal{O}_{K_i}} k$ (by assumption these coincide with the irreducible constituents of $\overline{\rho}_{\mathfrak{m}}^{\vee,c}(1-2n)|_{G_{F_v}}$). We deduce that the image of $\widetilde{e}$ in $\mathrm{End}_{\mathcal{O}_{K_i}}(\pi_i(T))$ is equal to zero, since it is an idempotent with image in $\varpi_{K_i}\pi_i(T)$. This shows that $\pi_i \circ \widetilde{e} = 0$, as claimed. Similarly, the image of $\widetilde{e}$ in $\mathrm{End}_k((V_i^0 \cap T) \otimes_{\mathcal{O}_{K_i}} k)$ is an idempotent isomorphism, hence the identity, and so the image of $\widetilde{e}$ in $\mathrm{End}_{\mathcal{O}_{K_i}}(V_i^0 \cap T)$ itself is an idempotent isomorphism, hence equal to the identity. □

In the preceding lemma, we constructed an idempotent $\widetilde{e}$, compatible with both the local 'P-ordinary' decomposition and the residual global decomposition $D_{\widetilde{\rho}_{\mathfrak{m}}} \otimes_{\widetilde{A}} k = D_{\overline{\rho}_{\mathfrak{m}}} D_{\overline{\rho}_{\mathfrak{m}}^{\vee,c}(1-2n)}$. Following [BC09, Theorem 1.4.4] (and its generalization [Che14, Theorem 2.22]), we can now use the idempotents $e_1 := \widetilde{e}, e_2 := 1-\widetilde{e}$ to equip $\widetilde{B}$ with a *generalized matrix algebra* structure compatible with the determinant $D_{\widetilde{\rho}_{\mathfrak{m}}}$. We now explain in detail what this means (see also [ANT20, §2]). By [Che14, Lem. 2.4], we have determinants of dimension n

$$\begin{aligned} D_{\widetilde{\rho}_{\mathfrak{m}},i} : e_i\widetilde{B}e_i &\to \widetilde{A} \\ x &\mapsto D_{\widetilde{\rho}_{\mathfrak{m}}}(x+1-e_i) \end{aligned}$$

for $i = 1, 2$, with $(D_{\widetilde{\rho}_{\mathfrak{m}},1} \otimes_{\widetilde{A}} k)(e_1xe_1) = D_{\overline{\rho}_{\mathfrak{m}}}(x)$ and $(D_{\widetilde{\rho}_{\mathfrak{m}},2} \otimes_{\widetilde{A}} k)(e_2xe_2) = D_{\overline{\rho}_{\mathfrak{m}}^{\vee,c}(1-2n)}(x)$. The $e_i\widetilde{B}e_i$ are equipped with $\widetilde{A}$-algebra isomorphisms

$$\psi_i : e_i\widetilde{B}e_i \cong M_n(\widetilde{A}) \tag{3.2.1}$$

with $\det \circ \psi_i = D_{\widetilde{\rho}_{\mathfrak{m}},i}$. Moreover, the map $(\psi_1, \psi_2) : e_1\widetilde{B}e_1 \oplus e_2\widetilde{B}e_2 \to M_n(\widetilde{A}) \times M_n(\widetilde{A})$ lifts $\iota : k[G_F]/\ker(D_{\widetilde{\rho}_{\mathfrak{m}}} \otimes_{\widetilde{A}} k) \cong M_n(k) \times M_n(k)$.

Define idempotents $E_i \in e_i\widetilde{B}e_i$ by asking for $\psi_i(E_i)$ to be the matrix with a one in the top left entry and zeroes elsewhere. We can now define $\mathcal{A}$-submodules $\mathcal{A}_{i,j} := E_i\widetilde{B}E_j \subset \widetilde{B}$. For each $1 \le i, j, k \le 2$ we write $\mathcal{A}_{i,j}\mathcal{A}_{j,k}$ for the $\widetilde{A}$-module generated by products xy for $x \in \mathcal{A}_{i,j}, y \in \mathcal{A}_{j,k}$. It is a submodule of $\mathcal{A}_{i,k}$. The ψ_i induce isomorphisms $\mathcal{A}_{i,i} \cong \widetilde{A}$, so we have a multiplication $\mathcal{A}_{i,j} \otimes_{\widetilde{A}} \mathcal{A}_{j,i} \to \widetilde{A}$ and we identify $\mathcal{A}_{i,j}\mathcal{A}_{j,i}$ with an ideal of $\widetilde{A}$. If $i \neq j$, then $e_i\widetilde{B}e_j\widetilde{B}e_i$ maps to $\ker(D_{\widetilde{\rho}_{\mathfrak{m}}} \otimes_{\widetilde{A}} k)$, and therefore $\mathcal{A}_{i,j}\mathcal{A}_{j,i} \subset \mathfrak{m}_{\widetilde{A}}$.

Putting everything together gives us isomorphisms

$$\widetilde{B} \xrightarrow{\sim} \begin{pmatrix} e_1\widetilde{B}e_1 & e_1\widetilde{B}e_2 \\ e_2\widetilde{B}e_1 & e_2\widetilde{B}e_2 \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} M_n(\widetilde{A}) & M_n(\mathcal{A}_{1,2}) \\ M_n(\mathcal{A}_{2,1}) & M_n(\widetilde{A}) \end{pmatrix}$$

with multiplication of the last matrix algebra given by using matrix multiplication and the maps $\mathcal{A}_{i,j} \otimes_{\widetilde{A}} \mathcal{A}_{j,i} \to \widetilde{A}$. The reducibility of the determinant $D_{\widetilde{\rho}_{\mathfrak{m}}}$ is reflected in the GMA structure. More precisely, we apply [ANT20, Proposition 2.5] in our setting (a mild generalization of [BC09, Proposition 1.5.1]) to deduce:

Proposition 3.2.3. *An ideal $J \subset \widetilde{A}$ contains $\mathcal{A}_{1,2}\mathcal{A}_{2,1}$ if and only if there are determinants $D_1, D_2 : \widetilde{B} \otimes_{\widetilde{A}} \widetilde{A}/J \to \widetilde{A}/J$ such that*

$$\begin{aligned} D_{\widetilde{\rho}_{\mathfrak{m}}} \otimes_{\widetilde{A}} \widetilde{A}/J &= D_1D_2, \\ D_1 \otimes_{\widetilde{A}/J} k &= D_{\overline{\rho}_{\mathfrak{m}}}, \\ \textit{and } D_2 \otimes_{\widetilde{A}/J} k &= D_{\overline{\rho}_{\mathfrak{m}}^{\vee,c}(1-2n)}. \end{aligned}$$

If this property holds, D_1 and D_2 are uniquely determined and satisfy $\ker(D_{\widetilde{\rho}_{\mathfrak{m}}} \otimes_{\widetilde{A}} \widetilde{A}/J) \subset \ker(D_i)$ *for $i = 1, 2$.*

Proof. This follows immediately from [ANT20, Proposition 2.5]. Note that the property '(COM)' in [BC09, Lemma 1.3.5] (which follows from the fact that the trace of a determinant satisfies the identity $\mathrm{Tr}(xy) = \mathrm{Tr}(yx)$) means that the ideals $\mathcal{A}_{1,2}\mathcal{A}_{2,1}$ and $\mathcal{A}_{2,1}\mathcal{A}_{1,2}$ are equal. □

We can now state and prove the key proposition of this section, where we combine the GMA structure defined above with the fact that our idempotent $\widetilde{e}$ was chosen to have good local properties at v.

Proposition 3.2.4. *Choose an idempotent $\widetilde{e} \in \widetilde{B_v}$ as in the third part of Lemma 3.2.2.*

(1) *The map*

$$\begin{aligned} A[G_F] &\to \widetilde{e}\widetilde{B}\widetilde{e} \otimes_{\widetilde{A}} A \\ x &\mapsto \widetilde{e}x\widetilde{e} \otimes 1 \end{aligned}$$

is a homomorphism and it induces the determinant $D_{\rho_{\mathfrak{m}}}$ when we compose with $\widetilde{e}\widetilde{B}\widetilde{e} \otimes_{\widetilde{A}} A \xrightarrow{\psi_1 \otimes \mathrm{id}} M_n(A)$ and the usual determinant (see (3.2.1) *for ψ_1).*

(2) *The map*

$$\begin{aligned} \widetilde{A}[G_{F_v}] &\to \widetilde{e}\widetilde{B}\widetilde{e} \\ x &\mapsto \widetilde{e}x\widetilde{e} \end{aligned}$$

is also a homomorphism and it induces the representation $\prod_{i=1}^r \widetilde{\rho}_{i,\mathfrak{m}}^0$ when we compose with the natural inclusion $\widetilde{e}\widetilde{B}\widetilde{e} \subset \mathrm{End}_{\widetilde{A}[1/p]}(\prod_i V_i^0)$ (see the final part of Lemma 3.2.2 for why we have this inclusion).

(3) *There is an $\widetilde{A}$-valued lift of the representation $\rho_{\mathfrak{m}}|_{G_{F_v}}$ which becomes isomorphic to $\prod_{i=1}^r \widetilde{\rho}_{i,\mathfrak{m}}^0$ when we invert p.*

Proof. For the first part, we use Proposition 3.2.3. Since the determinant $D_{\widetilde{\rho}_{\mathfrak{m}}} \otimes_{\widetilde{A}} A = D_{\rho_{\mathfrak{m}}} D_{\rho_{\mathfrak{m}}^{\vee,c}(1-2n)}$ is reducible, this tells us that the kernel J of $\widetilde{A} \to A$ contains the reducibility ideal $\mathcal{A}_{1,2}\mathcal{A}_{2,1}$. It follows that the map

$$\begin{aligned} A[G_F] &\to \widetilde{e}\widetilde{B}\widetilde{e} \otimes_{\widetilde{A}} A \\ x &\mapsto \widetilde{e}x\widetilde{e} \otimes 1 \end{aligned}$$

is a homomorphism, and the determinant induced by $D_{\widetilde{\rho}_{\mathfrak{m}},1} \otimes_{\widetilde{A}} A$ is equal to $D_{\rho_{\mathfrak{m}}}$ by the uniqueness part of Proposition 3.2.3.

For the second part, we can check that we have a homomorphism in $\mathrm{End}_{\widetilde{A}[1/p]}(\prod_i V_i^0)$, where it follows from the fact that $\prod_i V_i^0$ is G_{F_v}-stable. The identification of the representation with $\prod_{i=1}^r \widetilde{\rho}_{i,\mathfrak{m}}^0$ is now clear.

For the third part, since $\rho_{\mathfrak{m}}$ is absolutely irreducible as a G_F-representation, it follows from Skolem–Noether (see e.g. [Mil80, Proposition IV.1.4]) that, after conjugating ψ_1 by an element of $\mathrm{GL}_n(\widetilde{A})$, we can assume that the representation $A[G_F] \to M_n(A)$ given by the first part is equal to $\rho_{\mathfrak{m}}$. By the second part, ψ_1 also induces a *local* representation $\widetilde{A}[G_{F_v}] \to M_n(\widetilde{A})$ which clearly lifts $\rho_{\mathfrak{m}}|_{G_{F_v}}$. It also follows from the second part that after inverting p this representation becomes isomorphic to $\prod_{i=1}^r \widetilde{\rho}_{i,\mathfrak{m}}^0$. □

3.3. **Local deformation rings.** In this section we fix a place $v \in S_p$ in F and a residual local Galois representation

$$\overline{\rho}_v : G_{F_v} \to \mathrm{GL}_n(k).$$

Definition 3.3.1. *Let B be a finite E-algebra, and let $\lambda_v = (\lambda_{\tau,1} \geq \cdots \geq \lambda_{\tau,n})_{\tau \in \mathrm{Hom}(F_v,E)}$ be a dominant weight for $(\mathrm{Res}_{F_v/\mathbb{Q}_p} \mathrm{GL}_n)_E$.*

(1) *A continuous representation $\rho : G_{F_v} \to \mathrm{GL}_n(B)$ is semistable-ordinary of weight λ_v if it is conjugate to an upper triangular representation*

$$\begin{pmatrix} \chi_1 & * & \cdots & * \\ 0 & \chi_2 & \cdots & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & \cdots & \chi_n \end{pmatrix}$$

where for each $1 \leq j \leq n$ and $\sigma \in I_{F_v}$ we have

$$\chi_j(\sigma) = \prod_{\tau \in \mathrm{Hom}(F_v, E)} \tau(\mathrm{Art}_{F_v}^{-1}(\sigma))^{-\lambda_{\tau,n+1-j}-(j-1)}$$

(cf. [Ger19, Definition 3.8]*.)*

(2) *We define a p-adic Hodge type (in the sense of* [Kis08, §2.6]*) $\mathbf{v}_{\lambda_v}$ associated to λ_v as in* [Ger19, §3.3]. *This is an n-dimensional E-vector space D_E with a decreasing filtration on $D_E \otimes_{\mathbb{Q}_p} F_v$ by $E \otimes_{\mathbb{Q}_p} F_v$-submodules. More precisely $D_E \otimes_{\mathbb{Q}_p} F_v$ is isomorphic as a filtered $E \otimes_{\mathbb{Q}_p} F_v$-module to $D_{\mathrm{dR}}(\rho)$ where $\rho : G_{F_v} \to \mathrm{GL}_n(E)$ is a de Rham representation with labelled Hodge–Tate weights $(\lambda_{\tau,1} + n - 1 > \lambda_{\tau,2} + n - 2 > \cdots > \lambda_{\tau,n})_{\tau \in \mathrm{Hom}(F_v,E)}$.*

(3) *A semistable continuous representation $\rho : G_{F_v} \to \mathrm{GL}_n(B)$ has p-adic Hodge type $\mathbf{v}_{\lambda_v}$ if for each i there is an isomorphism of $B \otimes_{\mathbb{Q}_p} F_v$-modules*

$$\mathrm{gr}^i(\rho \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_{F_v}} \cong B \otimes_E (\mathrm{gr}^i D_E \otimes_{\mathbb{Q}_p} F_v).$$

We recall ([Ger19, Lemma 3.9]) that a semistable-ordinary representation of weight λ_v is semistable with p-adic Hodge type $\mathbf{v}_{\lambda_v}$.

Lemma 3.3.2. *Let B be a finite local E-algebra and suppose $\rho_B : G_{F_v} \to \mathrm{GL}_n(B)$ is semistable of Hodge type $\mathbf{v}_{\lambda_v}$ with $\rho := \rho_B \otimes_B (B/\mathfrak{m}_B)$ semistable-ordinary of weight λ_v. Then ρ_B is semistable-ordinary of weight λ_v.*

Proof. Extending E if necessary, we assume B has residue field E. Let $v_1 \leq v_2 \leq \cdots \leq v_n$ denote the valuations of the eigenvalues of geometric Frobenius acting on $\mathrm{WD}(\rho)$ (with algebraic multiplicities). It follows from the proof of [HKV20, Lemma 6.4.1] that in fact this sequence of slopes is strictly increasing and we have an equality

$$v_i = \frac{1}{e_v} \sum_\tau (\lambda_{\tau,n+1-i} + i - 1) \tag{3.3.1}$$

for each $1 \leq i \leq n$. The ordinary filtration on ρ corresponds to a filtration $\mathcal{F}^\bullet$ of $D_{\mathrm{st}}(\rho)$ by admissible filtered (ϕ, N)-modules which are free over $F_{v,0} \otimes_{\mathbb{Q}_p} E$. For $i = 1, \ldots, n$, $\mathcal{F}^i \subset D_{\mathrm{st}}(\rho)$ is the E-vector subspace spanned by the generalized eigenspaces of ϕ^{f_v} with eigenvalues of valuation $\leq v_i$. The (admissible) filtered (ϕ, N)-module $D_{\mathrm{st}}(\rho_B)$ is a successive extension of copies of $D_{\mathrm{st}}(\rho)$. We define $\mathcal{F}_B^i$ to be the B-submodule spanned by the generalized eigenspaces of ϕ^{f_v} with eigenvalues of valuation $\leq v_i$. This defines a filtration of $D_{\mathrm{st}}(\rho_B)$ by free $F_{v,0} \otimes_{\mathbb{Q}_p} B$-submodules, stable under the actions of ϕ and N. Weak admissibility of $D_{\mathrm{st}}(\rho_B)$ and the equalities (3.3.1) imply that each $\mathcal{F}_B^i$ is weakly admissible and $\mathcal{F}_B^i / \mathcal{F}_B^{i+1} = D_{\mathrm{st}}(\chi_{i,B})$ for a crystalline character $\chi_{i,B} : G_{F_v} \to E^\times$ with labelled Hodge–Tate weights $(\lambda_{\tau,n+1-i} + i - 1)_{\tau \in \mathrm{Hom}(F_v,E)}$. □

We consider the lifting ring $R^{\square}_{\overline{\rho}_v}$ with the universal lift $\rho_v^{\mathrm{univ}} : G_{F_v} \to \mathrm{GL}_n(R^{\square}_{\overline{\rho}_v})$, and recall some results of Bellovin, Geraghty, Hartl–Hellmann and Kisin:

Theorem 3.3.3. *Let* $\lambda_v = (\lambda_{\tau,1} \geq \cdots \geq \lambda_{\tau,n})_{\tau \in \mathrm{Hom}(F_v,E)}$ *be a dominant weight for* $(\mathrm{Res}_{F_v/\mathbb{Q}_p} \mathrm{GL}_n)_E$*.*

(1) *There is a unique* $\mathcal{O}$*-flat quotient* $R^{\mathrm{st},\lambda_v}_{\overline{\rho}_v}$ *of* $R^{\square}_{\overline{\rho}_v}$ *with the following property:*
  - *If* B *is a finite* E*-algebra, an* $\mathcal{O}$*-algebra map* $\zeta : R^{\square}_{\overline{\rho}_v} \to B$ *factors through* $R^{\mathrm{st},\lambda_v}_{\overline{\rho}_v}$ *if and only* $\rho_v^{\mathrm{univ}} \otimes_{R^{\mathrm{st},\lambda_v}_{\overline{\rho}_v},\zeta} B$ *is semistable with* p*-adic Hodge type* $\mathbf{v}_{\lambda_v}$*.*

(2) $R^{\mathrm{st},\lambda_v}_{\overline{\rho}_v}$ *is reduced.*

(3) *There is a unique* $\mathcal{O}$*-flat quotient* $R^{\mathrm{cris},\lambda_v}_{\overline{\rho}_v}$ *of* $R^{\square}_{\overline{\rho}_v}$ *with the following property:*
  - *If* B *is a finite* E*-algebra, an* $\mathcal{O}$*-algebra map* $\zeta : R^{\square}_{\overline{\rho}_v} \to B$ *factors through* $R^{\mathrm{cris},\lambda_v}_{\overline{\rho}_v}$ *if and only* $\rho_v^{\mathrm{univ}} \otimes_{R^{\mathrm{st},\lambda_v}_{\overline{\rho}_v},\zeta} B$ *is crystalline with* p*-adic Hodge type* $\mathbf{v}_{\lambda_v}$*.*

(4) $R^{\mathrm{cris},\lambda_v}_{\overline{\rho}_v}[\frac{1}{p}]$ *is regular (in particular,* $R^{\mathrm{cris},\lambda_v}_{\overline{\rho}_v}$ *is reduced).*

(5) *There is a unique* $\mathcal{O}$*-flat quotient* $R^{\triangle,\lambda_v}_{\overline{\rho}_v}$ *of* $R^{\square}_{\overline{\rho}_v}$ *with the following property:*
  - *If* B *is a finite* E*-algebra, an* $\mathcal{O}$*-algebra map* $\zeta : R^{\square}_{\overline{\rho}_v} \to B$ *factors through* $R^{\triangle,\lambda_v}_{\overline{\rho}_v}$ *if and only* $\rho_v^{\mathrm{univ}} \otimes_{R^{\mathrm{st},\lambda_v}_{\overline{\rho}_v},\zeta} B$ *is semistable-ordinary of weight* λ_v*.*

(6) $\mathrm{Spec}(R^{\triangle,\lambda_v}_{\overline{\rho}_v}[\frac{1}{p}])$ *is an open and closed subspace of* $\mathrm{Spec}(R^{\mathrm{st},\lambda_v}_{\overline{\rho}_v})[\frac{1}{p}]$*. In particular,* $R^{\triangle,\lambda_v}_{\overline{\rho}_v}$ *is reduced.*

Proof. The first and third parts are [Kis08, Theorem 2.7.6, Corollary 2.7.7]. The fourth part is a consequence of [Kis08, Theorem 3.3.8]. The second part is a consequence of (a very special case of) [BG19, Theorem 3.3.3] (see also [Bel16, HH20]). The fifth part is [Ger19, Lemma 3.10]. The sixth part follows from Lemma 3.3.2. □

Remark 3.3.4. We have emphasised the reducedness of our local deformation rings because in some parts of the literature the maximal reduced quotients of Kisin's deformation rings are introduced, characterised by their morphisms to finite field extensions of E (e.g. in [BLGGT14]). It won't make any difference to us in practice, because in the end we will be considering maps from local deformation rings to reduced finite flat $\mathcal{O}$-algebras $\widetilde{A}$ as in §3.2.

3.3.5. *Fixed determinant deformation rings.* It is often useful to consider deformation rings with fixed determinant. Let λ_v be a dominant weight for $(\mathrm{Res}_{F_v/\mathbb{Q}_p} \mathrm{GL}_n)_E$ and suppose $\psi : G_{F_v} \to \mathcal{O}^\times$ is a crystalline character with τ-labelled Hodge–Tate weights $\sum_{i=1}^n \lambda_{\tau,i} + (n-i)$ for each $\tau : F \hookrightarrow E$. Suppose moreover that $\det \overline{\rho}_v$ coincides with $\overline{\psi} : G_{F_v} \to k^\times$, the reduction of ψ. Then we have a quotient $R^{\square,\psi}_{\overline{\rho}_v}$ of $R^{\square}_{\overline{\rho}_v}$, classifying liftings with determinant ψ (composed with the structure map from $\mathcal{O}$ to the test $\mathcal{O}$-algebra).

We define quotients $R^{\mathrm{cris},\lambda_v,\psi}_{\overline{\rho}_v} = R^{\mathrm{cris},\lambda_v}_{\overline{\rho}_v} \otimes_{R^{\square}_{\overline{\rho}_v}} R^{\square,\psi}_{\overline{\rho}_v}$ and $R^{\triangle,\lambda_v,\psi}_{\overline{\rho}_v} = R^{\mathrm{cris},\lambda_v}_{\overline{\rho}_v} \otimes_{R^{\square}_{\overline{\rho}_v}} R^{\square,\psi}_{\overline{\rho}_v}$ of our local deformation rings.

Lemma 3.3.6. *Suppose* $p \nmid n$*. Let* λ_v *and* ψ *be as above. Let* $R = R^{\mathrm{cris},\lambda_v}_{\overline{\rho}_v}$ *or* $R^{\triangle,\lambda_v}_{\overline{\rho}_v}$*, and* $R^\psi = R^{\mathrm{cris},\lambda_v,\psi}_{\overline{\rho}_v}$ *or* $R^{\triangle,\lambda_v,\psi}_{\overline{\rho}_v}$ *respectively. Then there is a section to the* *quotient map* $R \to R^\psi$ *extending to an isomorphism* $R^\psi[\![X]\!] \cong R$*. In particular,* R^ψ *is* $\mathcal{O}$*-flat and reduced and can be characterised using the properties of Theorem 3.3.3 parts* (3) *or* (5) *respectively for maps from* $R^{\square,\psi}_{\bar\rho_v}$ *to finite* E*-algebras.*

Proof. This is essentially [EG14, Lemma 4.3.1]. If we consider the universal lifting $\rho^{\mathrm{univ}} : G_{F_v} \to \mathrm{GL}_n(R)$, then the composition of the character $\psi^{-1}\det\rho^{\mathrm{univ}}$ with any map $R \to B$ to a finite E-algebra is crystalline with all labelled Hodge–Tate weights equal to zero. In other words these compositions are unramified. Since R is $\mathcal{O}$-flat and Noetherian, it follows that $\psi^{-1}\det\rho^{\mathrm{univ}}$ is itself unramified. This character is also residually trivial. Since $p \nmid n$, Hensel's lemma implies that there is an unramified character $\alpha : G_{F_v} \to 1 + \mathfrak{m}_R$ with $\alpha^n = \psi^{-1}\det\rho^{\mathrm{univ}}$. The representation $\alpha^{-1} \otimes \rho^{\mathrm{univ}}$ has determinant ψ and defines a section of the quotient map $R \to R^\psi$. This extends to a map $R^\psi[\![X]\!] \to R$ sending X to $\alpha(\mathrm{Frob}_v) - 1$. We can identify $R^\psi[\![X]\!]$ with a quotient of $R^{\square}_{\bar\rho_v}$ (in fact, of R) using the lifting $\rho^{\mathrm{univ},\psi} \otimes \mathrm{ur}(1+X)$, the twist of the universal lifting to R^ψ with the unramified character taking Frob_v to $1+X$. Composing this lifting with our map $R^\psi[\![X]\!] \to R$ gives ρ^{univ}, and it follows that our map is an isomorphism $R^\psi[\![X]\!] \cong R$. □

4. Local-global compatibility in the crystalline case

4.1. **A computation of boundary cohomology.** The goal of this section (Corollary 4.1.9) is to describe, in terms of the cohomology of the G-locally symmetric spaces, a particular direct summand of the completed cohomology for the boundary of the $\widetilde{G}$-locally symmetric spaces. It will be one of the key ingredients allowing us to describe Hecke algebras acting on cohomology for G in an arbitrary fixed cohomological degree in terms of the middle degree cohomology for $\widetilde{G}$.

4.1.1. *Notation.* Let $\bar{T} \supseteq \bar{S}_p$ be a finite set of finite places of F^+ with preimage the finite set of finite places T of F. Let $\bar{S} \subseteq \bar{S}_p$ be a set of primes of F^+ dividing p with preimage $S \subseteq S_p$. For $\mathrm{G} = \widetilde{G}, P, U$, or G, we set $\mathrm{G}^0_{\bar{S}} = \prod_{\bar{v}\in\bar{S}} \mathrm{G}(\mathcal{O}_{F^+_{\bar{v}}})$.

Given $\tilde\lambda \in (\mathbb{Z}^{2n}_+)^{\mathrm{Hom}(F^+,E)}$ a dominant weight for $\widetilde{G}$, we define

$$\mathcal{V}_{\tilde\lambda_{\bar{S}}} := \bigotimes_{\bar{v}\in\bar{S}} \bigotimes_{\tau\in\mathrm{Hom}(F^+_{\bar{v}},E),\mathcal{O}} \mathcal{V}_{\tilde\lambda_\tau}.$$

We also define the object $\mathcal{V}_U(\tilde\lambda_{\bar{S}}, m)$ in $D^b(\mathrm{Sh}_{G^T\times K_T}(\overline{\mathfrak{X}}_G, \mathcal{O}/\varpi^m))$ corresponding to the object $R\Gamma(U^0_{\bar{S}}, \mathcal{V}_{\tilde\lambda_{\bar{S}}}/\varpi^m)$ of $D^b_{\mathrm{sm}}(K_S, \mathcal{O}/\varpi^m)$ (after inflation to $D^b_{\mathrm{sm}}(G^T \times K_T, \mathcal{O}/\varpi^m)$). The boundedness is a consequence of the finite cohomological dimension of the torsion-free compact p-adic analytic group $U^0_{\bar{S}}$ cf. [Ser65].

We let $\mathcal{V}^j_U(\tilde\lambda_{\bar{S}}, m)$ denote its cohomology sheaves; note that these are non-zero precisely when j ranges from 0 to $n^2\sum_{\bar{v}\in\bar{S}}[F^+_{\bar{v}} : \mathbb{Q}_p]$, by the Künneth formula for group cohomology and by Lemma 2.3.17.

If $K \subset G(\mathbb{A}_{F^+,f})$ is a good subgroup, each $\mathcal{V}_U(\tilde\lambda_{\bar{S}}, m)$ descends to an object in $D^b(\mathrm{Sh}(X_K, \mathcal{O}/\varpi^m))$ with locally constant cohomology sheaves. Taking a homotopy limit gives $\mathcal{V}_U(\tilde\lambda_{\bar{S}}) \in D^b(\mathrm{Sh}(X_K, \mathcal{O}))$ again with locally constant cohomology sheaves (since X_K is locally contractible we can find opens over which the $\mathcal{V}^j_U(\tilde\lambda_{\bar{S}}, m)$ are constant for all $m \geq 1$). By passing to a limit over m, the cohomology $R\Gamma(X_K, \mathcal{V}_U(\tilde\lambda_{\bar{S}}) \otimes \mathcal{V}_{\lambda_{\bar{S}_p\setminus\bar{S}}})$ comes with whatever actions (e.g. of $\mathbb{T}^T$ or a monoid acting at $\bar{S}_p\setminus\bar{S}$) we have on $R\Gamma(X_K, \mathcal{V}_U(\tilde\lambda_{\bar{S}}, m) \otimes \mathcal{V}_{\lambda_{\bar{S}_p\setminus\bar{S}}}/\varpi^m)$.

We will often have an action of a commutative $\mathcal{O}$-algebra $\mathbb{T}$ (one of our Hecke algebras) on an $\mathcal{O}$-module or, more generally, an object in C in $D(\mathcal{O})$. We will then write $\mathbb{T}(C)$ for the image of $\mathbb{T}$ in $\mathrm{End}_{D(\mathcal{O})}(C)$.

Given $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F,E)}$ a dominant weight for G, we define

$$\mathcal{V}_{\lambda_S} := \bigotimes_{v\in S} \bigotimes_{\tau\in\mathrm{Hom}(F_v,E),\mathcal{O}} \mathcal{V}_{\lambda_\tau}.$$

4.1.2. *Boundary cohomology.*

Theorem 4.1.3. *Assume that $\widetilde{K} \subset \widetilde{G}(\mathbb{A}_{F^+,f})$ is a good subgroup that is decomposed with respect to P, and with the property that, for each $\bar{v} \in \bar{S}_p$, $\widetilde{K}_{U,\bar{v}} = U^0_{\bar{v}}$. Let $\mathfrak{m} \subset \mathbb{T}^T$ be a non-Eisenstein maximal ideal and let $\widetilde{\mathfrak{m}} := \mathcal{S}^*(\mathfrak{m}) \subset \widetilde{\mathbb{T}}^T$.*

Choose a partition

$$\bar{S}_p = \bar{S}_1 \sqcup \bar{S}_2$$

of the set $\bar{S}_p$ of primes of F^+ lying above p. Let $\tilde{\lambda}$ be a dominant weight for $\widetilde{G}$ satisfying $\tilde{\lambda}_{\bar{v}} = 0$ for all $\bar{v} \in \bar{S}_2$. Then

$$\mathcal{S}^* \circ \mathrm{Ind}_{P_{\bar{S}_2}}^{\widetilde{G}_{\bar{S}_2}} R\Gamma\left(K^{\bar{S}_2}, R\Gamma\left(\overline{\mathfrak{X}}_G, \mathcal{V}_U(\tilde{\lambda}_{\bar{S}_1}, m)\right)\right)_{\mathfrak{m}}$$

is a $\widetilde{\mathbb{T}}^T$-equivariant direct summand of

$$R\Gamma\left(\widetilde{K}^{\bar{S}_2}, R\Gamma\left(\partial\mathfrak{X}_{\widetilde{G}}, \mathcal{V}_{\tilde{\lambda}}/\varpi^m\right)\right)_{\widetilde{\mathfrak{m}}}$$

in $D^+_{\mathrm{sm}}(\widetilde{G}_{\bar{S}_2}, \mathcal{O}/\varpi^m)$.

Proof. By combining Proposition 4.1.4 and Lemma 4.1.5, we obtain a $\widetilde{\mathbb{T}}^T$-equivariant isomorphism in $D^+_{\mathrm{sm}}(\widetilde{G}_{\bar{S}_2}, \mathcal{O}/\varpi^m)$

(4.1.1)
$$R\Gamma\left(\widetilde{K}^{\bar{S}_2}, R\Gamma\left(\partial\mathfrak{X}_{\widetilde{G}}, \mathcal{V}_{\tilde{\lambda}}/\varpi^m\right)\right)_{\widetilde{\mathfrak{m}}} \xrightarrow{\sim} R\Gamma\left(\widetilde{K}^{\bar{S}_2}, \mathrm{Ind}_{P^{\bar{S}_1}\times P^0_{\bar{S}_1}}^{\widetilde{G}^{\bar{S}_1}\times\widetilde{G}^0_{\bar{S}_1}} R\Gamma\left(\overline{\mathfrak{X}}_P, \mathcal{V}_{\tilde{\lambda}}/\varpi^m\right)\right)_{\widetilde{\mathfrak{m}}}.$$

Using the analogue for smooth representations of [NT16, Corollary 2.6], we can rewrite the RHS of (4.1.1) as r_P^* applied to

$$\bigoplus_g \mathrm{Ind}_{P_{\bar{S}_2}}^{\widetilde{G}_{\bar{S}_2}} R\Gamma\left(g\widetilde{K}^{\bar{S}_2}g^{-1} \cap P(\mathbb{A}_{F^+,f}), R\Gamma(\overline{\mathfrak{X}}_P, \mathcal{V}_{\tilde{\lambda}}/\varpi^m)\right)_{r_G^*\mathfrak{m}}, \tag{4.1.2}$$

where g runs over the double cosets

$$(P_{T\setminus\bar{S}_p} \times P^0_{\bar{S}_1})\backslash(\widetilde{G}_{T\setminus\bar{S}_p} \times \widetilde{G}^0_{\bar{S}_1})/(\widetilde{K}_{T\setminus\bar{S}_p} \times \widetilde{K}_{\bar{S}_1}),$$

and we view each $R\Gamma\left(g\widetilde{K}^{\bar{S}_2}g^{-1} \cap P(\mathbb{A}_{F^+,f}), R\Gamma(\overline{\mathfrak{X}}_P, \mathcal{V}_{\tilde{\lambda}}/\varpi^m)\right)$ as a $\mathbb{T}_P^T$-module in $D^+_{\mathrm{sm}}(P_{\bar{S}_2}, \mathcal{O}/\varpi^m)$. We restrict to the direct summand corresponding to $g = 1$ in (4.1.2). We now conclude by Proposition 4.1.8. □

Proposition 4.1.4.

(1) *We have $\widetilde{G}(\mathbb{A}_{F^+,f})$-equivariant closed immersion*

$$(\overline{\mathfrak{X}}_P \times \widetilde{G}(\mathbb{A}_{F^+,f}))/P(\mathbb{A}_{F^+,f}) \hookrightarrow \partial\mathfrak{X}_{\widetilde{G}}{}^{18},$$

[18]Here, we consider the groups $\widetilde{G}(\mathbb{A}_{F^+,f})$ and $P(\mathbb{A}_{F^+,f})$ as endowed with their natural profinite topologies.

whose complement is a disjoint union of locally closed subspaces of the form $(\mathfrak{X}_Q \times \widetilde{G}(\mathbb{A}_{F^+,f}))/Q(\mathbb{A}_{F^+,f})$ *with* $Q \subset \widetilde{G}$ *a standard* F^+*-rational parabolic such that* $Q \not\subseteq P$.

(2) *With all assumptions as in Theorem 4.1.3, the natural pullback map induces a* $\widetilde{\mathbb{T}}^T$*-equivariant isomorphism in* $D^+_{\mathrm{sm}}(\widetilde{G}_{\bar{S}_2}, \mathcal{O}/\varpi^m)$

$$R\Gamma\left(\widetilde{K}^{\bar{S}_2}, R\Gamma\left(\partial\mathfrak{X}_{\widetilde{G}}, \mathcal{V}_{\tilde{\lambda}}/\varpi^m\right)\right)_{\widetilde{\mathfrak{m}}} \xrightarrow{\sim} R\Gamma\left(\widetilde{K}^{\bar{S}_2}, R\Gamma\left((\overline{\mathfrak{X}}_P \times \widetilde{G}(\mathbb{A}_{F^+,f}))/P(\mathbb{A}_{F^+,f}), \mathcal{V}_{\tilde{\lambda}}/\varpi^m\right)\right)_{\widetilde{\mathfrak{m}}}.$$

Proof. The first part follows from the description of the boundary of the Borel–Serre compactification in [NT16, §3.1.2], see especially Lemma 3.10 of *loc. cit.* This reference uses the 'discrete' versions $\mathfrak{X}_Q^{\mathrm{dis}}$ of the spaces $\mathfrak{X}_Q$, but we can compare the two situations after taking quotients by compact open subgroups, and the space $(\mathfrak{X}_Q \times \widetilde{G}(\mathbb{A}_{F^+,f}))/Q(\mathbb{A}_{F^+,f})$ is equal to the inverse limit of its quotients by compact open subgroups $\widetilde{K}$ of $\widetilde{G}(\mathbb{A}_{F^+,f})$ (we can use the fact that $\widetilde{G}(\mathbb{A}_{F^+,f}) \to Q(\mathbb{A}_{F^+,f})\backslash\widetilde{G}(\mathbb{A}_{F^+,f})$ is a locally trivial $Q(\mathbb{A}_{F^+,f})$-torsor, as in the proof of Proposition 2.3.3). These quotients can in turn be computed as quotients of $(\mathfrak{X}_Q^{\mathrm{dis}} \times \widetilde{G}(\mathbb{A}_{F^+,f})^{\mathrm{dis}})/Q(\mathbb{A}_{F^+,f})^{\mathrm{dis}}$, because $Q(\mathbb{A}_{F^+,f})\backslash\widetilde{G}(\mathbb{A}_{F^+,f})/\widetilde{K}$ is a finite set for each compact open $\widetilde{K}$.

For the second part, we first note that we can check whether a map is an isomorphism on the level of cohomology groups. For a standard F^+-rational proper parabolic subgroup $Q \subset \widetilde{G}$ and a good subgroup $\widetilde{K} \subset \widetilde{G}(\mathbb{A}_{F^+,f})$, set

$$\widetilde{X}^Q_{\widetilde{K}} := (\mathfrak{X}_Q \times \widetilde{G}(\mathbb{A}_{F^+,f}))/Q(\mathbb{A}_{F^+,f})\widetilde{K},$$

which is a disjoint union of finitely many locally symmetric spaces for Q. Using excision and Lemma 2.1.7, we see that it is enough to show that, for $Q \subset \widetilde{G}$ a standard F^+-rational parabolic with $Q \not\subseteq P$, we have

$$H^i_c(\widetilde{X}^Q_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}}/\varpi^m)_{\widetilde{\mathfrak{m}}} = 0,$$

for any $i \in \mathbb{Z}_{\geq 0}$. This is standard by now, see for example the proof of [ACC+18, Theorem 3.4.2]. □

Lemma 4.1.5. *Keep the assumption on* $\tilde{\lambda}$ *from the statement of Theorem 4.1.3. There is a natural isomorphism in* $D^+_{\mathrm{sm}}(\widetilde{G}^{\bar{S}_1} \times \widetilde{G}^0_{\bar{S}_1}, \mathcal{O}/\varpi^m)$

$$R\Gamma\left((\overline{\mathfrak{X}}_P \times \widetilde{G}(\mathbb{A}_{F^+,f}))/P(\mathbb{A}_{F^+,f}), \mathcal{V}_{\tilde{\lambda}}/\varpi^m\right) \xrightarrow{\sim} \mathrm{Ind}_{P^{\bar{S}_1} \times P^0_{\bar{S}_1}}^{\widetilde{G}^{\bar{S}_1} \times \widetilde{G}^0_{\bar{S}_1}} R\Gamma\left(\overline{\mathfrak{X}}_P, \mathcal{V}_{\tilde{\lambda}}/\varpi^m\right).$$

Proof. The case of constant coefficients $\mathcal{O}/\varpi^m$ follows from Lemma 2.3.14 combined with the Iwasawa decomposition at primes in $\bar{S}_1$. For coefficients in a local system $\mathcal{V}_{\tilde{\lambda}}/\varpi^m$, where by assumption the action is non-trivial only at the primes in $\bar{S}_1$, we use the projection formula in Lemma 2.1.8 and the tensor identity in Lemma 2.3.15 (at primes in $\bar{S}_1$) to reduce to the case of constant coefficients. □

Lemma 4.1.6. *There is a natural isomorphism*

$$\mathrm{Inf}_{G(\mathbb{A}_{F^+,f})}^{P(\mathbb{A}_{F^+,f})} R\Gamma\left(\overline{\mathfrak{X}}_G, \mathcal{O}/\varpi^m\right) \xrightarrow{\sim} R\Gamma\left(\overline{\mathfrak{X}}_P, \mathcal{O}/\varpi^m\right)$$

in $D^+_{\mathrm{sm}}(P(\mathbb{A}_{F^+,f}), \mathcal{O}/\varpi^m)$.

Proof. The map is given by pullback along the $P(\mathbb{A}_{F^+,f})$-equivariant projection $\overline{\mathfrak{X}}_P \twoheadrightarrow \overline{\mathfrak{X}}_G$. To show that the map is an isomorphism, it is enough to check that it induces an isomorphism after applying the forgetful functor to $D^+(\mathcal{O}/\varpi^m)$, and then it is enough to consider it on the level of cohomology groups. By Lemma 2.1.7, this reduces to establishing the isomorphism

$$\varinjlim_{K} H^i(X_K, \mathcal{O}/\varpi^m) \xrightarrow{\sim} \varinjlim_{K_P} H^i(X^P_{K_P}, \mathcal{O}/\varpi^m),$$

where $K_P \subset P(\mathbb{A}_{F^+,f})$ runs over compact open subgroups of the form $K \ltimes K_U$ with $K \subset G(\mathbb{A}_{F^+,f})$ and $K_U \subset U(\mathbb{A}_{F^+,f})$ compact open subgroups. We use the Leray–Serre spectral sequence for the fibration $X^U_{K_U} \to X^P_{K_P} \to X_K$. It is enough to check that

$$\varinjlim_{K_U} H^j(X^U_{K_U}, \mathcal{O}/\varpi^m) = \varinjlim_{\Gamma_U := U(F^+)\cap K_U} H^j(\Gamma_U \backslash U(F^+ \otimes_{\mathbb{Q}} \mathbb{R}), \mathcal{O}/\varpi^m) = \begin{cases} \mathcal{O}/\varpi^m & \text{for } j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The first equality follows by strong approximation for U, the second by direct computation. We note that this computation should be thought of in the category of $\mathcal{O}/\varpi^m[\![K]\!]$-modules, with a trivial action of K on the RHS. □

Lemma 4.1.7. *With the notation as in the proof of Theorem 4.1.3, there exists a natural isomorphism*

$$\mathrm{Inf}^{P^{T\setminus \bar{S}_2}}_{G^{T\setminus \bar{S}_2}} R\Gamma\left(K_{T\setminus \bar{S}_2}, R\Gamma\left(\overline{\mathfrak{X}}_G, \mathcal{V}_U(\tilde{\lambda}_{\bar{S}_1}, m)\right)\right) \xrightarrow{\sim} R\Gamma\left(K_{P,T\setminus \bar{S}_2}, R\Gamma(\overline{\mathfrak{X}}_P, \mathcal{V}_{\tilde{\lambda}}/\varpi^m)\right)$$

in $D^+_{\mathrm{sm}}(P^{T\setminus \bar{S}_2}, \mathcal{O}/\varpi^m)$.

Proof. By Lemma 2.1.9, we have isomorphisms

$$R\Gamma\left(\overline{\mathfrak{X}}_G, \mathcal{V}_U(\tilde{\lambda}_{\bar{S}_1}, m)\right) \xrightarrow{\sim} R\Gamma(\overline{\mathfrak{X}}_G, \mathcal{O}/\varpi^m) \otimes^{\mathbb{L}}_{\mathcal{O}/\varpi^m} R\Gamma(K_{U,\bar{S}_1}, \mathcal{V}_{\tilde{\lambda}}/\varpi^m)$$

$$\xrightarrow{\sim} R\Gamma(\overline{\mathfrak{X}}_G, \mathcal{O}/\varpi^m) \otimes^{\mathbb{L}}_{\mathcal{O}/\varpi^m} R\Gamma(K_{U,T\setminus \bar{S}_2}, \mathcal{V}_{\tilde{\lambda}}/\varpi^m)$$

in $D^+_{\mathrm{sm}}(G^{T\setminus \bar{S}_2} \times K_{T\setminus \bar{S}_2}, \mathcal{O}/\varpi^m)$. We have a natural isomorphism

$$\mathrm{Inf}^{P^{T\setminus \bar{S}_2}}_{G^{T\setminus \bar{S}_2}} \left(R\Gamma(\overline{\mathfrak{X}}_G, \mathcal{O}/\varpi^m) \otimes^{\mathbb{L}}_{\mathcal{O}/\varpi^m} R\Gamma(K_{U,T\setminus \bar{S}_2}, \mathcal{V}_{\tilde{\lambda}}/\varpi^m)\right) \xrightarrow{\sim}$$

$$R\Gamma\left(K_{U,T\setminus \bar{S}_2}, \mathrm{Inf}^{P(\mathbb{A}_{F^+,f})}_{G(\mathbb{A}_{F^+,f})} R\Gamma(\overline{\mathfrak{X}}_G, \mathcal{O}/\varpi^m) \otimes^{\mathbb{L}}_{\mathcal{O}/\varpi^m} \mathcal{V}_{\tilde{\lambda}}/\varpi^m\right)$$

in $D^+_{\mathrm{sm}}(P^{T\setminus \bar{S}_2} \times K_{T\setminus \bar{S}_2}, \mathcal{O}/\varpi^m)$ because an arbitrary colimit of injective representations of $K_{U,T\setminus \bar{S}_2}$ is injective. We conclude by Lemma 4.1.6, by Hochschild–Serre applied to $K_{P,T\setminus \bar{S}_2} = K_{T\setminus \bar{S}_2} \ltimes K_{U,T\setminus \bar{S}_2}$, and by Lemma 2.1.8 applied to $\overline{\mathfrak{X}}_P$. □

Proposition 4.1.8. *With the notation as in the proof of Theorem 4.1.3, we have a* $\mathbb{T}^T_P$*-equivariant isomorphism*

$$R\Gamma\left(K_P^{\bar{S}_2}, R\Gamma(\overline{\mathfrak{X}}_P, \mathcal{V}_{\tilde{\lambda}}/\varpi^m)\right) \xrightarrow{\sim} r_G^* \circ \mathrm{Inf}^{P_{\bar{S}_2}}_{G_{\bar{S}_2}} R\Gamma\left(K^{\bar{S}_2}, R\Gamma\left(\overline{\mathfrak{X}}_G, \mathcal{V}_U(\tilde{\lambda}_{\bar{S}_1}, m)\right)\right)$$

in $D^+_{\mathrm{sm}}(P_{\bar{S}_2}, \mathcal{O}/\varpi^m)$.

Proof. We separate the set of finite places of F^+ away from $\bar{S}_2$ into the union of the set of finite places away from T and the set of finite places $T \setminus \bar{S}_2$, which contains

$\bar{S}_1$. By the analogue for smooth representations of [NT16, Corollary 2.8], there is a $\mathbb{T}_P^T$-equivariant natural transformation in $D^+_{\mathrm{sm}}(P_{\bar{S}_2}, \mathcal{O}/\varpi^m)$ between

$$r_G^* R\Gamma\left(K^T, \mathrm{Inf}_{G_{\bar{S}_2}}^{P_{\bar{S}_2}} R\Gamma(K_{T\setminus\bar{S}_2}, R\Gamma\left(\overline{\mathfrak{X}}_G, \mathcal{V}_U(\tilde{\lambda}_{\bar{S}_1}, m)\right)\right)$$

and

$$R\Gamma\left(K_P^T, \mathrm{Inf}_{G^{T\setminus\bar{S}_2}}^{P^{T\setminus\bar{S}_2}} R\Gamma(K_{T\setminus\bar{S}_2}, R\Gamma\left(\overline{\mathfrak{X}}_G, \mathcal{V}_U(\tilde{\lambda}_{\bar{S}_1}, m)\right)\right).$$

If we can show that this natural transformation is an isomorphism, we can conclude by Lemma 4.1.7. To prove that the natural transformation is an isomorphism, we can forget the $\mathbb{T}_P^T$-action and work in $D^+(\mathcal{O}/\varpi^m)$. There, it reduces to the statement that taking K_U^T-invariants is an exact functor, which holds true because K_U^T is a profinite abelian group with trivial p-part. $\square$

The following is a result in the style of [ACC+18, Theorems 4.2.1 and 5.4.1]. This is the only part of this subsection that we will use in what follows.

Corollary 4.1.9. *Assume that $\widetilde{K} \subset \widetilde{G}(\mathbb{A}_{F^+,f})$ is a good subgroup that is decomposed with respect to P, and with the property that, for each $\bar{v} \in \bar{S}_p$, $\widetilde{K}_{U,\bar{v}} = U^0_{\bar{v}}$. Let $\mathfrak{m} \subset \mathbb{T}^T$ be a non-Eisenstein maximal ideal and let $\widetilde{\mathfrak{m}} := \mathcal{S}^*(\mathfrak{m}) \subset \widetilde{\mathbb{T}}^T$.*

Choose a partition

$$\bar{S}_p = \bar{S}_1 \sqcup \bar{S}_2 \sqcup \bar{S}_3$$

of the set $\bar{S}_p$ of primes of F^+ lying above p. Let $\tilde{\lambda}$ and λ be dominant weights for $\widetilde{G}$ and G, respectively. Assume that the following conditions are satisfied:

1. *For each $\tau : F^+ \hookrightarrow E$ inducing a place $\bar{v} \in \bar{S}_1$, $\tilde{\lambda}_\tau = (-\lambda_{\tilde{\tau}c}, \lambda_{\tilde{\tau}})$ (identification as in (2.1.4));*
2. *For each $\tau : F^+ \hookrightarrow E$ inducing a place $\bar{v} \in \bar{S}_2 \sqcup \bar{S}_3$, $\tilde{\lambda}_\tau = 0$.*

Then

$$\mathcal{S}^* \circ \mathrm{Ind}_{P_{\bar{S}_3}}^{\widetilde{G}_{\bar{S}_3}} R\Gamma\left(K^{\bar{S}_3}, R\Gamma\left(\overline{\mathfrak{X}}_G, \mathcal{V}_{\lambda_{\bar{S}_1}}/\varpi^m \otimes \mathcal{V}_U(\tilde{\lambda}_{\bar{S}_2}, m)\right)\right)_{\mathfrak{m}}$$

is a $\widetilde{\mathbb{T}}^T$-equivariant direct summand of

$$R\Gamma\left(\widetilde{K}^{\bar{S}_3}, R\Gamma\left(\partial\mathfrak{X}_{\widetilde{G}}, \mathcal{V}_{\tilde{\lambda}}/\varpi^m\right)\right)_{\widetilde{\mathfrak{m}}}$$

in $D^+_{\mathrm{sm}}(\widetilde{G}_{\bar{S}_3}, \mathcal{O}/\varpi^m)$.

Proof. This follows from Theorem 4.1.3 applied with $\bar{S}'_1 := \bar{S}_1 \sqcup \bar{S}_2$ and $\bar{S}'_2 := \bar{S}_3$, as long as we can show that $\mathcal{V}_{\lambda_{\bar{S}_1}}/\varpi^m$ is a $G^T \times K_T$-equivariant direct summand of $\mathcal{V}_U(\tilde{\lambda}_{\bar{S}_1}, m)$. This follows from [NT16, Corollary 2.11], as in the proof of [ACC+18, Theorem 4.2.1] (and the proof of [ACC+18, Theorem 2.4.4]). $\square$

4.2. **The integral case.** In this section we prove our main results on local–global compatibility: Proposition 4.2.13 and Theorem 4.2.15.

4.2.1. *Degree shifting.* Let $\bar{S} \subset \bar{S}_p$. Recall that, for each $\bar{v} \in \bar{S}_p$, we have chosen a place $\tilde{v} \mid \bar{v}$ of F, with complex conjugate $\tilde{v}^c$. The isomorphism $\iota_{\tilde{v}} : \widetilde{G}(F^+_{\bar{v}}) \cong \mathrm{GL}_{2n}(F_{\tilde{v}})$ identifies the Levi subgroup $G(F^+_{\bar{v}}) = \mathrm{GL}_n(F_{\tilde{v}}) \times \mathrm{GL}_n(F_{\tilde{v}^c})$ with block diagonal matrices in $\mathrm{GL}_{2n}(F_{\tilde{v}})$ via $(A_{\tilde{v}}, A_{\tilde{v}^c}) \mapsto \begin{pmatrix} (\Psi_n\, {}^tA_{\tilde{v}^c}^{-1} \Psi_n)^c & 0 \\ 0 & A_{\tilde{v}} \end{pmatrix}$.

We have standard parabolics $Q_{\bar{v}} \subset P_{\bar{v}}$ for each $\bar{v} \in \bar{S}$ and we set $K_{\bar{v}} := \mathcal{Q}_{\bar{v}} \cap G_{\bar{v}}(F^+_{\bar{v}})$, a parahoric subgroup of $G_{\bar{v}}(F^+_{\bar{v}})$. The conjugate parahoric $(K_{\bar{v}})^{w_0^P}$ can also be viewed as the intersection $\overline{\mathcal{Q}}_{\bar{v}}^{w_0^P} \cap G_{\bar{v}}(F^+_{\bar{v}})$, where $\overline{Q}_{\bar{v}}^{w_0^P}$ is the standard parabolic with Levi subgroup $Q_{\bar{v}}^{w_0^P} \cap G_{\bar{v}}(F^+_{\bar{v}})$.

We introduce the following notation on the level of abstract Hecke algebras:

$$\mathbb{T}^{\mathcal{Q}_{\bar{S}},\bar{S}-\mathrm{ord}} := \mathbb{T}^T \otimes_{\mathbb{Z}} \left(\bigotimes_{\bar{v}\in\bar{S}} \mathcal{H}(\Delta_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}, K_{\bar{v}})\right), \mathbb{T}^{\mathcal{Q}_{\bar{S}},\bar{S}-\mathrm{ord}}_{w_0^P} := \mathbb{T}^T \otimes_{\mathbb{Z}} \left(\bigotimes_{\bar{v}\in\bar{S}} \mathcal{H}((\Delta_{\bar{v}}^{\mathcal{Q}_{\bar{v}}})^{w_0^P}, K_{\bar{v}}^{w_0^P})\right)$$

and $\widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}},\bar{S}-\mathrm{ord}} := \widetilde{\mathbb{T}}^T \otimes_{\mathbb{Z}} \left(\bigotimes_{\bar{v}\in\bar{S}} \mathcal{H}(\widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}, \mathcal{Q}_{\bar{v}})[\widetilde{U}_{\tilde{v},n}^{-1}]\right)$.

The unnormalised Satake transform $\mathcal{S} : \widetilde{\mathbb{T}} \to \mathbb{T}$ extends to a morphism denoted by

$$\mathcal{S}^{w_0^P} : \widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}},\bar{S}-\mathrm{ord}} \to \mathbb{T}^{\mathcal{Q}_{\bar{S}},\bar{S}-\mathrm{ord}}_{w_0^P},$$

given by $[\mathcal{Q}_{\bar{v}}\nu(\varpi_{\bar{v}})\mathcal{Q}_{\bar{v}}] \mapsto [K_{\bar{v}}^{w_0^P}\nu(\varpi_{\bar{v}})^{w_0^P}K_{\bar{v}}^{w_0^P}]$. In particular, $\widetilde{U}_{\tilde{v},n}$ maps to $U_{\tilde{v}} = [K_{\bar{v}}^{w_0^P}\operatorname{diag}(\varpi_{\tilde{v}}, \cdots, \varpi_{\tilde{v}})K_{\bar{v}}^{w_0^P}]$ and $\widetilde{U}_{\tilde{v},2n}$ maps to

$$U_{\tilde{v}}U_{\tilde{v}^c}^{-1} = [K_{\bar{v}}^{w_0^P}\left(\operatorname{diag}(\varpi_{\tilde{v}}, \cdots, \varpi_{\tilde{v}}), (\operatorname{diag}(\varpi_{\tilde{v}}, \cdots, \varpi_{\tilde{v}})^{-1})^c\right)K_{\bar{v}}^{w_0^P}].$$

To make use of Poincaré duality, we will need to introduce Hecke algebras twisted by the duality involutions $\iota, \tilde{\iota}$ which are defined by

$$\iota[K^{w_0^P}gK^{w_0^P}] = [K^{w_0^P}g^{-1}K^{w_0^P}] \text{ and } \tilde{\iota}[\widetilde{K}g\widetilde{K}] = [\widetilde{K}g^{-1}\widetilde{K}]$$

(see [ACC+18, §2.2.19]). They give isomorphisms

$$\iota : \mathbb{T}^{\mathcal{Q}_{\bar{S}},\bar{S}-\mathrm{ord}}_{w_0^P} \xrightarrow{\sim} \mathbb{T}^{\mathcal{Q}_{\bar{S}},\bar{S}-\mathrm{ord},\iota}_{w_0^P} := \mathbb{T}^T \otimes_{\mathbb{Z}} \left(\bigotimes_{\bar{v}\in\bar{S}} \mathcal{H}(\left((\Delta_{\bar{v}}^{\mathcal{Q}_{\bar{v}}})^{w_0^P}\right)^{-1}, K_{\bar{v}}^{w_0^P})\right)$$

and

$$\tilde{\iota} : \widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}},\bar{S}-\mathrm{ord}} \xrightarrow{\sim} \widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}},\bar{S}-\mathrm{ord},\tilde{\iota}} := \widetilde{\mathbb{T}}^T \otimes_{\mathbb{Z}} \left(\bigotimes_{\bar{v}\in\bar{S}} \mathcal{H}(\left(\widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}\right)^{-1}, \mathcal{Q}_{\bar{v}})[[\mathcal{Q}_{\bar{v}}\tilde{u}_{\tilde{v},n}^{-1}\mathcal{Q}_{\bar{v}}]^{-1}]\right).$$

We will also make use of an untwisted Hecke algebra for the Levi, defined by $\mathbb{T}^{\mathcal{Q}_{\bar{S}},\bar{S}-\mathrm{ord},\iota} := \mathbb{T}^T \otimes_{\mathbb{Z}} \left(\bigotimes_{\bar{v}\in\bar{S}} \mathcal{H}(\left(\Delta_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}\right)^{-1}, K_{\bar{v}})\right)$.

We define $\mathcal{S}^{\iota} : \widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}},\bar{S}-\mathrm{ord},\tilde{\iota}} \to \mathbb{T}^{\mathcal{Q}_{\bar{S}},\bar{S}-\mathrm{ord},\iota}$ extending $\mathcal{S}$ according to the formula

$$[\mathcal{Q}_{\bar{v}}\nu(\varpi_{\bar{v}})^{-1}\mathcal{Q}_{\bar{v}}] \mapsto [K_{\bar{v}}\nu(\varpi_{\bar{v}})^{-1}K_{\bar{v}}].$$

Proposition 4.2.2. *Assume that $\widetilde{K} \subset \widetilde{G}(\mathbb{A}_{F^+,f})$ is a good subgroup that is decomposed with respect to P and with the property that, for each $\bar{v} \in \bar{S}_p$, $\widetilde{K}_{U,\bar{v}} = U^0_{\bar{v}}$. Let $\mathfrak{m} \subset \mathbb{T}^T$ be a non-Eisenstein maximal ideal, let $\widetilde{\mathfrak{m}} := \mathcal{S}^*(\mathfrak{m}) \subset \widetilde{\mathbb{T}}^T$, and assume that $\bar{\rho}_{\widetilde{\mathfrak{m}}}$ is decomposed generic in the sense of Definition 2.1.27.*

Choose a partition

$$\bar{S}_p = \bar{S}_1 \sqcup \bar{S}_2 \sqcup \bar{S}_3$$

of the set $\bar{S}_p$ of primes of F^+ lying above p, together with standard parabolic subgroups $Q_{\bar{v}} \subset P_{\bar{v}}$ for each $\bar{v} \in \bar{S}_3$. Let $\tilde{\lambda}$ and λ be dominant weights for $\widetilde{G}$ and G, respectively. Assume that the following conditions are satisfied:

(1) *For each* $\tau : F^+ \hookrightarrow E$ *inducing a place* $\bar{v} \in \bar{S}_1$, $\tilde{\lambda}_\tau = (-w_{0,n}\lambda_{\tilde{\tau}c}, \lambda_{\tilde{\tau}})$ *(identification as in* (2.1.4)*);*
(2) *For each* $\tau : F^+ \hookrightarrow E$ *inducing a place* $\bar{v} \in \bar{S}_2$, $\tilde{\lambda}_\tau = 0$.
(3) *For each* $\bar{v} \in \bar{S}_3$, $\widetilde{K}_{\bar{v}} = \mathcal{Q}_{\bar{v}}$. *For each* $\tau : F^+ \hookrightarrow E$ *inducing such a place* $\bar{v}$, *we also have the standard identification* $\tilde{\lambda}_\tau = (-w_{0,n}\lambda_{\tilde{\tau}c}, \lambda_{\tilde{\tau}})$

Then the unnormalised Satake transform $\mathcal{S}^{w_0^P} : \widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}_3}, \bar{S}_3-\mathrm{ord}} \to \mathbb{T}_{w_0^P}^{\mathcal{Q}_{\bar{S}_3}, \bar{S}_3-\mathrm{ord}}$ *descends to a homomorphism*

$$\widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}_3}, \bar{S}_3-\mathrm{ord}}\left(H^d\left(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}}\right)^{\mathrm{ord}}_{\tilde{\mathfrak{m}}}\right) \to \mathbb{T}_{w_0^P}^{\mathcal{Q}_{\bar{S}_3}, \bar{S}_3-\mathrm{ord}}\left(\mathbb{H}^d\left(X_{K^{\bar{S}_3}K_{\bar{S}_3}^{w_0^P}}, \mathcal{V}_{\lambda_{\bar{S}_1}} \otimes \mathcal{V}_U(\tilde{\lambda}_{\bar{S}_2}) \otimes \mathcal{V}_{\lambda_{\bar{S}_3}}\right)_{\mathfrak{m}}\right),$$

where $\mathbb{H}^d$ *denotes the degree* d *hypercohomology. The Hecke action on the source is defined in* (2.1.9) *and the one on the target is defined in* (2.1.10). *The Hecke operators* $\widetilde{U}_{\tilde{v},n}$ *for* $\bar{v} \in \bar{S}$ *are invertible on the source because we are considering the ordinary part of cohomology at the primes in* $\bar{S}$.

Moreover, we have an injection

$$\widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}_3}, \bar{S}_3-\mathrm{ord}}\left(H^d\left(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}}\right)^{\mathrm{ord}}_{\tilde{\mathfrak{m}}}\right) \hookrightarrow \widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}_3}, \bar{S}_3-\mathrm{ord}}\left(H^d\left(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}}[1/p]\right)^{\mathrm{ord}}_{\tilde{\mathfrak{m}}}\right).$$

Proof. We will apply Theorem 2.1.28, noting that taking the ordinary part preserves injectivity and surjectivity of the maps in that statement. This reduces us to proving that $\mathcal{S}^{w_0^P}$ descends to a homomorphism

$$\widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}_3}, \bar{S}_3-\mathrm{ord}}\left(H^d\left(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}}\right)^{\mathrm{ord}}_{\tilde{\mathfrak{m}}}\right) \to \mathbb{T}_{w_0^P}^{\mathcal{Q}_{\bar{S}_3}, \bar{S}_3-\mathrm{ord}}\left(\mathbb{H}^d\left(X_{K^{\bar{S}_3}K_{\bar{S}_3}^{w_0^P}}, \mathcal{V}_{\lambda_{\bar{S}_1}} \otimes \mathcal{V}_U(\tilde{\lambda}_{\bar{S}_2}) \otimes \mathcal{V}_{\lambda_{\bar{S}_3}}\right)_{\mathfrak{m}}\right).$$

Combining Proposition 2.2.17 and Lemma 2.2.12 shows that $H^d(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}}/\varpi^m)^{\mathrm{ord}}_{\tilde{\mathfrak{m}}}$ is isomorphic as a $\widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}_3}, \bar{S}_3-\mathrm{ord}}$-module with

$$R^d\Gamma\left(K_{\bar{S}_3}, \pi_\partial^{\mathrm{ord}}(\widetilde{K}^{\bar{S}_3}, \tilde{\lambda}^{\bar{S}_3}, m) \otimes \mathcal{V}_{\lambda_{\bar{S}_3}}^{w_0^P}/\varpi^m\right).$$

For each $\bar{v} \in \bar{S}_3$, the action of $\mathcal{H}(\widetilde{\Delta}_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}, \mathcal{Q}_{\bar{v}})[\widetilde{U}_{\tilde{v},n}^{-1}]$ is via its isomorphism to $\mathcal{H}(\Delta_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}, K_{\bar{v}})$ (cf. Lemma 2.1.15).

Using Corollary 4.1.9 and interchanging the P-ordinary part with the tensor product by $\mathcal{V}_{\lambda_{\bar{S}_3}}^{w_0^P}$, we find a $\widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}_3}, \bar{S}_3-\mathrm{ord}}$-equivariant direct summand

(4.2.1)
$$\mathrm{ord}_0\,\mathbb{H}^d\left(K_{\bar{S}_3} \ltimes U_{\bar{S}_3}^0, \mathrm{Ind}_{P_{\bar{S}_3}}^{\widetilde{G}_{\bar{S}_3}} R\Gamma\left(K^{\bar{S}_3}, R\Gamma\left(\overline{\mathfrak{X}}_G, \mathcal{V}_{\lambda_{\bar{S}_1}}/\varpi^m \otimes \mathcal{V}_U(\tilde{\lambda}_{\bar{S}_2}, m)\right)\right) \otimes \mathcal{V}_{\lambda_{\bar{S}_3}}^{w_0^P}/\varpi^m\right)_{\tilde{\mathfrak{m}}}.$$

We find a subquotient of the term (4.2.1) using the w_0^P-case of Proposition 2.3.11. We claim that it is $\widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}_3}, \bar{S}_3-\mathrm{ord}}$-equivariantly isomorphic to

$$\mathbb{H}^d\left(X_{K^{\bar{S}_3}K_{\bar{S}_3}^{w_0^P}}, \mathcal{V}_{\lambda_{\bar{S}_1}}/\varpi^m \otimes \mathcal{V}_U(\tilde{\lambda}_{\bar{S}_2}, m) \otimes \mathcal{V}_{\lambda_{\bar{S}_3}}/\varpi^m\right)_{\tilde{\mathfrak{m}}}, \tag{4.2.2}$$

where the action of $\widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}_3}, \bar{S}_3-\mathrm{ord}}$ on (4.2.2) is via the extension $\mathcal{S}^{w_0^P}$ of the unnormalised Satake transform defined above. This follows from the fact that the action of $G_{\bar{S}_3}$ on $R\Gamma\left(K^{\bar{S}_3}, R\Gamma\left(\overline{\mathfrak{X}}_G, \mathcal{V}_{\lambda_{\bar{S}_1}}/\varpi^m \otimes \mathcal{V}_U(\tilde{\lambda}_{\bar{S}_2}, m)\right)\right)$ needs to be pre-conjugated by w_0^P when applying Proposition 2.3.11. Note that this is compatible with the rescaled Hecke actions, since the fact that w_0^P commutes with w_0^G implies that we have the equality

$$\alpha_{\lambda}^{\overline{\mathcal{Q}}_{\bar{v}}^{w_0^P}}(\nu(\varpi_{\bar{v}})^{w_0^P}) = \alpha_{\lambda^{w_0^P}}^{\mathcal{Q}_{\bar{v}}}(\nu(\varpi_{\bar{v}})).$$

Taking stock, we obtain a homomorphism

$$\widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}_3},\bar{S}_3-\mathrm{ord}}\left(H^d\left(\partial\widetilde{X}_{\widetilde{K}},\mathcal{V}_{\tilde{\lambda}}/\varpi^m\right)^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}}\right) \to \widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}_3},\bar{S}_3-\mathrm{ord}}\left(\mathbb{H}^d\left(X_K,\mathcal{V}_{\lambda_{\bar{S}_1}}/\varpi^m\otimes\mathcal{V}_U(\tilde{\lambda}_{\bar{S}_2},m)\otimes\mathcal{V}_{\lambda_{\bar{S}_3}}/\varpi^m\right)_{\widetilde{\mathfrak{m}}}\right),$$

where the action of $\widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}_3},\bar{S}_3-\mathrm{ord}}$ on the RHS is via $\mathcal{S}^{w_0^P}$.

Finally, these morphisms are compatible as m varies and all the above cohomology groups are finitely generated $\mathcal{O}$-modules, so we can take inverse limits with respect to m to obtain a homomorphism

$$\widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}_3},\bar{S}_3-\mathrm{ord}}\left(H^d\left(\partial\widetilde{X}_{\widetilde{K}},\mathcal{V}_{\tilde{\lambda}}\right)^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}}\right) \to \widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}_3},\bar{S}_3-\mathrm{ord}}\left(\mathbb{H}^d\left(X_K,\mathcal{V}_{\lambda_{\bar{S}_1}}\otimes\mathcal{V}_U(\tilde{\lambda}_{\bar{S}_2})\otimes\mathcal{V}_{\lambda_{\bar{S}_3}}\right)_{\widetilde{\mathfrak{m}}}\right).$$

We conclude because $\mathcal{S}^*(\mathfrak{m}) = \widetilde{\mathfrak{m}}$, so $\mathbb{H}^d\left(X_K,\mathcal{V}_{\lambda_{\bar{S}_1}}\otimes\mathcal{V}_U(\tilde{\lambda}_{\bar{S}_2})\otimes\mathcal{V}_{\lambda_{\bar{S}_3}}\right)_{\mathfrak{m}}$ is a Hecke-equivariant direct summand of $\mathbb{H}^d\left(X_K,\mathcal{V}_{\lambda_{\bar{S}_1}}\otimes\mathcal{V}_U(\tilde{\lambda}_{\bar{S}_2})\otimes\mathcal{V}_{\lambda_{\bar{S}_3}}\right)_{\widetilde{\mathfrak{m}}}$. □

We have a similar statement to the above which will be useful after applying Poincaré duality. Most things go through very similarly. At places in $\bar{S}_2$, the weight $\tilde{\lambda}_{\bar{S}_2}$ is trivial so nothing changes when we take a dual. At places in $\bar{S}_3$ we use the results of §2.2.18 and Corollary 2.3.12. However, we need to modify things at places in $\overline{S}_1$, because $R\Gamma(U^0_{\bar{S}},\mathcal{V}^\vee_{\tilde{\lambda}_{\bar{S}}}/\varpi^m)$ may not admit $\mathcal{V}^\vee_{\lambda_{\bar{S}}}/\varpi^m$ as a $K_{\bar{S}}$-equivariant direct summand. This causes a problem for the analogue of Corollary 4.1.9 on the dual side. The following lemma will act as a replacement.

Lemma 4.2.3. *Let $\bar{S}\subset\bar{S}_p$, let $\tilde{\lambda}$ and λ be dominant weights for $\widetilde{G}$ and G respectively. Assume that the following condition is satisfied:*

(1) *For each $\tau: F^+\hookrightarrow E$ inducing a place $\bar{v}\in\bar{S}$, $\tilde{\lambda}_\tau = (\lambda_{\tilde{\tau}}, -w_{0,n}\lambda_{\tilde{\tau}c})$ (note that this is the w_0^P-conjugate of the standard identification).*

Let $m\in\mathbb{Z}_{\geq 1}$ be an integer. Then $R\Gamma\left(U^0_{\bar{S}},\mathcal{V}^\vee_{\tilde{\lambda}_{\bar{S}}}/\varpi^m\right)$ admits $\mathcal{V}^\vee_{\lambda_{\bar{S}}}/\varpi^m$ as a $K_{\bar{S}}$-equivariant direct summand.

Proof. Taking duals and applying the proof of [ACC+18, Theorem 2.4.4], it suffices to show that there is a $\mathcal{V}_{\tilde{\lambda}_{\bar{S}}}$ surjective $P(\mathcal{O}_{F^+,\bar{S}})$-equivariant map

$$\mathcal{V}_{\tilde{\lambda}_{\bar{S}}}\to\mathcal{V}_{\lambda_{\bar{S}}}$$

with a $K_{\bar{S}}$-equivariant splitting. It follows from [NT16, Proposition 2.10] that there is a $K_{\bar{S}}$-equivariant decomposition $\mathcal{V}_{\tilde{\lambda}_{\bar{S}}} = \mathcal{V}_{\lambda_{\bar{S}}}\oplus W$, with $\mathcal{V}_{\lambda_{\bar{S}}}$ invariant under the action of the unipotent subgroup $\overline{U}(\mathcal{O}_{F^+,\bar{S}})$ in the parabolic opposite to P. Moreover, it follows from the main theorem of [Cab84] that, after extending scalars to $\overline{E}$, $W\otimes_{\mathcal{O}}\overline{E}$ is identified with the $P(F^+_{\bar{S}})$-stable subspace $(1-U(F^+_{\bar{S}}))\mathcal{V}_{\tilde{\lambda}_{\bar{S},\overline{E}}}\subset\mathcal{V}_{\tilde{\lambda}_{\bar{S},\overline{E}}}$ (thanks to Lambert A'Campo for pointing this out). This implies that W is $P(\mathcal{O}_{F^+,\bar{S}})$-stable, so quotienting out by W gives the desired map $\mathcal{V}_{\tilde{\lambda}_{\bar{S}}}\to\mathcal{V}_{\lambda_{\bar{S}}}$. □

We can now state our version of Proposition 4.2.2 with dual coefficients.

Proposition 4.2.4. *Assume that $\widetilde{K} \subset \widetilde{G}(\mathbb{A}_{F^+,f})$ is a good subgroup that is decomposed with respect to P, and with the property that, for each $\bar{v} \in \bar{S}_p$, $\widetilde{K}_{U,\bar{v}} = U^0_{\bar{v}}$. Let $\mathfrak{m} \subset \mathbb{T}^T$ be a non-Eisenstein maximal ideal and assume that $\bar{\rho}_{\mathcal{S}^*(\mathfrak{m}^\vee)}$ is decomposed generic in the sense of Definition 2.1.27.*

Choose a partition

$$\bar{S}_p = \bar{S}_1 \sqcup \bar{S}_2 \sqcup \bar{S}_3$$

of the set $\bar{S}_p$ of primes of F^+ lying above p, together with standard parabolic subgroups $Q_{\bar{v}} \subset P_{\bar{v}}$ for each $\bar{v} \in \bar{S}_3$. Let $\tilde{\lambda}$ and λ be dominant weights for $\widetilde{G}$ and G, respectively. Assume that the following conditions are satisfied:

1. *For each $\tau : F^+ \hookrightarrow E$ inducing a place $\bar{v} \in \bar{S}_1$, $\tilde{\lambda}_\tau = (\lambda_{\tilde{\tau}}, -w_{0,n}\lambda_{\tilde{\tau}c})$ (note the change compared to Proposition 4.2.2);*
2. *For each $\tau : F^+ \hookrightarrow E$ inducing a place $\bar{v} \in \bar{S}_2$, $\tilde{\lambda}_\tau = 0$.*
3. *For each $\bar{v} \in \bar{S}_3$, $\widetilde{K}_{\bar{v}} = \mathcal{Q}_{\bar{v}}$. For each $\tau : F^+ \hookrightarrow E$ inducing such a place $\bar{v}$, we also have $\tilde{\lambda}_\tau = (\lambda_{\tilde{\tau}}, -w_{0,n}\lambda_{\tilde{\tau}c})$*

Then the unnormalised Satake transform $\mathcal{S}^\iota : \widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}_3}, \bar{S}_3-\mathrm{ord}, \tilde{\iota}} \to \mathbb{T}^{\mathcal{Q}_{\bar{S}_3}, \bar{S}_3-\mathrm{ord}, \iota}$ descends to a homomorphism

$$\widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}_3}, \bar{S}_3-\mathrm{ord}, \tilde{\iota}} \left(H^d \left(\widetilde{X}_{\widetilde{K}}, \mathcal{V}^\vee_{\tilde{\lambda}} \right)^{\mathrm{ord}^\vee}_{\mathcal{S}^*(\mathfrak{m}^\vee)} \right) \to \mathbb{T}^{\mathcal{Q}_{\bar{S}_3}, \bar{S}_3-\mathrm{ord}, \iota} \left(\mathbb{H}^d \left(X_K, \mathcal{V}^\vee_{\lambda_{\bar{S}_1}} \otimes \mathcal{V}_U(\tilde{\lambda}_{\bar{S}_2}) \otimes \mathcal{V}^\vee_{\lambda_{\bar{S}_3}} \right)_{\mathfrak{m}^\vee} \right),$$

where $\mathbb{H}^d$ denotes the degree d hypercohomology. Moreover, we have an injection

$$\widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}_3}, \bar{S}_3-\mathrm{ord}, \tilde{\iota}} \left(H^d \left(\widetilde{X}_{\widetilde{K}}, \mathcal{V}^\vee_{\tilde{\lambda}} \right)^{\mathrm{ord}^\vee}_{\mathcal{S}^*(\mathfrak{m}^\vee)} \right) \hookrightarrow \widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}_3}, \bar{S}_3-\mathrm{ord}, \tilde{\iota}} \left(H^d \left(\widetilde{X}_{\widetilde{K}}, \mathcal{V}^\vee_{\tilde{\lambda}}[1/p] \right)^{\mathrm{ord}^\vee}_{\mathcal{S}^*(\mathfrak{m}^\vee)} \right)$$

and the target is isomorphic, via Poincaré duality, to $\widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}_3}, \bar{S}_3-\mathrm{ord}} \left(H^d \left(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}}[1/p] \right)^{\mathrm{ord}}_{\tilde{\iota}^*\mathcal{S}^*(\mathfrak{m}^\vee)} \right)$.

Proof. With the ingredients mentioned in the preamble, this is a straightforward modification of the proof of Proposition 4.2.2. □

Note that we have $\bar{\rho}_{\tilde{\iota}^*\mathcal{S}^*(\mathfrak{m}^\vee)} = \bar{\rho}_{\mathfrak{m}}(-n) \oplus \bar{\rho}^{\vee,c}_{\mathfrak{m}}(1-n)$.

Assume we are given a non-Eisenstein maximal ideal $\mathfrak{m} \subset \mathbb{T}$ with $\widetilde{\mathfrak{m}} := \mathcal{S}^*(\mathfrak{m})$, and a subset $\bar{S} \subseteq \bar{S}_p$. We will use the following notation:

$$A(K, \lambda, q) := \mathbb{T}^{\overline{\mathcal{Q}}^{w_0^P}_{\bar{S}}, \bar{S}-\mathrm{ord}}_{w_0^P} \left(H^q(X_K, \mathcal{V}_\lambda)_{\mathfrak{m}} \right),$$

$$A(K, \lambda, q, m) := \mathbb{T}^{\overline{\mathcal{Q}}^{w_0^P}_{\bar{S}}, \bar{S}-\mathrm{ord}}_{w_0^P} \left(H^q(X_K, \mathcal{V}_\lambda / \varpi^m)_{\mathfrak{m}} \right), \text{ and}$$

$$\widetilde{A}(\widetilde{K}, \tilde{\lambda}, \bar{S}) := \widetilde{\mathbb{T}}^{\overline{\mathcal{Q}}^{w_0^P}_{\bar{S}}, \bar{S}-\mathrm{ord}} \left(H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}} \right).$$

Given a neat compact open subgroup $K \subset \mathrm{GL}_n(\mathbb{A}_{F,f})$ and an integer $m \in \mathbb{Z}_{\geq 1}$, define the subgroup $K(m, \bar{S}) \subset K$ by setting

$$K(m, \bar{S})_v := K_v \cap \{(1_n) \bmod \varpi_v^m\} \subset \mathrm{GL}_n(\mathcal{O}_{F_v})$$

if v is a p-adic place of F which lies above a place in $\bar{S}$, and $K(m, \bar{S})_v := K_v$ otherwise. Also, given a good subgroup $\widetilde{K} \subset \widetilde{G}(\mathbb{A}_{F^+,f})$ and an integer $m \in \mathbb{Z}_{\geq 1}$, define the good subgroup $\widetilde{K}(m, \bar{S}) \subset \widetilde{K}$ by setting

$$\widetilde{K}(m, \bar{S})_{\bar{v}} := \widetilde{K}_{\bar{v}} \cap \left\{ \begin{pmatrix} 1_n & * \\ 0 & 1_n \end{pmatrix} \bmod \varpi_{\bar{v}}^m \right\} \subset \mathrm{GL}_{2n}(\mathcal{O}_{F_{\tilde{v}}}) = \widetilde{G}(\mathcal{O}_{F^+_{\bar{v}}})$$

if $\bar{v}$ is a p-adic place of F^+ contained in $\bar{S}$, and $\widetilde{K}(m,\bar{S})_{\bar{v}} := \widetilde{K}_{\bar{v}}$ otherwise.

Lemma 4.2.5. *Assume that the subset $\bar{S} \subset \bar{S}_p$ has the following property: $\sum_{\bar{v}\notin\bar{S}}[F^+_{\bar{v}} : \mathbb{Q}_p] \geq \frac{1}{2}[F^+ : \mathbb{Q}]$. Let $q \in \left[\left\lfloor \frac{d}{2}\right\rfloor, d-1\right]$. Then $d - q \leq \sum_{\bar{v}\notin\bar{S}} n^2[F^+_{\bar{v}} : \mathbb{Q}_p]$.*

Proof. If d is odd, then so is $[F^+ : \mathbb{Q}]$, and we have

$$d - q \leq \frac{d+1}{2} \leq \frac{d+n^2}{2} \leq \sum_{\bar{v}\notin\bar{S}} n^2[F^+_{\bar{v}} : \mathbb{Q}_p].$$

If d is even, then $d - q \leq \frac{d}{2} \leq \sum_{\bar{v}\notin\bar{S}} n^2[F^+_{\bar{v}''} : \mathbb{Q}_p]$. ☐

Proposition 4.2.6. *Let $\bar{v}$, $\bar{v}'$ be two distinct places of $\bar{S}_p$. Let $\bar{S}_1 := \{\bar{v}'\}$, $\bar{S}_3 := \{\bar{v}\}$ and $\bar{S}_2$ be their complement in $\bar{S}_p$. Let $\lambda \in \left(\mathbb{Z}^n_+\right)^{\mathrm{Hom}(F,E)}$ be a highest weight for G. Let $m \in \mathbb{Z}_{\geq 1}$ be an integer and $\widetilde{K} \subset \widetilde{G}(\mathbb{A}_{F^+,f})$ be a good subgroup. Assume that the following conditions are satisfied.*

(1) *We have*

$$\sum_{\substack{\bar{v}''\in\bar{S}_p\\ \bar{v}''\neq\bar{v},\bar{v}'}} [F^+_{\bar{v}''} : \mathbb{Q}_p] \geq \frac{1}{2}[F^+ : \mathbb{Q}].$$

(2) *For each p-adic place $\bar{v}''$ of F^+ not equal to $\bar{v}$ (including $\bar{v}'' = \bar{v}'$), we have $U(\mathcal{O}_{F^+_{\bar{v}''}}) \subset \widetilde{K}_{\bar{v}''}$ and $\widetilde{K}_{\bar{v}''} = \widetilde{K}(m, \bar{S}_1 \cup \bar{S}_2)_{\bar{v}''}$; in other words $\widetilde{K}_{\bar{v}''} \subset \left\{\begin{pmatrix} 1_n & * \\ 0 & 1_n \end{pmatrix} \bmod \varpi^m_{\bar{v}''}\right\}$ for each of these places. Finally, we have $\widetilde{K}_{\bar{v}} = \overline{\mathcal{Q}}^{w_0^P}_{\bar{v}}$ corresponding to the standard parabolic $\overline{Q}^{w_0^P}_{\bar{v}} \subset P_{F^+_{\bar{v}}}$ with Levi subgroup $Q^{w_0^P}_{\bar{v}} \cap G(F^+_{\bar{v}})$.*

(3) *For each embedding $\tau : F \hookrightarrow E$ inducing the place $\bar{v}$ or $\bar{v}'$ of F^+, we have $-\lambda_{\tau c,1} - \lambda_{\tau,1} \geq 0$.*

(4) *$\mathfrak{m} \subset \mathbb{T}$ is a non-Eisenstein maximal ideal such that $\bar{\rho}_{\widetilde{\mathfrak{m}}}$ is decomposed generic.*

Define a weight $\tilde{\lambda} \in (\mathbb{Z}^{2n}_+)^{\mathrm{Hom}(F^+,E)}$ as follows: if $\tau : F^+ \hookrightarrow E$ does not induce either $\bar{v}$ or $\bar{v}'$, set $\tilde{\lambda}_\tau = 0$. If τ induces $\bar{v}$ or $\bar{v}'$, set $\tilde{\lambda}_\tau = (-\lambda_{\tilde{\tau}c}, \lambda_{\tilde{\tau}})$ (identification as in (2.1.4))[19]*. Set $K := (\widetilde{K}^{\bar{v}} \cap G(\mathbb{A}^{\bar{v}}_{F^+,f})) \cdot (\mathcal{Q}_{\bar{v}} \cap G(F^+_{\bar{v}}))$, identified in the usual way with a neat subgroup of $\mathrm{GL}_n(\mathbb{A}_{F,f})$.*

Let $q \in \left[\left\lfloor \frac{d}{2}\right\rfloor, d-1\right]$. Then there exists an integer $m' \geq m$, an integer $N \geq 1$, a nilpotent ideal $J \subset A(K,\lambda,q,m)$ satisfying $J^N = 0$, and a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathbb{T}}^{\overline{\mathcal{Q}}^{w_0^P}_{\bar{v}},\{\bar{v}\}-\mathrm{ord}} & \longrightarrow & \widetilde{A}\left(\widetilde{K}(m',\bar{S}_2),\tilde{\lambda},\bar{v}\right) \\ \big\downarrow \mathcal{S}^{w_0^P} & & \big\downarrow \\ \mathbb{T}^{\overline{\mathcal{Q}}^{w_0^P}_{\bar{v}},\{\bar{v}\}-\mathrm{ord}}_{w_0^P} & \longrightarrow & A(K,\lambda,q,m)/J. \end{array}$$

Moreover, the integer N can be chosen to only depend on n and on $[F^+ : \mathbb{Q}]$.

[19]The condition (3) in the statement of the Proposition guarantees that $\tilde{\lambda}$ will be dominant.

Proof. To simplify notation, we set $\mathbb{T} := \mathbb{T}_{w_0^P}^{\overline{\mathcal{Q}}_{\bar{v}}^{w_0^P},\{\bar{v}\}-\mathrm{ord}}$, $\widetilde{\mathbb{T}} := \widetilde{\mathbb{T}}^{\overline{\mathcal{Q}}_{\bar{v}}^{w_0^P},\{\bar{v}\}-\mathrm{ord}}$. We would like to show that there exist non-negative integers $m' \geq m$ and N such that

$$\mathcal{S}^{w_0^P}\left(\mathrm{Ann}_{\widetilde{\mathbb{T}}}\ H^d(\widetilde{X}_{\widetilde{K}(m',\bar{S}_2)}, \mathcal{V}_{\tilde{\lambda}})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}}\right)^N \subseteq \mathrm{Ann}_{\mathbb{T}}\ H^q(X_K, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}}.$$

This is similar to [ACC+18, Prop. 4.4.1]; we will apply Proposition 4.2.2 repeatedly, which plays the same role as Proposition 4.3.1 in *loc. cit.*. The argument is subtle, for two reasons.

- We need to work with $\mathcal{O}$-coefficients in order to access the Hecke algebras $\widetilde{A}\left(\widetilde{K}(m', \bar{S}_2), \tilde{\lambda}, \bar{v}\right)$, whilst the Hecke algebra $A(K, \lambda, q, m)$ acts on cohomology with torsion coefficients.
- The spectral sequence computing the hypercohomology in Proposition 4.2.2 is not known to degenerate.

The second issue does not occur in [ACC+18]. To deal with both these issues, we will argue by descending induction on the degree q. The induction hypothesis is the following.

Hypothesis 4.2.7. Let $q \in \left[\left\lfloor \frac{d}{2} \right\rfloor, d-1\right]$. Then the Proposition holds for every cohomological degree $i \in [q+1, d-1]$ and for every $m \in \mathbb{Z}_{\geq 1}$. Moreover, the integer N can be chosen to depend only on n, $[F^+ : \mathbb{Q}]$ and q.

Note that the induction hypothesis is satisfied automatically for $q = d-1$. Assume that the hypothesis is satisfied for some $q \in \left[\left\lfloor \frac{d}{2} \right\rfloor, d-1\right]$. We will prove the Proposition for q, which will imply the induction hypothesis for $q-1$.

Fix m, $\widetilde{K}$ and λ as in the statement. Let $M = M(m) \geq m$ be the integer guaranteed by Lemma 2.3.17. We first increase the level, going from X_K to $X_{K(M,\bar{S}_2)}$. This uses the same argument as in the proof of [ACC+18, Prop. 4.4.1], that we briefly recall here. Firstly, Poincaré duality gives an equality

$$\mathrm{Ann}_{\mathbb{T}}\ H^q(X_K, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}} = \iota\left(\mathrm{Ann}_{\mathbb{T}^\iota}\ H^{d-1-q}(X_K, \mathcal{V}_\lambda^\vee/\varpi^m)_{\mathfrak{m}^\vee}\right),$$

where $\mathfrak{m}^\vee = \iota(\mathfrak{m}) \subset \mathbb{T}^T$ and $\mathbb{T}^\iota := \mathbb{T}_{w_0^P}^{\mathcal{Q}_{\bar{S}_3},\bar{S}_3-\mathrm{ord},\iota}$.

The Hochschild–Serre spectral sequence gives an inclusion

$$\prod_{i=0}^{d-q-1} \mathrm{Ann}_{\mathbb{T}^\iota}\ H^i(X_{K(M,\bar{S}_2)}, \mathcal{V}_\lambda^\vee/\varpi^m)_{\mathfrak{m}^\vee} \subset \mathrm{Ann}_{\mathbb{T}^\iota}\ H^{d-1-q}(X_K, \mathcal{V}_\lambda^\vee/\varpi^m)_{\mathfrak{m}^\vee},$$

and Poincaré duality gives

$$\prod_{i=q}^{d-1} \mathrm{Ann}_{\mathbb{T}}\ H^i(X_{K(M,\bar{S}_2)}, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}} \subset \mathrm{Ann}_{\mathbb{T}}\ H^q(X_K, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}}.$$

We deal with the terms for $i \geq q+1$ using induction. (See the last paragraph of the proof for more details on how one applies the induction hypothesis.) Therefore, we are left with the term for $i = q$ and we may assume that $K = K(M, \bar{S}_2)$.

Now, note that the $\mathbb{T}$-algebra $A(K, \lambda, q, m)$ does not depend on $\lambda_{\bar{v}''}$ for $\bar{v}'' \neq \bar{v}$, because the level $K_{\bar{v}''}$ is deep enough that the action on $\mathcal{V}_\lambda/\varpi^m$ is trivial. Therefore, we replace $\mathcal{V}_\lambda/\varpi^m$ by

$$\mathcal{V}_{\lambda_{\bar{v}}} \otimes \mathcal{V}_{\lambda_{\bar{v}'}} \otimes \mathcal{V}_U^{d-q}(\tilde{\lambda}_{\bar{S}_2}, m).$$

This is non-zero by Lemmas 2.3.17 and 4.2.5. More generally, for any non-negative integer $j \leq \sum_{\bar{v}'' \in \bar{S}_2} n^2[F^+_{\bar{v}''} : \mathbb{Q}_p]$, set $\mathcal{V}^j := \mathcal{V}_{\lambda_{\bar{v}}} \otimes \mathcal{V}_{\lambda_{\bar{v}'}} \otimes \mathcal{V}^j_U(\tilde{\lambda}_{\bar{S}_2})$. We have a short exact sequence of $\mathbb{T}$-modules

$$0 \to H^q(X_K, \mathcal{V}^{d-q})_{\mathfrak{m}}/\varpi^m \to H^q(X_K, \mathcal{V}^{d-q}/\varpi^m)_{\mathfrak{m}} \to H^{q+1}(X_K, \mathcal{V}^{d-q})_{\mathfrak{m}}[\varpi^m] \to 0,$$

where we are interested in understanding the Hecke algebra $A(K, \lambda, q, m)$ acting on the term in the middle. We can understand the Hecke algebra acting on the ϖ^m-torsion in $H^{q+1}(X_K, \mathcal{V}^{d-q})_{\mathfrak{m}}$ using the induction hypothesis: the argument is identical to the argument used in the proof of [ACC+18, Prop. 4.4.1]. Therefore, we are left with understanding the faithful quotient of $\mathbb{T}$ acting on $H^q(X_K, \mathcal{V}^{d-q})_{\mathfrak{m}}/\varpi^m$.

There is a $\mathbb{T}$-equivariant spectral sequence

$$E_2^{i,j}(\mathcal{O}) := H^i(X_K, \mathcal{V}^j)_{\mathfrak{m}} \Rightarrow \mathbb{H}^{i+j}(X_K, \mathcal{V}_{\lambda_{\bar{v}}} \otimes \mathcal{V}_{\lambda_{\bar{v}'}} \otimes \mathcal{V}_U(\tilde{\lambda}_{\bar{S}_2}))_{\mathfrak{m}}. \tag{4.2.3}$$

If we knew that this spectral sequence degenerates on the E_2 page, we would deduce that $H^q(X_K, \mathcal{V}^{d-q})_{\mathfrak{m}}$ is a $\mathbb{T}$-equivariant subquotient of $\mathbb{H}^d(X_K, \mathcal{V}_{\lambda_{\bar{S}_1}} \otimes \mathcal{V}_{\lambda_{\bar{S}_3}} \otimes \mathcal{V}_U(\tilde{\lambda}_{\bar{S}_2}))_{\mathfrak{m}}$, and we would win by Proposition 4.2.2. However, it is not clear, in this generality, whether the spectral sequence (4.2.3) degenerates. Instead, we compare it to the following $\mathbb{T}$-equivariant spectral sequence, whose terms are $\mathcal{O}/\varpi^m$-modules

$$E_2^{i,j}(\mathcal{O}/\varpi^m) := H^i(X_K, \mathcal{V}^j/\varpi^m)_{\mathfrak{m}} \Rightarrow \mathbb{H}^{i+j}(X_K, \mathcal{V}_{\lambda_{\bar{v}}} \otimes \mathcal{V}_{\lambda_{\bar{v}'}} \otimes \mathcal{V}_U(\tilde{\lambda}_{\bar{S}_2}, m))_{\mathfrak{m}}. \tag{4.2.4}$$

Since $K = K(M, \bar{S}_2)$, Lemma 2.3.17 implies that all the differentials in (4.2.4) are zero. Let

$$\phi_r^{i,j} : E_r^{i,j}(\mathcal{O}) \to E_r^{i,j}(\mathcal{O}/\varpi^m)$$

be the natural, $\mathbb{T}$-equivariant map between the spectral sequences (4.2.3) and (4.2.4). Let $F_r^{i,j} := \mathrm{Im}(\phi_r^{i,j})$. When $r = 2$, we have $F_2^{i,j} = E_2^{i,j}(\mathcal{O})/\varpi^m$. For $r \geq 3$, we at least have a surjection $E_r^{i,j}(\mathcal{O})/\varpi^m \twoheadrightarrow F_r^{i,j}$, because $F_r^{i,j}$ is an $\mathcal{O}/\varpi^m$-module.

With this new notation, we are interested in relating $\mathrm{Ann}_{\mathbb{T}}(F_2^{q,d-q})$ to $\mathrm{Ann}_{\mathbb{T}}(E_\infty^d(\mathcal{O}))$. For any $r \geq 2$, let $d_r^- : E_r^{q-r,d-q+1-r} \to E_r^{q,d-q}$ and $d_r^+ : E_r^{q,d-q} \to E_r^{q+r,d-q+1-r}$ denote the rth differentials. Because all the differentials in (4.2.4) are zero, we have that $\mathrm{Im}(d_r^-) \subseteq \mathrm{Ker}(\phi_r^{q,d-q})$. Since $\phi_r^{q,d-q}$ induces $\phi_{r+1}^{q,d-q}$, we deduce that we have an injection

$$F_{r+1}^{q,d-q} = \mathrm{Ker}(d_r^+)/\left(\mathrm{Im}(d_r^-) + \mathrm{Ker}(d_r^+) \cap \mathrm{Ker}(\phi_r^{q,d-q})\right) \hookrightarrow F_r^{q,d-q} = E_r^{q,d-q}/\mathrm{Ker}(\phi_r^{q,d-q}).$$

Moreover, the cokernel of this injection becomes identified, under the map induced by d_r^+, with $\mathrm{Im}(d_r^+)/d_r^+(\mathrm{Ker}(\phi_r^{q,d-q}))$. Since $(\varpi^m) \subseteq \mathrm{Ker}(\phi_r^{q,d-q})$, the latter is a quotient of $\mathrm{Im}(d_r^+)/\varpi^m$. By Lemma 2.3.18, there exists some $m'_r \geq m$ such that the latter is a subquotient of $E_2^{q+r,d-q+1-r}(\mathcal{O})/\varpi^{m'_r}$, or even of $E_2^{q+r,d-q+1-r}(\mathcal{O}/\varpi^{m'_r})$. We therefore have an inclusion

$$\mathrm{Ann}_{\mathbb{T}}\ E_\infty^d(\mathcal{O}) \cdot \prod_{r=2}^{d-q-1} \mathrm{Ann}_{\mathbb{T}}\ E_2^{q+r,d+1-q-r}(\mathcal{O}/\varpi^{m'_r}) \subseteq \mathrm{Ann}_{\mathbb{T}}\ F_2^{q,d-q}.$$

For the $E_\infty^d(\mathcal{O})$ term, Proposition 4.2.2 implies that there is an inclusion

$$\mathcal{S}^{w_0^P}\left(\mathrm{Ann}_{\widetilde{\mathbb{T}}}\ H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}})^{\mathrm{ord}}_{\tilde{\mathfrak{m}}}\right) \subseteq \mathrm{Ann}_{\mathbb{T}}\ E_\infty^d(\mathcal{O}).$$

For each $E_2^{q+r,d+1-q-r}(\mathcal{O}/\varpi^{m'_r}) = H^{q+r}(X_K, \mathcal{V}^{d+1-q-r}/\varpi^{m'_r})_{\mathfrak{m}}$, we use the argument above via Poincaré duality and the Hochschild–Serre spectral sequence to increase the level to $K(m'_r, \bar{S}_2)$. We then apply the induction hypothesis.

Each time we apply the induction hypothesis, we find some integer $m'_i \geq m$ and some nilpotence degree N_i, which can be bounded in terms of $\dim(X_K)$, for i running over some finite index set I whose size can also be bounded in terms of $\dim(X_K)$. To find a common $m' \geq m$, we let $m' := \sup_{i\in I} m'_i$. For each i, we have

$$\mathrm{Ann}_{\widetilde{\mathbb{T}}}\, H^d(\widetilde{X}_{\widetilde{K}(m',\bar{S}_2)}, \mathcal{V}_{\tilde{\lambda}})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}} \subseteq \mathrm{Ann}_{\widetilde{\mathbb{T}}}\, H^d(\widetilde{X}_{\widetilde{K}(m'_i,\bar{S}_2)}, \mathcal{V}_{\tilde{\lambda}})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}},$$

because this is true rationally and the cohomology groups are torsion-free. We then let J denote the image of the ideal $\mathcal{S}^{w_0^P}\left(\mathrm{Ann}_{\widetilde{\mathbb{T}}}\, H^d(\widetilde{X}_{\widetilde{K}(m',\bar{S}_2)}, \mathcal{V}_{\tilde{\lambda}})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}}\right)$ in $A(K,\lambda,q,m)$. To find an appropriate nilpotence degree N, we set $N = 1 + \sum_i N_i$. $\square$

Again we have a similar statement with dual coefficients. We introduce some more notation, depending on a decomposition $\bar{S}_p = \bar{S}_1 \cup \bar{S}_2 \cup \bar{S}_3$:

$$A^\vee(K,\lambda,q) := \mathbb{T}^{\mathcal{Q}_{\bar{S}_3},\bar{S}_3-\mathrm{ord},\iota}\left(H^q(X_K,\mathcal{V}_\lambda^\vee)_{\mathfrak{m}^\vee}\right),$$
$$A^\vee(K,\lambda,q,m) := \mathbb{T}^{\mathcal{Q}_{\bar{S}_3},\bar{S}_3-\mathrm{ord},\iota}\left(H^q(X_K,\mathcal{V}_\lambda^\vee/\varpi^m)_{\mathfrak{m}^\vee}\right), \text{ and}$$
$$\widetilde{A}^\vee(\widetilde{K},\tilde{\lambda},\bar{S}_3) := \widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{S}_3},\bar{S}_3-\mathrm{ord},\tilde{\iota}}\left(H^d(\widetilde{X}_{\widetilde{K}},\mathcal{V}_{\tilde{\lambda}}^\vee)^{\mathrm{ord}^\vee}_{\mathcal{S}^*\mathfrak{m}^\vee}\right).$$

Proposition 4.2.8. *Let $\bar{v}$, $\bar{v}'$ be two distinct places of $\bar{S}_p$. Let $\bar{S}_1 := \{\bar{v}'\}$, $\bar{S}_3 := \{\bar{v}\}$ and $\bar{S}_2$ be their complement in $\bar{S}_p$. Let $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F,E)}$ be a highest weight for G. Let $m \in \mathbb{Z}_{\geq 1}$ be an integer and $\widetilde{K} \subset \widetilde{G}(\mathbb{A}_{F^+,f})$ be a good subgroup. Assume that the following conditions are satisfied.*

(1) *We have*
$$\sum_{\substack{\bar{v}''\in\bar{S}_p\\ \bar{v}''\neq\bar{v},\bar{v}'}} [F^+_{\bar{v}''}:\mathbb{Q}_p] \geq \frac{1}{2}[F^+:\mathbb{Q}].$$
(2) *For each p-adic place $\bar{v}''$ of F^+ not equal to $\bar{v}$ (including $\bar{v}'' = \bar{v}'$), we have $U(\mathcal{O}_{F^+_{\bar{v}''}}) \subset \widetilde{K}_{\bar{v}''}$ and $\widetilde{K}_{\bar{v}''} = \widetilde{K}(m,\bar{S}_1\cup\bar{S}_2)_{\bar{v}''}$; in other words $\widetilde{K}_{\bar{v}''} \subset \left\{\begin{pmatrix} 1_n & * \\ 0 & 1_n \end{pmatrix} \bmod \varpi^m_{\bar{v}''}\right\}$ for each of these places. Finally, we have $\widetilde{K}_{\bar{v}} = \mathcal{Q}_{\bar{v}}$ corresponding to the standard parabolic $Q_{\bar{v}} \subset P_{F^+_{\bar{v}}}$.*
(3) *For each embedding $\tau : F \hookrightarrow E$ inducing the place $\bar{v}$ or $\bar{v}'$ of F^+, we have $\lambda_{\tau c,n} + \lambda_{\tau,n} \geq 0$.*
(4) *$\mathfrak{m} \subset \mathbb{T}$ is a non-Eisenstein maximal ideal such that $\bar{\rho}_{\mathcal{S}^*(\mathfrak{m}^\vee)}$ is decomposed generic.*

Define a weight $\tilde{\lambda} \in (\mathbb{Z}^{2n}_+)^{\mathrm{Hom}(F^+,E)}$ as follows: if $\tau : F^+ \hookrightarrow E$ does not induce either $\bar{v}$ or $\bar{v}'$, set $\tilde{\lambda}_\tau = 0$. If τ induces $\bar{v}$ or $\bar{v}'$, set $\tilde{\lambda}_\tau = (\lambda_{\tilde{\tau}}, -\lambda_{\tilde{\tau}c})$. Set $K := \widetilde{K} \cap G(\mathbb{A}_{F^+,f})$, identified in the usual way with a neat subgroup of $\mathrm{GL}_n(\mathbb{A}_{F,f})$.

Let $q \in \left[\left\lfloor \frac{d}{2}\right\rfloor, d-1\right]$. Then there exists an integer $m' \geq m$, an integer $N \geq 1$, a nilpotent ideal $J \subset A^\vee(K,\lambda,q,m)$ satisfying $J^N = 0$, and a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{v}},\{\bar{v}\}-\mathrm{ord},\tilde{\iota}} & \longrightarrow & \widetilde{A}^\vee\left(\widetilde{K}(m',\bar{S}_2),\tilde{\lambda},\bar{v}\right) \\ \downarrow{\scriptstyle \mathcal{S}^\iota} & & \downarrow \\ \mathbb{T}^{\mathcal{Q}_{\bar{v}},\{\bar{v}\}-\mathrm{ord},\iota} & \longrightarrow & A^\vee(K,\lambda,q,m)/J. \end{array}$$

Moreover, the integer N can be chosen to only depend on n and on $[F^+ : \mathbb{Q}]$.

We will now be able to reduce questions about Galois representations with coefficients in the torsion Hecke algebras $A(K, \lambda, q, m)$ to understanding the properties of the Galois representations with coefficients in the p-torsion free Hecke algebras $\widetilde{A}(\cdots)$ and $\widetilde{A}^\vee(\cdots)$. To do this, we need some results about automorphic Galois representations in characteristic 0.

For the statement of the next proposition, recall that we have introduced Hecke operators $\widetilde{U}_v^k$ for a place $v \in S_p$ in §3.1.

Proposition 4.2.9. *Let $\bar{v}$ be a p-adic place of F^+, let $\mathfrak{m} \subset \mathbb{T}_{w_0^P}^{\overline{\mathcal{Q}}_{\bar{v}}^{w_0^P}, \bar{v}-\mathrm{ord}}$ be a non-Eisenstein maximal ideal in the support of some $H^*(X_K, \mathcal{V}_\lambda)$, and set $\widetilde{\mathfrak{m}} := (\mathcal{S}^{w_0^P})^*(\mathfrak{m})$, a maximal ideal of $\widetilde{\mathbb{T}}^{\overline{\mathcal{Q}}_{\bar{v}}^{w_0^P}, \bar{v}-\mathrm{ord}}$. Fix $v|\bar{v}$ in F and suppose that $\widetilde{U}_v^k \notin \widetilde{\mathfrak{m}}$ for $1 \le k \le t$.*

Assume that π is a cuspidal automorphic representation of $\widetilde{G}(\mathbb{A}_{F^+})$, ι is an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ and π is ι-$\overline{Q}_{\bar{v}}^{w_0^P}$-ordinary of weight $\tilde{\lambda}$. Suppose moreover that the Hecke eigenvalues on $(\iota^{-1}\pi^\infty)^{\widetilde{K}, \overline{\mathcal{Q}}_{\bar{v}}^{w_0^P}-\mathrm{ord}}$ come from a map

$$f : \widetilde{\mathbb{T}}_{\widetilde{\mathfrak{m}}}^{\overline{\mathcal{Q}}_{\bar{v}}^{w_0^P}-\mathrm{ord}} \to \overline{\mathbb{Q}}_p,$$

where the superscript '$\overline{\mathcal{Q}}_{\bar{v}}^{w_0^P} - \mathrm{ord}$' denotes that we replace the local factor $\pi_{\bar{v}}^{\overline{\mathcal{Q}}_{\bar{v}}^{w_0^P}}$ with its one-dimensional $\overline{\mathcal{Q}}_{\bar{v}}^{w_0^P}$-ordinary subspace.

We have the associated p-adic Galois representation $r_\iota(\pi) : G_F \to \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_p)$ (the existence of f implies that we have an isomorphism of semi-simplified reductions $\bar{r}_\iota(\pi) \cong \bar{\rho}_{\widetilde{\mathfrak{m}}}$). Consider the (n, n)-block decomposition

$$r_\iota(\pi)|_{G_{F_{\bar{v}}}} \simeq \begin{pmatrix} r_1(\pi) & * \\ 0 & r_2(\pi) \end{pmatrix}$$

guaranteed by Theorem 3.1.2 (noting that $r_1(\pi), r_2(\pi)$ may be futher decomposed according to the shape of $\overline{\mathcal{Q}}_{\bar{v}}^{w_0^P}$). For $i = 1, 2$, assume that E is large enough that $r_i(\pi)$ can be defined over it, via the embedding $E \hookrightarrow \overline{\mathbb{Q}}_p$ coming from f, and let $\overline{r_i(\pi)}$ be the semi-simplification of the reduction modulo ϖ of $r_i(\pi)$. Then

$$\det \overline{r_1(\pi)}\,(\mathrm{Art}_{F_v}(\varpi_v)) = \det \bar{\rho}_{\mathfrak{m}}(\mathrm{Art}_{F_v}(\varpi_v)) \text{ and}$$

$$\det \overline{r_2(\pi)}\,(\mathrm{Art}_{F_v}(\varpi_v)) = \det (\bar{\rho}_{\mathfrak{m}}^{\vee,c}(1-2n))\,(\mathrm{Art}_{F_v}(\varpi_v)).$$

Proof. By Theorem 3.1.2, $\det r_1(\pi)\,(\mathrm{Art}_{F_{\bar{v}}}(\varpi_{\bar{v}})) \in \mathcal{O}^\times$ is equal to $\epsilon_p^{\frac{n(1-n)}{2}}(\mathrm{Art}_{F_v}(\varpi_v))$ times the eigenvalue of $\widetilde{U}_{v,n}$ acting on the $\overline{\mathcal{Q}}_{\bar{v}}^{w_0^P}$-ordinary subspace of $\iota^{-1}\pi^{\overline{\mathcal{Q}}_{\bar{v}}^{w_0^P}}$. By the description of the map $\mathcal{S}^{w_0^P}$, the reduction of this eigenvalue modulo ϖ is equal to the image of U_v in $\mathbb{T}_{w_0^P}^{\overline{\mathcal{Q}}_{\bar{v}}^{w_0^P}, \bar{v}-\mathrm{ord}}/\mathfrak{m}$. By Lemma 2.1.21, $\det \bar{\rho}_{\mathfrak{m}}(\mathrm{Art}_{F_v}(\varpi_v))$ is equal to $\bar{\epsilon}_p^{\frac{n(1-n)}{2}}(\mathrm{Art}_{F_v}(\varpi_v))$ times the eigenvalue of U_v acting on $H^*(X_K, \mathcal{V}_\lambda/\varpi)_{\mathfrak{m}}$, so we obtain the first equation. For the second equation, let $\overline{r_\iota(\pi)}$ be the semi-simplification of the reduction modulo ϖ of $r_\iota(\pi)$. The same line of reasoning implies that $\det \overline{r_\iota(\pi)}(\mathrm{Art}_{F_v}(\varpi_v))$ is equal to $\bar{\epsilon}_p^{n(1-2n)}(\mathrm{Art}_{F_v}(\varpi_v))$ times the image of $U_v \cdot U_{v^c}^{-1}$ in $\mathbb{T}^{\bar{v}-\mathrm{ord}}/\mathfrak{m}$. We conclude by the first equation and by another application of Lemma 2.1.21. ☐

To proceed, we recall the notion of a CTG weight from [ACC+18, Def. 4.3.5].

Definition 4.2.10. *A weight $\tilde{\lambda} \in (\mathbb{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$ is CTG ("cohomologically trivial for G") if it satisfies the following condition*

- *Given $w \in W^P$, define $\lambda_w = w(\tilde{\lambda} + \rho) - \rho$, viewed as an element of $(\mathbb{Z}_+^n)^{\mathrm{Hom}(F,E)}$ in the usual way. For each $w \in W^P$ and $i_0 \in \mathbb{Z}$, there exists $\tau \in \mathrm{Hom}(F,E)$ such that $\lambda_{w,\tau} - \lambda_{w,\tau c}^\vee \neq (i_0, i_0, \ldots, i_0)$.*

An important application of the CTG assumption is the following variant of [ACC+18, Theorem 2.4.11]:

Proposition 4.2.11. *Let $\mathfrak{m} \subset \mathbb{T}^T$ be a non-Eisenstein maximal ideal. Fix a place $\bar{v} \in \bar{S}_p$ and a standard parabolic $Q_{\bar{v}} \subset P_{\bar{v}}$ and suppose $\widetilde{\mathfrak{m}}$ is a maximal ideal of $\widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{v}},\{\bar{v}\}-\mathrm{ord}}$ which extends $\mathcal{S}^*(\mathfrak{m})$. Let $\widetilde{K} \subset \widetilde{G}(\mathbb{A}_{F^+,f})$ be a good subgroup such that $\widetilde{\mathfrak{m}}$ is in the support of $H^*(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}})^{\mathrm{ord}}$ for a CTG weight $\tilde{\lambda}$. Suppose that $\widetilde{U}_v^k \notin \widetilde{\mathfrak{m}}$ for $1 \leq k \leq t$ (in other words, these Hecke operators act with unit eigenvalues on $H^*(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}}$). Let $d = \frac{1}{2}\dim_{\mathbb{R}} X^{\widetilde{G}} = n^2[F^+ : \mathbb{Q}]$.*

Then $H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}}[1/p]$ is a semisimple $\widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{v}},\{\bar{v}\}-\mathrm{ord}}[1/p]$-module, and for every homomorphism

$$f : \widetilde{\mathbb{T}}^{\mathcal{Q}_{\bar{v}},\{\bar{v}\}-\mathrm{ord}}(H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\tilde{\lambda}})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}}) \to \overline{\mathbb{Q}}_p,$$

and isomorphism $\iota : \overline{\mathbb{Q}}_p \to \mathbb{C}$ there exists a cuspidal automorphic representation π of $\widetilde{G}(\mathbb{A}_{F^+})$ which is ι-$Q_{\bar{v}}$-ordinary of weight $\tilde{\lambda}$ such that f is associated to the Hecke eigenvalues on $(\iota^{-1}\pi^\infty)^{\widetilde{K},\mathcal{Q}_{\bar{v}}-\mathrm{ord}}$, where '$\mathcal{Q}_{\bar{v}} - \mathrm{ord}$' indicates that we replace the local factor $\pi_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}$ with its one-dimensional $\mathcal{Q}_{\bar{v}}$-ordinary subspace.

Proof. This follows from combining (the proof of) [ACC+18, Theorem 2.4.11] with the fact that the $\mathcal{Q}_{\bar{v}}$-ordinary subspace of $\pi_{\bar{v}}^{\mathcal{Q}_{\bar{v}}}$ (which is all that contributes to cohomology localised at $\widetilde{\mathfrak{m}}$) is one-dimensional, which is part of Theorem 3.1.2. ☐

Remark 4.2.12. In Proposition 4.2.6, we may assume that the weight $\tilde{\lambda}$ is CTG, without changing the Hecke algebra $A(K, \lambda, q, m)$. This is because [ACC+18, Lemma 4.3.6] shows that a weight can be ensured to be CTG by modifying it at only one embedding $\tau : F^+ \hookrightarrow E$. Choose a τ which induces the place $\bar{v}'$ of F^+. Because the level at $\bar{v}'$ is assumed to be deep enough in Proposition 4.2.6, we may modify $\tilde{\lambda}_\tau = (-\lambda_{\tilde{\tau}c}, \lambda_{\tilde{\tau}})$ without changing the Hecke algebra $A(K, \lambda, q, m)$.

We can now apply the results of §3 to obtain the main result of this section.

Proposition 4.2.13. *Assume that p splits in an imaginary quadratic subfield of F. Let $K \subset G(\mathbb{A}_{F^+,f}) = \mathrm{GL}_n(\mathbb{A}_{F,f})$ be a good subgroup and fix distinct places $\bar{v}, \bar{v}' \in S_p$. Let λ be a dominant weight for G.*

Let $m \in \mathbb{Z}_{\geq 1}$ be an integer. Fix a standard parabolic $Q_{\bar{v}} \subset P_{\bar{v}}$, suppose that $K_{\bar{v}} = \mathcal{Q}_{\bar{v}} \cap G(F_{\bar{v}}^+)$ and let $\mathfrak{m} \subset \mathbb{T}^{\mathcal{Q}_{\bar{v}},\{\bar{v}-\mathrm{ord}\}}$ be a maximal ideal in the support of $H^*(X_K, \mathcal{V}_\lambda/\varpi^m)$.

Fix $\tilde{v}|\bar{v}$ in F. Using $\iota_{\tilde{v}}$, we identify $Q_{\bar{v}}$ with a standard block-upper-triangular parabolic subgroup of GL_{2n} corresponding to a decomposition $(n_1, \ldots, n_t)$ of $2n$. Suppose that $n = n_1 + \cdots + n_r$.

Assume that:

(1) *We have*

$$\sum_{\substack{\bar{v}''\in\bar{S}_p\\ \bar{v}''\neq\bar{v},\bar{v}'}}[F^+_{\bar{v}''}:\mathbb{Q}_p]\geq\frac{1}{2}[F^+:\mathbb{Q}].$$

(2) $\mathfrak{m}$ *is a non-Eisenstein maximal ideal such that* $\bar{\rho}_{\mathfrak{m}}$ *is decomposed generic.*
(3) *Let* $v\notin T$ *be a finite place, with residue characteristic* l. *Then either* T *contains no* l*-adic places and* l *is unramified in* F, *or there is an imaginary quadratic subfield of* F *in which* l *splits.*
(4) *For all* $\nu\in X_{Q_{\bar{v}}}$, *the Hecke operator* $[K_{\bar{v}}\nu(\varpi_{\bar{v}})K_{\bar{v}}]$ *is not contained in* $\mathfrak{m}$.

Then for each $q\in[0,d-1]$ *there exists an integer* $N\geq 1$, *depending only on* n *and* $[F^+:\mathbb{Q}]$, *a nilpotent ideal* J *of* $\mathbb{T}^{\mathcal{Q}_{\bar{v}},\{\bar{v}-\mathrm{ord}\}}(H^q(X_K,\mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}})$ *with* $J^N=0$ *and a continuous* n*-dimensional representation*

$$\rho_{\mathfrak{m}}:G_{F,T}\to\mathrm{GL}_n(\mathbb{T}^{\mathcal{Q}_{\bar{v}},\{\bar{v}-\mathrm{ord}\}}(H^q(X_K,\mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}})/J)$$

such that the following conditions are satisfied:

(1) *For each place* $v\notin T$ *of* F, *the characteristic polynomial of* $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ *is equal to the image of* $P_v(X)$.
(2) *For* $v|\bar{v}$, *the representation* $\rho_{\mathfrak{m}}|_{G_{F_v}}$ *has a lift to* $\tilde{\rho}_v:G_{F_v}\to\mathrm{GL}_n(\widetilde{A})$, *where* $\widetilde{A}$ *is a finite flat local* $\mathcal{O}$*-algebra equipped with a morphism*

$$f:\widetilde{A}\to\mathbb{T}^{\mathcal{Q}_{\bar{v}},\{\bar{v}-\mathrm{ord}\}}(H^q(X_K,\mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}})/J.$$

(3) *Inverting* p, *the lift* $\tilde{\rho}_v[1/p]$ *is semistable with labelled Hodge–Tate weights* $(\lambda_{\tau,n}<\cdots<\lambda_{\tau,1}+n-1)_{\tau:F_v\hookrightarrow E}$.
(4) *Furthermore, these semistable lifts satisfy*

$$\tilde{\rho}_{\bar{v}}[1/p]\simeq\begin{pmatrix}\tilde{\rho}_{\bar{v},r+1}&*&\cdots&*\\0&\tilde{\rho}_{\bar{v},r+2}&\cdots&*\\0&0&\ddots&*\\0&0&\cdots&\tilde{\rho}_{\bar{v},t}\end{pmatrix}\text{ and }\tilde{\rho}_{\bar{v}^c}[1/p]\simeq\begin{pmatrix}\tilde{\rho}_{\bar{v}^c,r}&*&\cdots&*\\0&\tilde{\rho}_{\bar{v}^c,r-1}&\cdots&*\\0&0&\ddots&*\\0&0&\cdots&\tilde{\rho}_{\bar{v}^c,1}\end{pmatrix}$$

where the representations $\tilde{\rho}_{v,j}:G_{F_v}\to\mathrm{GL}_{n_j}(\widetilde{A}[1/p])$ *are crystalline with labelled Hodge–Tate weights determined by the requirement that they are increasing from top left to bottom right.*
(5) *For* $j=r+1,\ldots,t$, *the characters* $\det\tilde{\rho}_{\bar{v},j}$ *take values in* $\tilde{A}$ *and their image under* f *is given by characters* ψ_j *determined by:*
   - $\prod_{j=r+1}^{k}\psi_j(\mathrm{Art}_{F_{\bar{v}}}(u))=\prod_{i=1}^{n_{r+1}+\cdots+n_k}\prod_{\tau:F_{\bar{v}}\hookrightarrow\overline{\mathbb{Q}}_p}\tau(u)^{-\lambda_{\tau,n-i+1}-i+1}$ *for* $u\in\mathcal{O}_{F_{\bar{v}}}^\times$.
   - $\prod_{j=r+1}^{k}\psi_j(\mathrm{Art}_{F_{\bar{v}}}(\varpi_{\bar{v}}))$ *is equal to* $\epsilon_p^{\sum_{i=1}^{n_{r+1}+\cdots+n_k}(1-i)}(\mathrm{Art}_{F_{\bar{v}}}(\varpi_{\bar{v}}))\widetilde{U}_v^{k-r}$.

Proof. We already know the existence of $\rho_{\mathfrak{m}}$ satisfying the first condition (local–global compatibility at unramified places), so we are free to enlarge T. As explained in the proof of [ACC+18, Corollary 4.4.8], we may assume (applying a twisting argument) that $\overline{\rho}_{\widetilde{\mathfrak{m}}}=\overline{\rho}_{\mathfrak{m}}\oplus\overline{\rho}_{\mathfrak{m}}^{\vee,c}(1-2n)$ is decomposed generic, not just that $\bar{\rho}_{\mathfrak{m}}$ is decomposed generic. We will use a similar twisting argument later in this proof. We can also use Hochschild–Serre to reduce to the case when

$K_{\bar{v}''} \subset \left\{ \begin{pmatrix} 1_n & * \\ 0 & 1_n \end{pmatrix} \bmod \varpi_{\bar{v}''}^m \right\}$ for each $\bar{v}'' \in \overline{S}_p - \{\bar{v}\}$. This means we can moreover assume that $\lambda_{\bar{v}''} = 0$ if $\bar{v}'' \in \bar{S}_p - \{\bar{v}\}$.

Now we let $\widetilde{K} \subset \widetilde{G}(\mathbb{A}_{F^+,f})$ be a good subgroup satisfying:

- $\widetilde{K} \cap G(\mathbb{A}_{F^+,f}) = K$.
- $\widetilde{K}^T = \widetilde{G}(\widehat{\mathcal{O}}_{F^+}^T)$.
- For each $\bar{v}'' \in \overline{S}_p - \{\bar{v}\}$, we have $U(\mathcal{O}_{F^+_{\bar{v}''}}) \subset \widetilde{K}_{\bar{v}''}$ and $\widetilde{K}_{\bar{v}''} \subset \left\{ \begin{pmatrix} 1_n & * \\ 0 & 1_n \end{pmatrix} \bmod \varpi_{\bar{v}''}^m \right\}$.
- $\widetilde{K}_{\bar{v}} = \overline{\mathcal{Q}}_{\bar{v}}^{w_0^P}$ (the corresponding standard parabolic has block sizes $(n_{r+1}, \ldots, n_t, n_1, \ldots, n_r)$).

Next, we use a twisting argument to reduce to the case when $-\lambda_{\tilde{\tau}c,1} - \lambda_{\tau,1} \geq 0$. Indeed, twisting by ϵ_p^μ moves us from the weight λ to the weight $\lambda' := (\lambda_{\tau,1} - \mu, \ldots, \lambda_{\tau,n} - \mu)_{\tau : F \hookrightarrow E}$ (cf. [ACC+18, Proposition 2.2.22]), and we satisfy the desired condition if we take μ to be sufficiently positive.

At this point we assume that $q \geq \lfloor \frac{d}{2} \rfloor$. We will handle small q at the end of the proof using Poincaré duality. We are now in a position to apply Proposition 4.2.6. Following Remark 4.2.12, we are free to modify $\lambda_{\bar{v}'}$ so that the weight $\tilde{\lambda}$ is CTG. We have $A(K, \lambda, q, m) = \mathbb{T}^{\mathcal{Q}_{\bar{v}}, \{\bar{v}-\mathrm{ord}\}}(H^q(X_K, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}})$.

Suppose we have a continuous character $\overline{\psi} : G_F \to k^\times$ (perhaps after extending $\mathcal{O}$), which is unramified at S_p, and let $\psi : G_F \to \mathcal{O}^\times$ denote the Teichmüller lift of $\overline{\psi}$. Choose a finite set $T' \supset T$ (closed under complex conjugation) which also contains all the places where ψ is ramified, and a good normal subgroup $K' \subset K$ satisfying:

- $(K')^{T'-T} = K^{T'-T}$.
- K'/K is abelian of order prime to p.
- For each place v of F, the restriction of $\psi|_{G_{F_v}} \circ \mathrm{Art}_{F_v}$ to $\det(K'_v)$ is trivial.
- T' satisfies assumption (3) from the Proposition.

We will then consider the Hecke algebras for the twist

$$A(K', \lambda, q, m, \psi) := \mathbb{T}^{\mathcal{Q}_{\bar{v}}, \{\bar{v}-\mathrm{ord}\}}(H^q(X_{K'}, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}(\psi)}),$$

(see [ACC+18, §2.2.19] for the definition of $\mathfrak{m}(\psi)$ and note that $\overline{\rho}_{\mathfrak{m}(\psi)} = \overline{\rho}_{\mathfrak{m}} \otimes \overline{\psi}$). Establishing the proposition for any of these twists will imply it for the Hecke algebra $A(K, \lambda, q, m)$. We will always assume that $\overline{\psi}$ is chosen so that $\overline{\rho}_{\widetilde{\mathfrak{m}(\psi)}}$ remains decomposed generic.

We can, twisting by a suitable ψ if necessary, assume that the isomorphism classes of the irreducible constituents of $\overline{\rho}_{\mathfrak{m}}|_{G_{F_{\bar{v}}}}$ are disjoint from those of $\overline{\rho}_{\mathfrak{m}}^{\vee,c}(1-2n)|_{G_{F_{\bar{v}}}}$.

Applying Proposition 4.2.6, for each ψ as above we have a finite flat $\mathcal{O}$-algebra $\widetilde{A}(\psi)$ and a nilpotent ideal J_ψ with a map $f_\psi : \widetilde{A}(\psi) \to A(K', \lambda, q, m, \psi)/J_\psi$. Using Propositions 4.2.9 and 4.2.11, we deduce that there is a $\overline{Q}_{\bar{v}}^{w_0^P}$-ordinary Galois representation $\tilde{\rho}_{\mathfrak{m}(\psi)} = \prod_{i=1}^r \tilde{\rho}^i_{\mathfrak{m}(\psi)} : G_F \to \mathrm{GL}_{2n}(\widetilde{A}(\psi)[1/p]) = \prod_{i=1}^r E$ such that, for each i, the factor $\tilde{\rho}^i_{\mathfrak{m}(\psi)}$ comes with a (n,n)-block decomposition

$$\tilde{\rho}^i_{\mathfrak{m}(\psi)}|_{G_{F_{\bar{v}}}} \simeq \begin{pmatrix} r^i_{1,\psi} & * \\ 0 & r^i_{2,\psi} \end{pmatrix}$$

We can take a semisimplified reduction to the residue field and then, by Proposition 4.2.9, we have

$$\det \overline{r^i_{1,\psi}}(\mathrm{Art}_{F_{\tilde{v}}}(\varpi_{\tilde{v}})) = \det(\overline{\rho}_{\mathfrak{m}} \otimes \overline{\psi})(\mathrm{Art}_{F_{\tilde{v}}}(\varpi_{\tilde{v}})). \tag{4.2.5}$$

Sub-lemma 1. *Possibly after enlarging $\mathcal{O}$, there is a continuous character $\overline{\psi} : G_F \to k^\times$, unramified at S_p and with $\overline{\rho}_{\widetilde{\mathfrak{m}(\psi)}}$ decomposed generic, such that*

(1) *the isomorphism classes of the irreducible constituents of $\overline{\rho}_{\mathfrak{m}(\psi)}|_{G_{F_{\tilde{v}}}}$ are disjoint from those of $\overline{\rho}^{\vee,c}_{\mathfrak{m}(\psi)}(1-2n)|_{G_{F_{\tilde{v}}}}$;*

(2) *for all $1 \le i \le r$ the isomorphism classes of the irreducible constituents of the residual representation $\overline{r^i_{1,\psi}}$ coincide with those of $\overline{\rho}_{\mathfrak{m}(\psi)}|_{G_{F_{\tilde{v}}}}$.*

Proof. We denote the irreducible constituents (with multiplicity) of $\overline{\rho}_{\mathfrak{m}}|_{G_{F_{\tilde{v}}}}$ by $S_1, \ldots, S_m$, and let $d_j = \dim S_j$ and $\delta_j = \det S_j$. The irreducible constituents of $\overline{\rho}^{\vee,c}_{\mathfrak{m}}(1-2n)|_{G_{F_{\tilde{v}}}}$ are given by $T_1, \ldots, T_m$ where $T_j = S_j^{\vee,c}(1-2n)$. For each ψ, the irreducible constituents of $\overline{r^i_{1,\psi}}$ are given by a multiset $\{S_j \otimes \bar{\psi} : j \in I_i\} \coprod \{T_j \otimes \bar{\psi}^{\vee,c} : j \in J_i\}$ for two subsets $I_i, J_i \subset \{1, \ldots, m\}$ with $\sum_{j\in I_i} d_j + \sum_{j \in J_i} d_j = n$.

Now comparing what this entails for $\det \overline{r^i_{1,\psi}}(\mathrm{Art}_{F_{\tilde{v}}}(\varpi_{\tilde{v}}))$ with the formula (4.2.5), we get:

$$\left(\bar{\psi}^n \prod_{j=1}^m \delta_j\right)(\mathrm{Art}_{F_{\tilde{v}}}(\varpi_{\tilde{v}})) = \left(\prod_{j\in I_i} \bar{\psi}^{d_j}\delta_j \prod_{j \in J_i} (\bar{\psi}^{\vee,c})^{d_j} \delta_j^{\vee,c} \bar{\epsilon}_p^{d_j(1-2n)}\right)(\mathrm{Art}_{F_{\tilde{v}}}(\varpi_{\tilde{v}}))$$

which rearranges to

$$\begin{aligned}(\bar{\psi}^{\vee,c})^{\sum_{j\in J_i} d_j}(\mathrm{Art}_{F_{\tilde{v}}}(\varpi_{\tilde{v}})) &= \left(\bar{\psi}^{n - \sum_{j \in I_i} d_j} \prod_{j=1}^m \delta_j \prod_{j\in I_i} \delta_j^\vee \prod_{j \in J_i} \delta_j^c \bar{\epsilon}_p^{d_j(2n-1)}\right)(\mathrm{Art}_{F_{\tilde{v}}}(\varpi_{\tilde{v}})) \\ &= \left(\bar{\psi}^{n-\sum_{j\in I_i} d_j} \cdot \delta(I_i, J_i)\right)(\mathrm{Art}_{F_{\tilde{v}}}(\varpi_{\tilde{v}})),\end{aligned}$$

where the character $\delta(I_i, J_i)$ only depends on the (finitely many) possible choices of I_i and J_i. Now we can choose $\overline{\psi}$ so that any equation of this form forces $\sum_{j\in J_i} d_j = 0$ and $\sum_{j \in I_i} d_j = n$ (whilst also preserving the first condition of the sub-lemma). Indeed, since Grunwald–Wang allows us to find $\overline{\psi}$ with specified behaviour at any finite set of places, we can choose $\overline{\psi}$ locally trivial at a prime which is decomposed generic for $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ and the pair $\overline{\psi}(\mathrm{Art}_{F_{\tilde{v}}}(\varpi_{\tilde{v}})), \overline{\psi}(\mathrm{Art}_{F_{\tilde{v}^c}}(\varpi_{\tilde{v}^c}))$ arbitrary in $(\overline{\mathbb{F}}_p^\times)^2$. We choose this pair of elements with orders bigger than n, coprime to each other, and coprime to the orders of the elements $\delta(I_i, J_i)(\mathrm{Art}_{F_{\tilde{v}}}(\varpi_{\tilde{v}}))$.

We conclude that $J_i = \emptyset$ and $I_i = \{1, \ldots, m\}$ for every i, so in other words the isomorphism classes of the irreducible constituents of the residual representation $\overline{r^i_{1,\psi}}$ coincide with those of $\overline{\rho}_{\mathfrak{m}(\psi)}|_{G_{F_{\tilde{v}}}}$. ☐

We may now assume that the isomorphism classes of the irreducible constituents of $\overline{\rho}_{\mathfrak{m}}|_{G_{F_{\tilde{v}}}}$ are disjoint from those of $\overline{\rho}^{\vee,c}_{\mathfrak{m}}(1-2n)|_{G_{F_{\tilde{v}}}}$ and that for all $1 \le i \le r$ the isomorphism classes of the irreducible constituents of the residual representation $\overline{r^i_1}$ coincide with those of $\overline{\rho}_{\mathfrak{m}}|_{G_{F_{\tilde{v}}}}$. Applying Proposition 3.2.4 and Theorem 3.1.2 we deduce the statement of the Proposition for the Hecke algebra $A(K, \lambda, q, m)$.

It remains to handle the case $q < \lfloor \frac{d}{2} \rfloor$. Poincaré duality gives an isomorphism:

$$\iota : A(K, \lambda, q, m) \cong A^\vee(K, \lambda, d-1-q, m).$$

We can now run the same argument as above, using Proposition 4.2.8. Since the Satake map in the dual degree shifting is untwisted, in this case we will have $2n$-dimensional $Q_{\bar{v}}$-ordinary representations $\tilde{\rho}_{\mathfrak{m}^\vee}$ lifting

$$\overline{\rho}_{\tilde{\iota}^* \mathcal{S}^*(\mathfrak{m}^\vee)} = \overline{\rho}_{\mathfrak{m}}(-n) \oplus \overline{\rho}_{\mathfrak{m}}^{\vee,c}(1-n)$$

with a decomposition

$$\tilde{\rho}_{\mathfrak{m}^\vee}|_{G_{F_{\tilde{v}}}} \simeq \begin{pmatrix} r_1 & * \\ 0 & r_2 \end{pmatrix}$$

such that the *lower right* block r_2 lifts $\overline{\rho}_{\mathfrak{m}}(-n)$. This is compatible with the following: in the dual case, we twist by a sufficiently *negative* power of cyclotomic to arrange that $\tilde{\lambda} = w_0^P \lambda$ is dominant for $\widetilde{G}$ (using our standard identification of weights for G and $\widetilde{G}$). Then $\tilde{\rho}_{\mathfrak{m}^\vee}$ is cohomological of weight $w_0^P \lambda$, so the τ-labelled Hodge–Tate weights of $\tilde{\rho}_{\mathfrak{m}^\vee}|_{G_{F_{\tilde{v}}}}$ are given by

$$-\lambda_{\tau c,1} < \cdots < -\lambda_{\tau c,n} + n - 1 < \lambda_{\tau,n} + n < \cdots < \lambda_{\tau,1} + 2n - 1.$$

In particular, the Hodge–Tate weights of $r_2(n)$ are as expected. This completes the proof for all values of q. □

4.2.14. *Local–global–compatibility using deformation rings.* We formulate a consequence of Proposition 4.2.13 in terms of Galois deformation rings. The local deformation rings we need were defined in §3.3.

Theorem 4.2.15. *Suppose that F is an imaginary CM field that contains an imaginary quadratic field. Let p be a prime which splits in an imaginary quadratic subfield of F. Let T be a finite set of finite places of F, which contains S_p and which is stable under complex conjugation, and such that the following condition is satisfied:*

- *Let $v \notin T$ be a finite place, with residue characteristic ℓ. Then either T contains no ℓ-adic places and ℓ is unramified in F, or there exists an imaginary quadratic subfield of F in which ℓ splits.*

Let $K \subset G(\mathbb{A}_{F^+,f}) = \mathrm{GL}_n(\mathbb{A}_{F,f})$ be a good subgroup with $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ $\forall v \notin T$. Fix distinct places $\bar{v}, \bar{v}' \in S_p$. Let λ be a dominant weight for G.

We fix a standard parabolic $Q_{\bar{v}} \subset P_{\bar{v}}$ and suppose we are in one of three cases:

(cr-ord) *$\iota_{\bar{v}}(Q_{\bar{v}})$ is the standard parabolic given by the partition $(n, 1, \ldots, 1)$ of $2n$.*
(ord) *$\iota_{\bar{v}}(Q_{\bar{v}}) = \mathrm{B}_{2n}$*
(cr) *$Q_{\bar{v}} = P_{\bar{v}}$.*

Suppose that $K_{\bar{v}} = \mathcal{Q}_{\bar{v}} \cap G(F_{\bar{v}}^+)$ and let $\mathfrak{m} \subset \mathbb{T}^{\mathcal{Q}_{\bar{v}},\{\bar{v}-\mathrm{ord}\}}$ be a maximal ideal in the support of $H^*(X_K, \mathcal{V}_\lambda)$.

Assume that:

(1) *We have*

$$\sum_{\substack{\bar{v}'' \in \bar{S}_p \\ \bar{v}'' \neq \bar{v}, \bar{v}'}} [F_{\bar{v}''}^+ : \mathbb{Q}_p] \geq \frac{1}{2}[F^+ : \mathbb{Q}].$$

(2) *$\mathfrak{m}$ is a non-Eisenstein maximal ideal such that $\bar{\rho}_{\mathfrak{m}}$ is decomposed generic.*
(3) *For all $\nu \in X_{Q_{\bar{v}}}$, the Hecke operator $[K_{\bar{v}} \nu(\varpi_{\bar{v}}) K_{\bar{v}}]$ is not contained in $\mathfrak{m}$.*

Then there exists an integer $N \geq 1$, depending only on n and $[F^+ : \mathbb{Q}]$, a nilpotent ideal J of $\mathbb{T}^{\mathcal{Q}_{\bar{v}},\{\bar{v}-\mathrm{ord}\}}(R\Gamma(X_K, \mathcal{V}_\lambda)_\mathfrak{m})$ with $J^N = 0$ and a continuous n-dimensional representation

$$\rho_\mathfrak{m} : G_{F,T} \to \mathrm{GL}_n(\mathbb{T}^{\mathcal{Q}_{\bar{v}},\{\bar{v}-\mathrm{ord}\}}(R\Gamma(X_K, \mathcal{V}_\lambda)_\mathfrak{m})/J)$$

satisfying

- *For each place $v \notin T$ of F, $\rho_\mathfrak{m}$ is unramified and the characteristic polynomial of $\rho_\mathfrak{m}(\mathrm{Frob}_v)$ is equal to the image of $P_v(X)$.*

Moreover, the induced map $t_{\rho_\mathfrak{m}} : R^{\square}_{\overline{\rho}_\mathfrak{m}} \to \mathbb{T}^{\mathcal{Q}_{\bar{v}},\{\bar{v}-\mathrm{ord}\}}(R\Gamma(X_K, \mathcal{V}_\lambda)_\mathfrak{m})/J$ has the following property in each of our three cases:

(cr-ord) *The restriction of $t_{\rho_\mathfrak{m}}$ to $R^{\square}_{\overline{\rho}_\mathfrak{m}|_{G_{F_{\tilde{v}}}}}$ factors through $R^{\triangle,\lambda_{\tilde{v}}}_{\overline{\rho}_\mathfrak{m}|_{G_{F_{\tilde{v}}}}}$ and the restriction to $R^{\square}_{\overline{\rho}_\mathfrak{m}|_{G_{F_{\tilde{v}^c}}}}$ factors through $R^{\mathrm{cris},\lambda_{\tilde{v}^c}}_{\overline{\rho}_\mathfrak{m}|_{G_{F_{\tilde{v}^c}}}}$.*

(ord) *For $v|\bar{v}$, restriction of $t_{\rho_\mathfrak{m}}$ to $R^{\square}_{\overline{\rho}_\mathfrak{m}|_{G_{F_v}}}$ factors through $R^{\triangle,\lambda_v}_{\overline{\rho}_\mathfrak{m}|_{G_{F_v}}}$.*

(cr) *For $v|\bar{v}$, restriction of $t_{\rho_\mathfrak{m}}$ to $R^{\square}_{\overline{\rho}_\mathfrak{m}|_{G_{F_v}}}$ factors through $R^{\mathrm{cris},\lambda_v}_{\overline{\rho}_\mathfrak{m}|_{G_{F_v}}}$.*

Proof. The first point is that is enough to prove our statement for Galois representations with coefficients in $\mathbb{T}^{\mathcal{Q}_{\bar{v}},\{\bar{v}-\mathrm{ord}\}}(R\Gamma(X_K, \mathcal{V}_\lambda/\varpi^m)_\mathfrak{m})$ for integers $m \in \mathbb{Z}_{\geq 1}$. This is because we have an isomorphism (by [NT16, Lemma 3.11])

$$\mathbb{T}^{\mathcal{Q}_{\bar{v}},\{\bar{v}-\mathrm{ord}\}}(R\Gamma(X_K, \mathcal{V}_\lambda)_\mathfrak{m}) \xrightarrow{\sim} \varprojlim_m \mathbb{T}^{\mathcal{Q}_{\bar{v}},\{\bar{v}-\mathrm{ord}\}}(R\Gamma(X_K, \mathcal{V}_\lambda/\varpi^m)_\mathfrak{m}).$$

In fact, we can prove the statement one cohomological degree at a time (cf. the proof of [ACC+18, Theorem 4.5.1]). Indeed, the kernel of the map

$$\mathbb{T}^{\mathcal{Q}_{\bar{v}},\{\bar{v}-\mathrm{ord}\}}(R\Gamma(X_K, \mathcal{V}_\lambda/\varpi^m)_\mathfrak{m}) \to \prod_q \mathbb{T}^{\mathcal{Q}_{\bar{v}},\{\bar{v}-\mathrm{ord}\}}(H^q(X_K, \mathcal{V}_\lambda/\varpi^m)_\mathfrak{m})$$

is a nilpotent ideal with vanishing dth power, and a Galois representation with coefficients in a quotient of $\prod_q \mathbb{T}^{\mathcal{Q}_{\bar{v}},\{\bar{v}-\mathrm{ord}\}}(H^q(X_K, \mathcal{V}_\lambda/\varpi^m)_\mathfrak{m})$ which satisfies condition (4.2.15) can be conjugated to take values in the image of $\mathbb{T}^{\mathcal{Q}_{\bar{v}},\{\bar{v}-\mathrm{ord}\}}(R\Gamma(X_K, \mathcal{V}_\lambda/\varpi^m)_\mathfrak{m})$ (by Carayol's lemma, cf. the proof of [ACC+18, Proposition 4.4.8]).

Now our statement follows from Proposition 4.2.13. This produces a lift of the map $R^{\square}_{\overline{\rho}_\mathfrak{m}|_{G_{F_v}}} \to \mathbb{T}^{\mathcal{Q}_{\bar{v}},\{\bar{v}-\mathrm{ord}\}}(H^q(X_K, \mathcal{V}_\lambda/\varpi^m)_\mathfrak{m})/J$ to a map with target a finite flat local $\mathcal{O}$-algebra $R^{\square}_{\overline{\rho}_\mathfrak{m}|_{G_{F_v}}} \to \widetilde{A}$. This map factors through the appropriate (crystalline or ordinary) quotient by the characterising property of this quotient. $\square$

We also need a small refinement which will help us in a 'fixed determinant' setting:

Corollary 4.2.16. *In the setting of Theorem 4.2.15, assume moreover that $p \nmid n$ and we have a quotient map $f : \mathbb{T}^{\mathcal{Q}_{\bar{v}},\{\bar{v}-\mathrm{ord}\}}(R\Gamma(X_K, \mathcal{V}_\lambda)_\mathfrak{m})/J \to A$ such that $\det(f_*(\rho_\mathfrak{m})) = \psi$ for a character $\psi : G_{F,T} \to \mathcal{O}^\times$ which is crystalline at all places in S_p with τ-labelled Hodge–Tate weights $\sum_{i=1}^n \lambda_{\tau,i} + (n-i)$ for each $\tau : F \hookrightarrow E$.*

Then for $v|\bar{v}$ the induced map $R^{\triangle,\lambda_v}_{\overline{\rho}_\mathfrak{m}|_{G_{F_v}}} \to A$ or $R^{\mathrm{cris},\lambda_v}_{\overline{\rho}_\mathfrak{m}|_{G_{F_v}}} \to A$ factors through the appropriate fixed determinant ψ lifting ring (cf. §3.3.5).

Proof. This follows from Lemma 3.3.6. $\square$

4.3. **The characteristic 0 case.** For simplicity, we restrict to the crystalline case here. For this subsection, we drop our running assumption that F contains an imaginary quadratic field.

Theorem 4.3.1. *Suppose π is cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, regular algebraic of weight λ, with F a totally real or CM field. Let v be a place of F dividing p and suppose $\pi^{\mathrm{GL}_n(\mathcal{O}_{F_v})}$ and $\pi^{\mathrm{GL}_n(\mathcal{O}_{F_{v^c}})}$ are both non-zero. (We allow the possibility that $v = v^c$, even in the CM case.) Let $\iota : \overline{\mathbb{Q}}_p \to \mathbb{C}$ be an isomorphism, and consider the continuous semisimple representation $r_\iota(\pi) : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ constructed in* [HLTT16]. *Assume that:*

(1) *$\overline{r_\iota(\pi)}$ is irreducible and decomposed generic.*

Then $r_\iota(\pi)|_{G_{F_v}}$ and $r_\iota(\pi)|_{G_{F_{v^c}}}$ are crystalline with τ-labelled Hodge–Tate weights $\lambda_{\iota\tau,n} < \cdots < \lambda_{\iota\tau,1} + n - 1$ for $\tau : F \to \overline{\mathbb{Q}}_p$ inducing v or v^c respectively.

Proof. Fix a prime ℓ such that $\overline{r_\iota(\pi)}$ satisfies the decomposed generic condition at ℓ. Using cyclic base change and Theorem 4.2.15, it suffices to find a cyclic CM extension F'/F with the following properties:

(1) F' is linearly disjoint from $\overline{F}^{\ker \overline{r_\iota(\pi)}}$ over F.
(2) F' contains an imaginary quadratic field.
(3) Every p-adic place of $(F')^+$ splits in F'.
(4) ℓ splits completely in F'.
(5) The places v, v^c split completely in F'.
(6) There is a place $\bar{w}$ of $(F')^+$, lying over the place $\bar{v}|v$ of F^+, and another p-adic place $\bar{w}'$ of $(F')^+$ such that

$$\sum_{\substack{\bar{w}''|p \text{ in } (F')^+ \\ \bar{w}'' \neq \bar{w}, \bar{w}'}} [(F')^+_{\bar{w}''} : \mathbb{Q}_p] \geq \frac{1}{2}[(F')^+ : \mathbb{Q}].$$

We can achieve the final property by choosing F' with $[(F')^+ : F^+] \geq 4$ and with $\bar{v}$ split completely in $(F')^+$, and then choosing $\bar{w}, \bar{w}'$ to be two distinct places of $(F')^+$ lying over $\bar{v}$. We conclude that it is possible to find such an extension F'. □

5. Automorphy lifting

5.1. **A potentially Barsotti–Tate modularity lifting theorem.** We begin by stating the main theorem in this section. In order to get an optimal result for applications to modularity of elliptic curves, we only consider Galois representations with inverse-cyclotomic determinant.

Theorem 5.2. *Let F be an imaginary CM field and let p be an odd prime. Suppose given a continuous representation $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ satisfying the following conditions:*

(1) *ρ is unramified almost everywhere and $\det(\rho) = \epsilon_p^{-1}$.*
(2) *For each place $v|p$ of F, the representation $\rho|_{G_{F_v}}$ is potentially semistable with all labelled Hodge–Tate weights equal to $(0, 1)$.*
(3) *$\overline{\rho}$ is decomposed generic (Definition 2.1.27) and $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible.*
(4) *If $p = 5$ and the projective image of $\overline{\rho}(G_{F(\zeta_5)})$ is conjugate to $\mathrm{PSL}_2(\mathbb{F}_5)$, we assume further that the extension of F cut out by the projective image of $\overline{\rho}$ does not contain ζ_5.*

(5) *There exists a cuspidal automorphic representation* π *of* $\mathrm{PGL}_2(\mathbf{A}_F)$ *and an isomorphism* $\iota : \overline{\mathbb{Q}}_p \cong \mathbb{C}$ *satisfying the following conditions:*
   (a) π *is regular algebraic of weight* 0.
   (b) *For each place* $v|p$ *where* $\rho|_{G_{F_v}}$ *is potentially crystalline,* $r_\iota(\pi)|_{G_{F_v}}$ *is potentially ordinary of weight* 0 *(in the sense of* [Ger19, §5.2]*) if and only if* $\rho|_{G_{F_v}}$ *is potentially ordinary of weight* 0*. We moreover assume that* $\mathrm{rec}_{F_v}(\pi_v)$ *has monodromy operator* 0.
   (c) *For each place* $v|p$ *where* $\rho|_{G_{F_v}}$ *is not potentially crystalline,* π *is* ι*-ordinary of weight* 0 *at* v *and* $r_\iota(\pi)|_{G_{F_v}}$ *is not potentially crystalline.*

Then ρ *is automorphic: there exists a cuspidal automorphic representation* Π *of* $\mathrm{PGL}_2(\mathbf{A}_F)$*, regular algebraic of weight* 0*, such that* $\rho \cong r_\iota(\Pi)$.

Remark 5.2.1. Using Theorem 4.2.15, we can replace the assumption that $r_\iota(\pi)|_{G_{F_v}}$ is potentially ordinary of weight 0 at certain places v with an assumption on π_v.

Remark 5.2.2. We restrict to Galois representations with inverse-cyclotomic determinant, as in [AKT19], so that we can handle the case where $p = 3$ and the image of $\overline{\rho}(G_{F(\zeta_3)})$ is $\mathrm{SL}_2(\mathbb{F}_3)$.

Remark 5.2.3. We could consider also the Fontaine–Laffaille case, for arbitrary dimension n Galois representations – namely, assume that p is unramified in F and that the weight λ satisfies the condition:

$$\lambda_{\tau,1} - \lambda_{\tau,n} < p - n \text{ for all } \tau \in \mathrm{Hom}(F, E). \tag{5.2.1}$$

Using the local-global compatibility result given by Theorem 4.2.15 instead of Theorem [ACC+18, Theorem 4.5.1], one could also prove an automorphy lifting theorem in this case that strenghtens [ACC+18, Theorem 6.1.1]. The restrictions on the Hodge–Tate weights in the automorphy lifting theorem in *loc. cit.* are stronger than the condition in (5.2.1) only because of restrictions in the corresponding result on local-global compatibility. The rest of the argument would go through verbatim.

5.3. **Galois deformation theory.** To prove our automorphy lifting theorem we apply the patching method in [ACC+18, §6], making modifications as in [AKT19] to avoid the assumption that there is a $\sigma \in G_F - G_{F(\zeta_p)}$ such that $\overline{\rho}(\sigma)$ is scalar and to include cases with $p = 3$ or 5 where $\overline{\rho}|_{G_{F(\zeta_p)}}$ does not have enormous image. In addition, we work with local lifting rings at p that have two irreducible components (either ordinary/non-ordinary or crystalline ordinary/non-crystalline ordinary), which is why we need assumption (5b) in this theorem. The fact that these local lifting rings have generically reduced special fibre, which we will recall shortly, is important for implementing (derived) Ihara avoidance in a situation where local lifting rings have more than one component.

We adopt all the terminology and notation of [ACC+18, §6.2.1], although the coefficient ring 'Λ' appearing there will always be $\mathcal{O}$ for us, and our Galois representations will all be two-dimensional. So, we fix a continuous and absolutely irreducible $\overline{\rho} : G_F \to \mathrm{GL}_2(k)$, and let S be a finite set of finite places of F containing S_p and all the places where $\overline{\rho}$ is ramified.

For each $v \in S$, we have a local lifting ring $R_v^\square$ for $\overline{\rho}_v := \overline{\rho}|_{G_{F_v}}$ and the notion of a local deformation problem $\mathcal{D}_v$: a set valued functor on $\mathrm{CNL}_\mathcal{O}$ satisfying some conditions which in particular imply that it is represented by a quotient R_v of $R_v^\square$.

A global deformation problem is a tuple

$$(\overline{\rho}, S, \{R_v\}_{v\in S}),$$

where for each $v \in S$, R_v is a quotient of $R_v^\square$ representing a local deformation problem for $\overline{\rho}_v$.

A global deformation problem with fixed determinant is a tuple

$$(\overline{\rho}, \psi, S, \{R_v\}_{v\in S}),$$

where $\psi : G_{F,S} \to \mathcal{O}^\times$ is a character which lifts $\det(\overline{\rho})$, $(\overline{\rho}, S, \{R_v\}_{v\in S})$ is a global deformation problem, and the lifts parameterized by each R_v have determinant $\psi|_{G_{F_v}}$.

If $\mathcal{S}$ is a global deformation problem (with or without fixed determinant), and T is a subset of S, we have the functor $\mathcal{D}_{\mathcal{S}}^T$ of T-framed deformations of type $\mathcal{S}$. When $T = \emptyset$, we denote the functor (of deformations of type $\mathcal{S}$) by $\mathcal{D}_{\mathcal{S}}$.

The functors $\mathcal{D}_{\mathcal{S}}$ and $\mathcal{D}_{\mathcal{S}}^T$ are represented by $\mathrm{CNL}_{\mathcal{O}}$-algebras $R_{\mathcal{S}}$ and $R_{\mathcal{S}}^T$, respectively.

For a global deformation problem $\mathcal{S} = (\overline{\rho}, S, \{R_v\}_{v\in S})$ (or one with fixed determinant, $\mathcal{S} = (\overline{\rho}, \psi, S, \{R_v\}_{v\in S})$) and $T \subset S$, we define $R_{\mathcal{S}}^{T,\mathrm{loc}} = \widehat{\otimes}_{v\in T} R_v$. There is a natural local $\mathcal{O}$-algebra map $R_{\mathcal{S}}^{T,\mathrm{loc}} \to R_{\mathcal{S}}^T$.

We are assuming that p is odd. We will make use of the following local deformation problems (which we identify in terms of their representing ring), where ψ always denotes a fixed determinant character.

- Fixed determinant lifting rings R_v^ψ, parameterizing lifts of $\overline{\rho}|_{G_{F_v}}$ with determinant $\psi|_{G_{F_v}}$. We will make use of these rings when $v \notin S_p$ and $H^2(F_v, \mathrm{ad}^0\overline{\rho}) = 0$, in which case R_v^ψ is formally smooth over $\mathcal{O}$ of relative dimension 3.
- Fixed determinant 'level raising' lifting rings $R_v^{\psi,\chi}$, for v with $q_v \equiv 1 \mod p$ and $\overline{\rho}|_{G_{F_v}}$ trivial, and a character $\chi : \mathcal{O}_{F_v}^\times \to \mathcal{O}^\times$ which is trivial modulo ϖ (cf. [AKT19, §A.1.2]). They classify lifts ρ with determinant ψ and characteristic polynomial

  $$\mathrm{char}_{\rho(\sigma)}(X) = (X - \chi(\mathrm{Art}_{F_v}^{-1}(\sigma)))(X - \chi^{-1}\psi(\mathrm{Art}_{F_v}^{-1}(\sigma)))$$

  for all $\sigma \in I_{F_v}$.
- Barsotti–Tate lifting rings $R_v^{\psi,\mathrm{BT}}$ for $v|p$. These are the rings $R_{\overline{\rho}_v}^{\mathrm{cris},(0,0)_{\tau\in\mathrm{Hom}(F_v,E)},\psi}$ in the notation of §3.3. Note that we assume that $\psi|_{G_{F_v}}$ is crystalline with all labelled Hodge–Tate weights equal to 1 and that it lifts $\det \overline{\rho}_v$ for this ring to be defined. In practice, we will take ψ to be the inverse of the cyclotomic character.

5.3.1. *Ordinary deformation rings.* We will also use, when $v \mid p$, the ordinary lifting ring $R_v^{\triangle,(0,0)_{\tau\in\mathrm{Hom}(F_v,E)},\psi}$ with fixed determinant and Hodge–Tate weights $(0, 1)$. For simplicity, we only consider the case $\psi = \epsilon_p^{-1}$, with $\overline{\rho}|_{G_{F_v}}$ trivial and $\overline{\epsilon}_p$ trivial on G_{F_v}, and set

$$R_v^\triangle := R_v^{\triangle,(0,0)_{\tau\in\mathrm{Hom}(F_v,E)},\psi}$$

in the notation of §3.3.

We recall some properties of $R_v^\triangle$, following [Sno18]. Note that twisting by the cyclotomic character and using the reducedness of $R_v^\triangle$ (Theorem 3.3.3) shows that our lifting ring can indeed be identified with the ring denoted by R in [Sno18, Proposition 4.3.2].

Proposition 5.3.2. (1) $\mathrm{Spec}(R_v^{\triangle})$ *is equidimensional of dimension* $[F_v : \mathbb{Q}_p]+4$*, with two irreducible components* $X^{cr} = \mathrm{Spec}(R_v^{\triangle,cr}), X^{st} = \mathrm{Spec}(R_v^{\triangle,st})$ *characterized by their points valued in finite extensions* E'/E*:*

- $x : R_v^{\triangle} \to E'$ *factors through* $R_v^{\triangle,cr}$ *if and only if* ρ_x *is crystalline.*
- $x : R_v^{\triangle} \to E'$ *factors through* $R_v^{\triangle,st}$ *if and only if* ρ_x *is conjugate to a representation of the form* $\begin{pmatrix} 1 & * \\ 0 & \epsilon_p^{-1} \end{pmatrix}$.

(2) *Each generic point of* $\mathrm{Spec}(R_v^{\triangle}/\varpi)$ *is the specialization of a unique generic point of* $\mathrm{Spec}(R_v^{\triangle})$.

Proof. The first part follows from [Sno18, Proposition 4.3.2]. The second part also follows from Snowden's results, as we now explain. It suffices to show that the dimension of $X^{cr} \cap X^{st} \cap \mathrm{Spec}(R_v^{\triangle}/\varpi)$ is $< [F_v : \mathbb{Q}_p] + 3$, as this shows that there is no point of large enough dimension to be a generic point of $\mathrm{Spec}(R_v^{\triangle}/\varpi)$ generalizing to both generic points of $\mathrm{Spec}(R_v^{\triangle})$.

Snowden defines another ring $\widetilde{R}_v^{\triangle}$ [Sno18, Proposition 4.4.3]. The ring $\widetilde{R}_v^{\triangle}$ is a quotient ($\mathcal{O}$-flat and reduced) of the finite $R_v^{\square,\epsilon_p^{-1}}$-algebra given by adjoining a root of the characteristic polynomial of a lift of Frobenius under the universal lifting of $\overline{\rho}|_{G_{F_v}}$. It comes with a finite morphism $\pi : \mathrm{Spec}(\widetilde{R}_v^{\triangle}) \to \mathrm{Spec}(R_v^{\triangle})$ which is an isomorphism after inverting p. In particular, π is surjective and induces a bijection between irreducible components. We denote the irreducible components of $\mathrm{Spec}(\widetilde{R}_v^{\triangle})$ lying over X^{cr} and X^{st} by $\widetilde{X}^{cr}$ and $\widetilde{X}^{st}$ respectively. Morevoer, π induces an isomorphism $\widetilde{X}^{st} \cong X^{st}$. From this, we deduce that π induces a finite surjective map $\widetilde{X}^{st} \cap \widetilde{X}^{cr} \to X^{st} \cap X^{cr}$ and hence a finite surjective map $\widetilde{X}^{st} \cap \widetilde{X}^{cr} \cap \mathrm{Spec}(\widetilde{R}_v^{\triangle}/\varpi) \to X^{cr} \cap X^{st} \cap \mathrm{Spec}(R_v^{\triangle}/\varpi)$. So we can bound the dimension of the target of this map by bounding the dimension of the source. Snowden explicitly describes the mod ϖ fibre $\mathrm{Spec}(\widetilde{R}_v^{\triangle}/\varpi)$ [Sno18, Theorem 4.6.1, Lemma 4.6.4]. Its irreducible components are the mod ϖ fibres of $\widetilde{X}^{st}$ and $\widetilde{X}^{cr}$, and they intersect in a proper closed subset (of dimension $[F_v : \mathbb{Q}_p] + 2$). ☐

Lemma 5.3.3. *Each generic point of* $\mathrm{Spec}(R_v^{\epsilon_p^{-1},\mathrm{BT}}/\varpi)$ *is the specialization of a unique generic point of* $\mathrm{Spec}(R_v^{\epsilon_p^{-1},\mathrm{BT}})$.

Proof. We let $R = R_v^{\epsilon_p^{-1},\mathrm{BT}}$. The lemma follows from generic reducedness of $\mathrm{Spec}(R/\varpi)$ and the fact that every generic point of $\mathrm{Spec}(R)$ has characteristic 0. The generic reducedness follows from [CEGS, Theorem 1.3]. This proposition applies to the lifting ring $R_v^{\mathrm{BT}} = R_v^{\psi,(0,1)_{\tau\in\mathrm{Hom}(F_v,\overline{\mathbb{Q}}_p)}}$ without fixed determinant, but since p is odd R_v^{BT} is formally smooth over R. So we deduce that R is also generically reduced. Every generic point of $\mathrm{Spec}(R)$ has characteristic 0 because R is $\mathcal{O}$-flat. To deduce the claim about generic points of $\mathrm{Spec}(R/\varpi)$, let $\mathfrak{p} \in \mathrm{Spec}(R)$ be the image of a generic point of $\mathrm{Spec}(R/\varpi)$. Since $\mathrm{Spec}(R/\varpi)$ is generically reduced, $R_{\mathfrak{p}}/\varpi R_{\mathfrak{p}} = (R/\varpi)_{\mathfrak{p}}$ is a field. Now we know that $R_{\mathfrak{p}}$ is a Noetherian local ring with a principal maximal ideal; it is therefore a local principal ideal ring with non-zero ideals generated by powers of ϖ. Since $\mathfrak{p}$ is not a generic point of $\mathrm{Spec}(R)$, $R_{\mathfrak{p}}$ has dimension 1 and (0) is its unique minimal prime (in particular, $R_{\mathfrak{p}}$ is a DVR). ☐

Lemma 5.3.4. *Let* v *be a place of* F *with* $v|p$*. Suppose that* $\overline{\rho}|_{G_{F_v}}$ *is trivial, the residue field* k_v *is not equal to* $\mathbb{F}_p$*, and* $R_v^{\epsilon_p^{-1},\mathrm{BT}}$ *is non-zero. Then* $\mathrm{Spec}(R_v^{\epsilon_p^{-1},\mathrm{BT}})$

has exactly two irreducible components, one whose points correspond to ordinary Galois representations and one whose points correspond to non-ordinary Galois representations.

Proof. This follows from results of Kisin [Kis09, Corollary 2.5.16] and Gee [Gee06, Proposition 2.3]. ☐

Definition 5.3.5. *A Taylor–Wiles datum for a global deformation problem $\mathcal{S}$ is a tuple $(Q, N, (\alpha_{v,1}, \alpha_{v,2})_{v \in Q})$ consisting of:*

- *A finite set of finite places Q of F, disjoint from S, and a positive integer N such that $q_v \equiv 1 \mod p^N$ for each $v \in Q$.*
- *For each $v \in Q$ and $i \in \{1, 2\}$, distinct unramified k-valued characters $\alpha_{v,1}, \alpha_{v,2}$ such that $\overline{\rho}|_{G_{F_v}} \cong \bigoplus_{i=1}^{2} \alpha_{v,i}$.*

We call N the level of the Taylor–Wiles datum.

If $(Q, N, (\alpha_{v,1}, \alpha_{v,2})_{v \in Q})$ is a Taylor–Wiles datum for $\mathcal{S}$, then we define a new global deformation problem

$$\mathcal{S}_Q = (\overline{\rho}, S \cup Q, \{R_v\}_{v \in S} \cup \{R_v^{\square}\}_{v \in Q})$$

(respectively, $\mathcal{S}_Q = (\overline{\rho}, \psi, S \cup Q, \{R_v\}_{v \in S} \cup \{R_v^{\psi}\}_{v \in Q})$ if $\mathcal{S}$ has fixed determinant).

5.4. **Patching.** Our set-up is very close to that of [ACC⁺18, §6]. First we give an axiomatic description of the kinds of objects which will be the output of the patching method, and deduce a formal modularity lifting result.

We assume given the following objects:

(1) A power series ring $S_\infty = \mathcal{O}[[X_1, \cdots, X_r]]$ with augmentation ideal $\mathfrak{a}_\infty = (X_1, \ldots, X_r)$.

(2) Perfect complexes C_∞, C'_∞ of S_∞-modules, and a fixed isomorphism

$$C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi \cong C'_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi$$

in $\mathbf{D}(S_\infty/\varpi)$.

(3) Two S_∞-subalgebras

$$T_\infty \subset \mathrm{End}_{\mathbf{D}(S_\infty)}(C_\infty)$$

and

$$T'_\infty \subset \mathrm{End}_{\mathbf{D}(S_\infty)}(C'_\infty),$$

which have the same image in

$$\mathrm{End}_{\mathbf{D}(S_\infty/\varpi)}(C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi) = \mathrm{End}_{\mathbf{D}(S_\infty/\varpi)}(C'_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi),$$

where these endomorphism algebras are identified using the fixed isomorphism in (2). Call this common image $\overline{T}_\infty$. Note that T_∞ and T'_∞ are finite S_∞-algebras.

(4) Two Noetherian complete local S_∞-algebras R_∞ and R'_∞ and surjections $R_\infty \twoheadrightarrow T_\infty/I_\infty$, $R'_\infty \twoheadrightarrow T'_\infty/I'_\infty$, where I_∞ and I'_∞ are nilpotent ideals. We write $\overline{I}_\infty$ and $\overline{I}'_\infty$ for the image of these ideals in $\overline{T}_\infty$. Note that it then makes sense to talk about the support of $H^*(C_\infty)$ and $H^*(C'_\infty)$ over R_∞, R'_∞, even though they are not genuine modules over these rings. These supports actually belong to the closed subsets of $\mathrm{Spec}\, R_\infty$, $\mathrm{Spec}\, R'_\infty$ given by $\mathrm{Spec}\, T_\infty$, $\mathrm{Spec}\, T'_\infty$, and hence are finite over $\mathrm{Spec}\, S_\infty$.

(5) An isomorphism $R_\infty/\varpi \cong R'_\infty/\varpi$ compatible with the S_∞-algebra structure and the actions (induced from T_∞ and T'_∞) on

$$H^*(C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi)/(\overline{I}_\infty + \overline{I}'_\infty) = H^*(C'_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi)/(\overline{I}_\infty + \overline{I}'_\infty),$$

where these cohomology groups are identified using the fixed isomorphism.

(6) Integers $q_0 \in \mathbf{Z}$ and $l_0 \in \mathbf{Z}_{\geq 0}$.

Assumption 5.4.1. Our set-up is assumed to satisfy the following:

(1) $\dim R_\infty = \dim R'_\infty = \dim S_\infty - l_0$, and $\dim R_\infty/\varpi = \dim R'_\infty/\varpi = \dim S_\infty - l_0 - 1$.

(2) (Behavior of components) Assume that each generic point of $\operatorname{Spec} R_\infty/\varpi$ is the specialization of unique generic points of $\operatorname{Spec} R_\infty$ and $\operatorname{Spec} R'_\infty$. Moreover, we assume that $\operatorname{Spec} R_\infty$ and $\operatorname{Spec} R'_\infty$ are $\mathcal{O}$-flat and equidimensional. These hypotheses imply that every generic point of $\operatorname{Spec} R_\infty$ and $\operatorname{Spec} R'_\infty$ has characteristic 0 (by $\mathcal{O}$-flatness) and $\operatorname{Spec} R_\infty/\varpi$, $\operatorname{Spec} R'_\infty/\varpi$ are equidimensional (by the principal ideal theorem).

(3) (Generic concentration) We have

$$H^*(C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\mathfrak{a}_\infty)[\frac{1}{p}] \neq 0,$$

and these groups are non-zero only for degrees in the interval $[q_0, q_0 + l_0]$.

(4) (Automorphic point) We fix a characteristic 0 point $x \in \operatorname{Spec}(T_\infty/\mathfrak{a}_\infty T_\infty)$.

Note that $\operatorname{Supp}_{R_\infty}(H^*(C_\infty)) = \operatorname{Spec} T_\infty$ and $\operatorname{Supp}_{R'_\infty}(H^*(C'_\infty)) = \operatorname{Spec} T'_\infty$. (This is because the kernel of $T_\infty \to \operatorname{End}_{S_\infty}(H^*(C_\infty))$ is nilpotent and the same for T'_∞ and C'_∞.)

Proposition 5.4.2. *Consider the automorphic subset of* $\operatorname{Spec} R_\infty$*:*

$$\operatorname{Supp}_{R_\infty}(H^*(C_\infty)) = \operatorname{Spec} T_\infty \subset \operatorname{Spec} R_\infty.$$

(1) *There exists an irreducible component $C_a \subset \operatorname{Spec} R_\infty$, containing the automorphic point x, with $C_a \subset \operatorname{Spec} T_\infty$.*

(2) *Let $C_a \subset \operatorname{Spec} T_\infty$ be an irreducible component of $\operatorname{Spec} R_\infty$ which contains x. Suppose $C \subset \operatorname{Spec} R_\infty$ is an irreducible component such that the subsets $C \cap \operatorname{Spec}(R_\infty/\varpi)$ and $C_a \cap \operatorname{Spec}(R_\infty/\varpi)$ of $\operatorname{Spec}(R_\infty/\varpi)$ contain generic points $\overline{x}_C$, $\overline{x}_a$ respectively, which generalize to the same generic point x' of $\operatorname{Spec} R'_\infty$. Then $C \subset \operatorname{Spec} T_\infty$.*

Proof. First we note that the pullback of x to S_∞ is $\mathfrak{a}_\infty$. We write $\tilde{x} \in \operatorname{Spec}(T_\infty \otimes_{S_\infty} S_{\infty,\mathfrak{a}_\infty})$ for the prime ideal extending x.

It follows from our assumptions and [CG18, Lemma 6.2], just as in the proof of Proposition [ACC+18, Proposition 6.3.8], that $H^*(C_{\infty,\mathfrak{a}_\infty})$ is non-zero exactly in degree q_0+l_0 and that $M_\infty := H^{q_0+l_0}(C_{\infty,\mathfrak{a}_\infty})$ is a Cohen–Macaulay $S_{\infty,\mathfrak{a}_\infty}$-module with depth and dimension equal to $\dim S_{\infty,\mathfrak{a}_\infty} - l_0 = \dim S_\infty - l_0 - 1$.

Since the image of $\mathfrak{a}_\infty S_{\infty,\mathfrak{a}_\infty}$ in $\operatorname{End}(H^{q_0+l_0}(C_{\infty,\mathfrak{a}_\infty})))$ is contained in the image of $\tilde{x}$, we deduce that $\operatorname{depth}(\tilde{x}, M_\infty) \geq \dim S_{\infty,\mathfrak{a}_\infty} - l_0$. An M_∞-regular sequence in $\tilde{x}$ remains regular on $M_{\infty,\tilde{x}}$, and the localization $(T_\infty \otimes_{S_\infty} S_{\infty,\mathfrak{a}_\infty})_{\tilde{x}}$ is equal to $T_{\infty,x}$. So $\operatorname{depth}_{T_{\infty,x}}(M_{\infty,\tilde{x}}) \geq \dim S_{\infty,\mathfrak{a}_\infty} - l_0$. In particular, $\dim T_{\infty,x} \geq \dim S_\infty - l_0 - 1$, so the one-dimensional prime x is contained in an irreducible component of $\operatorname{Spec} T_\infty$ of dimension at least $\dim S_\infty - l_0$. By dimension considerations, this irreducible component can be identified with an irreducible component of $\operatorname{Spec} R_\infty$. This shows the existence of an irreducible component C_a as in the first part.

For the second part, we let x_a be the generic point of C_a. Since the pullback of x_a to S_∞ is contained in $\mathfrak{a}_\infty$, the 'localization' C_{∞,x_a} defined following [ACC+18, Lemma 6.3.3] is quasi-isomorphic to the complex with M_{∞,x_a} in degree q_0+l_0 and zero elsewhere. In particular, with the length function on complexes defined in [ACC+18, §6.3.1], we have $\operatorname{lg}_{T_{\infty,x_a}}(C_{\infty,x_a}) \neq 0$. The generic point $\overline{x}_a$ of $\operatorname{Spec} R_\infty/(x_a,\varpi)$ given to us in the statement of the proposition has dimension $\dim S_\infty - l_0 - 1$ and is a generic point of $\operatorname{Spec} R_\infty/(\varpi)$ which lies in $\operatorname{Spec} T_\infty$. Let $\overline{x}'_a$ denote the corresponding point of $\operatorname{Spec} R'_\infty/(\varpi)$. It has a unique generalization $x' \in \operatorname{Spec} R'_\infty$.

Now let x_C be the generic point of C. We wish to show that it lies in $\operatorname{Spec} T_\infty$. We are given a generic point $\overline{x}_C$ of $\operatorname{Spec} R_\infty/(x_C,\varpi)$, which must have dimension $\dim S_\infty - l_0 - 1$ and be a generic point of $\operatorname{Spec} R_\infty/(\varpi)$. Let $\overline{x}'_C$ denote the corresponding point of $\operatorname{Spec} R'_\infty/(\varpi)$, which also generalizes to x' by hypothesis.

We now repeatedly use [ACC+18, Lemma 6.3.7]. As $\operatorname{lg}_{T_{\infty,x_a}}(C_{\infty,x_a}) \neq (0)$, we deduce that $\operatorname{lg}_{T_{\infty,\overline{x}_a}}((C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/(\varpi))_{\overline{x}_a}) \neq 0$. Hence $\overline{x}'_a \in \operatorname{Spec} T'_\infty$ and $\operatorname{lg}_{T'_{\infty,\overline{x}'_a}}((C'_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/(\varpi))_{\overline{x}'_a}) \neq 0$, from which we deduce that $x' \in \operatorname{Spec} T'_\infty$ and $\operatorname{lg}_{T'_{\infty,x'}}(C'_{\infty,x'}) \neq 0$. We further deduce that $\overline{x}'_C \in \operatorname{Spec} T'_\infty$ and $\operatorname{lg}_{T'_{\infty,\overline{x}'_C}}((C'_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/(\varpi))_{\overline{x}'_C}) \neq 0$. Hence $\overline{x}_C \in \operatorname{Spec} T_\infty$ and $\operatorname{lg}_{T_{\infty,\overline{x}_C}}((C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/(\varpi))_{\overline{x}_C}) \neq 0$, from which we finally deduce that $x_C \in \operatorname{Spec} T_\infty$ (and $\operatorname{lg}_{T_{\infty,x_C}}(C_{\infty,x_C}) \neq (0)$). □

Corollary 5.4.3. *Let C be an irreducible component of $\operatorname{Spec} R_\infty$ satisfying the assumption of Proposition 5.4.2(2) for some 'automorphic' component C_a. Let x be a point of C, and let y be the contraction of x in S_∞. Then the support of $H^*(C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/y)_y$ over $\operatorname{Spec} R_\infty$ contains x. If y is one-dimensional of characteristic 0 this says that x is in the support of $H^*(C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/y)[1/p]$.*

Proof. This follows from Proposition 5.4.2(2) by considering the Tor spectral sequence computing the cohomology of $C_{\infty,y} \otimes^{\mathbf{L}}_{S_{\infty,y}} S_{\infty,y}/y$, as in the proof of [ACC+18, Corollary 6.3.9]. □

5.5. Hecke algebras and cohomology of locally symmetric spaces for PGL_2.

We now go back to the constructions of §2.1.2, which we apply to $\mathrm{G} = \overline{G} = \mathrm{PGL}_{2,F}$ for an imaginary CM field F. We need to drop the assumption that $K = \prod_v K_v \subset \overline{G}(\mathbb{A}_{F,f})$ is neat. We assume for convenience that $K \subset \mathrm{PGL}_2(\widehat{\mathcal{O}}_F)$. Thanks to the results of [AKT19, §5], all the properties we need for cohomology of locally symmetric spaces for PGL_2 can be deduced from the case of GL_2 with neat level.

We fix a finite set S of finite places of F such that $S_p \subset S$ and $K_v = \mathrm{PGL}_2(\mathcal{O}_{F_v})$ for $v \notin S$. We assume that $R = \mathcal{O}$ or $\mathcal{O}/\varpi^m$ for some $m \in \mathbb{Z}_{\geq 1}$. Let $\mathcal{V}$ be a $R[K_S]$-module, finite free as an R-module, and such that $\mathcal{V}/\varpi^r$ is a smooth K_S-module for each $r \geq 1$.

We will make use of the Hecke algebra $\mathcal{H}(\overline{G}^S, K^S)$. For each finite place $v \notin S$ and $1 \leq i \leq 2$, we write $T_{v,i}$ for the image of $T_{v,i} \in \mathcal{H}(\mathrm{GL}_2(F_v), \mathrm{GL}_2(\mathcal{O}_{F_v}))$ in $\mathcal{H}(\mathrm{PGL}_2(F_v), \mathrm{PGL}_2(\mathcal{O}_{F_v}))$. In fact $T_{v,2} = 1$ in $\mathcal{H}(\overline{G}^S, K^S)$. We write $P_v(X)$ for the image of the polynomial (2.1.5) in $\mathcal{H}(\mathrm{PGL}_2(F_v), \mathrm{PGL}_2(\mathcal{O}_{F_v}))[X]$.

We consider the object

$$C^\bullet(K,\mathcal{V}) := \varprojlim_r R\Gamma(K, R\Gamma(\overline{\mathfrak{X}}_{\overline{G}}, \mathcal{V}/\varpi^r))$$

of $D^+(R)$ (we do not need to take a limit if R is finite), which comes equipped with an action of $\mathcal{H}(\overline{G}^S, K^S)$.

More generally, if $K' = \prod K'_v \subset K$ is an open normal subgroup with $(K')^S = K^S$, then we consider the object

$$C^\bullet(K/K',\mathcal{V}) = \varprojlim_r R\Gamma(K', R\Gamma(\overline{\mathfrak{X}}_{\overline{G}}, \mathcal{V}/\varpi^r))$$

of $D^+(R[K/K'])$, which again comes with an action of $\mathcal{H}(\overline{G}^S, K^S)$.

We can construct $C^\bullet(K/K',\mathcal{V})$, with its Hecke action, by taking a derived limit of the complexes $\mathrm{Hom}_{\mathbb{Z}[K']}(C_\bullet, \mathcal{V}/\varpi^r)$, where $C_\bullet$ denotes the complex of singular chains with $\mathbb{Z}$-coefficients on $\overline{\mathfrak{X}}_{\overline{G}}$. Commuting the derived limit with cohomology of K/K', using [Sta13, Tag 08U1], we see that $R\Gamma(K/K', C^\bullet(K/K',\mathcal{V})) = C^\bullet(K,\mathcal{V})$.

Lemma 5.5.1. *There are natural Hecke equivariant quasi-isomorphisms $A(K/K',\mathcal{V}) \cong C^\bullet(K/K',\mathcal{V})$, where $A(K/K',\mathcal{V})$ are the complexes constructed in* [AKT19, §5.1] *using the singular chains of $\overline{\mathfrak{X}}^{\mathrm{dis}}_{\overline{G}}$.*

Proof. We can reduce to the case where $\mathcal{V} = \mathcal{V}/\varpi^r$ and K' is neat. Pullback by the continuous map $\overline{\mathfrak{X}}^{\mathrm{dis}}_{\overline{G}} \to \overline{\mathfrak{X}}_{\overline{G}}$ induces a Hecke equivariant map $A(K/K',\mathcal{V}) \to C^\bullet(K/K',\mathcal{V})$ inducing the identity on the cohomology groups, which are identified with $H^*(\overline{X}^{\overline{G}}_{K'}, \mathcal{V})$ on both sides. □

Since K is not assumed to be neat, $C^\bullet(K,\mathcal{V})$ is not necessarily a perfect complex. However, using the Hochschild–Serre spectral sequence to compute its cohomology in terms of $C^\bullet(K',\mathcal{V})$ for $K' \subset K$ a neat open normal subgroup, we see that its cohomology groups are finitely generated R-modules [AKT19, Lemma 5.1].

A similar argument shows that the $\mathcal{O}$-algebras

$$\mathbb{T}^S_{\overline{G}}(C^\bullet(K/K',\mathcal{V})) \subset \mathrm{End}_{D^+(R[K/K'])}(C^\bullet(K/K',\mathcal{V}))$$

generated by the image of $\mathcal{H}(\overline{G}^S, K^S)$ are $\mathcal{O}$-finite [AKT19, Lemma 5.2].

As a consequence, for a maximal ideal $\mathfrak{m} \subset \mathbb{T}^S_{\overline{G}}(C^\bullet(K/K',\mathcal{V}))$, we have a direct summand $C^\bullet(K/K',\mathcal{V})_{\mathfrak{m}}$ cut out by an idempotent $e_{\mathfrak{m}}$ in the Hecke algebra [AKT19, Prop. 3.6]

Proposition 5.5.2. *Suppose that p is odd, and that S satisfies the following conditions:*

1. *S is stable under complex conjugation*
2. *F contains an imaginary quadratic field. Let $v \notin S$ be a finite place, with residue characteristic l. Then either S contains no l-adic places and l is unramified in F, or there exists an imaginary quadratic subfield of F in which l splits.*

Then for any maximal ideal $\mathfrak{m} \subset \mathbb{T}^S_{\overline{G}}(C^\bullet(K/K',\mathcal{V}))$, there exists a semisimple continuous representation

$$\overline{\rho}_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_2(\mathbb{T}^S_{\overline{G}}(C^\bullet(K/K',\mathcal{V}))/\mathfrak{m})$$

such that for each $v \notin S$,

$$\det(X - \overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)) = P_v(X) \mod \mathfrak{m}.$$

Suppose moreover that $\overline{\rho}_{\mathfrak{m}}$ *is absolutely irreducible. Then there exists an integer* $N \geq 1$, *depending only on* $[F : \mathbb{Q}]$, *an ideal* $J \subset \mathbb{T}^S_{\widetilde{G}}(C^\bullet(K/K', \mathcal{V}))$ *with* $J^N = 0$, *and a continuous representation*

$$\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_2(\mathbb{T}^S_{\widetilde{G}}(C^\bullet(K/K', \mathcal{V}))/J)$$

such that for each $v \notin S$,

$$\det(X - \rho_{\mathfrak{m}}(\mathrm{Frob}_v)) = P_v(X) \mod J.$$

In particular, we have $\det \rho_{\mathfrak{m}} = \epsilon_p^{-1}$.

Proof. This is essentially [AKT19, Cor. 5.7] (the assumption there that K/K' is abelian is not necessary). We let K_G and K'_G be the pre-images in $\mathrm{GL}_2(\widehat{\mathcal{O}}_F)$ of K and K' respectively. Then [AKT19, Corollary 5.5] identifies $\mathbb{T}^S_{\widetilde{G}}(H^*(C^\bullet(K/K', \mathcal{V})))$ as a quotient of a GL_2-Hecke algebra $\mathbb{T}^S_G(H^*(C^\bullet(K_G/K'_G, \mathcal{V})))$. We then reduce to the case where $K_G = K'_G$ is a neat level subgroup in GL_2, as in the proofs of [AKT19, Theorems 5.6, 5.8], and finally appeal to [ACC+18, Theorems 2.3.5, 2.3.7]. The determinant of $\rho_{\mathfrak{m}}$ is ϵ_p^{-1} by Chebotarev density, considering the constant terms of the polynomials $P_v(X)$. □

Proposition 5.5.3. *Let* $\mathfrak{m}$ *be a maximal ideal of* $\mathbb{T}^S_{\widetilde{G}}(C^\bullet(K, \mathcal{V}))$ *with residue field* k. *We make the following assumptions:*

1. $\mathcal{V} \otimes k \cong k$ *(with trivial action of* K_S*).*
2. p *is odd and* $\overline{\rho}_{\mathfrak{m}}$ *is absolutely irreducible.*
3. $\zeta_p \in F$.

Then the cohomology groups $H^i(C^\bullet(K, \mathcal{V}))_{\mathfrak{m}}$ *vanish for* $i > \dim_{\mathbb{R}} X^{\widetilde{G}}$. *In particular,* $C^\bullet(K, \mathcal{V})_{\mathfrak{m}}$ *is a perfect complex of* R*-modules.*

Proof. This follows from [AKT19, Thm. 5.11]. □

5.6. **The proof of Theorem 5.2.** We first prove a version of Theorem 5.2 with some additional assumptions. The general case will follow using solvable base change, as in [ACC+18, §6.5.12] and the proof of [AKT19, Theorem A.14].

We fix the following data:

1. An imaginary CM field F, an odd prime p and an isomorphism $\iota : \overline{\mathbb{Q}}_p \cong \mathbb{C}$.
2. A finite set S of finite places of F, including the places above p.
3. A (possibly empty) subset $R \subset S$ of places prime to p.
4. A decomposition $S_p = S_p^{cr} \coprod S_p^{st}$.
5. A cuspidal automorphic representation π of $\mathrm{PGL}_2(\mathbf{A}_F)$ which is regular algebraic of weight 0. We may identify π with a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ with trivial central character.

We assume the following conditions are satisfied:

(5) If l is a prime lying below an element of S, or which is ramified in F, then F contains an imaginary quadratic field in which l splits. In particular, each place of S is split over F^+ and the extension F/F^+ is everywhere unramified.

(6) For each $v \in S_p$, let $\overline{v}$ denote the place of F^+ lying below v. Then there exists a place $\overline{v}' \neq \overline{v}$ of F^+ such that $\overline{v}'|p$ and

$$\sum_{\overline{v}'' \neq \overline{v}, \overline{v}'} [F^+_{\overline{v}''} : \mathbf{Q}_p] > \frac{1}{2}[F^+ : \mathbf{Q}].$$

Moreover, we assume that the residue field of v is strictly bigger than $\mathbb{F}_p$.

(7) π_v is unramified for $v \notin R \cup S_p^{st}$.

(8) If $v \in R \cup S_p^{st}$, then $\pi_v^{\mathrm{Iw}_v} \neq 0$.

(9) If $v \in S_p^{st}$, then π is ι-ordinary of weight 0 at v and $r_\iota(\pi)|_{G_{F_v}}$ is non-crystalline ordinary.

(10) If $S = S_p \cup R$, then $\zeta_p \in F$.

(11) If $S \neq S_p \cup R$, then $S - (S_p \cup R)$ contains at least two places with distinct residue characteristics.

(12) If $v \in S - (R \cup S_p)$, then $v \notin R^c$ and $H^2(F_v, \mathrm{ad}^0 \overline{r}_{\pi,\iota}) = 0$.

(13) $\overline{r}_{\pi,\iota}$ is decomposed generic and $\overline{r}_{\pi,\iota}|_{G_{F(\zeta_p)}}$ is irreducible.

(14) $\overline{r}_{\pi,\iota}|_{G_{F_v}}$ is the trivial representation for $v \in S_p \cup R$. (In particular, by considering determinants, $q_v \equiv 1 \mod p$ for $v \in R$.)

(15) If $p = 5$ and the projective image of $\overline{r}_{\pi,\iota}(G_{F(\zeta_5)})$ is conjugate to $\mathrm{PSL}_2(\mathbb{F}_5)$, we assume further that the extension of F cut out by the projective image of $\overline{r}_{\pi,\iota}$ does not contain ζ_5.

Proposition 5.6.1. *With notation and assumptions as in (1)–(15), suppose given a continuous representation $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ satisfying the following conditions:*

(1) *We have $\overline{\rho} \cong \overline{r}_{\pi,\iota}$ and* $\det(\rho) = \epsilon_p^{-1}$.

(2) *For each place $v \in S_p^{cr}$, $\rho|_{G_{F_v}}$ is Barsotti–Tate.*

(3) *For each place $v \in S_p^{cr}$, $r_\iota(\pi)|_{G_{F_v}}$ is ordinary if and only if $\rho|_{G_{F_v}}$ is ordinary.*

(4) *For each place $v \in S_p^{st}$, $\rho|_{G_{F_v}}$ is a non-crystalline extension of ϵ_p^{-1} by the trivial character.*

(5) *For each finite place $v \notin S$ of F, $\rho|_{G_{F_v}}$ is unramified.*

(6) *For each place $v \in R$, $\rho|_{G_{F_v}}$ is unipotently ramified.*

Then ρ is automorphic: there exists a cuspidal automorphic representation Π of $\mathrm{PGL}_2(\mathbf{A}_F)$ of weight 0 such that $\rho \cong r_\iota(\Pi)$.

Proof. We define a compact open subgroup $K = \prod_v K_v$ of $\mathrm{PGL}_2(\widehat{\mathcal{O}}_F)$ as follows:

- If $v \notin S$ or $v \in S_p^{cr}$, then $K_v = \mathrm{PGL}_2(\mathcal{O}_{F_v})$.
- If $v \in R \cup S_p^{st}$, then $K_v = \mathrm{Iw}_v$.
- If $v \in S - (S_p \cup R)$, then $K_v = \mathrm{Iw}_{v,1}$ is the pro-v Iwahori subgroup of $\mathrm{PGL}_2(\mathcal{O}_{F_v})$.

If $S - (S_p \cup R)$ non-empty, K is neat, by the same argument as [ACC+18, Lemma 6.5.2]. We set $T = S \cup S^c$.

Recalling the unitary group $\widetilde{G}$ from §2.1.11 (for $n = 2$), we need to define standard parabolic subgroups $Q_{\bar{v}} \subset \widetilde{G}_{F^+_{\bar{v}}}$ for each $\bar{v} \in \overline{S}_p$. For each $\bar{v}$ we choose a place of F, $\tilde{v}|\bar{v}$. We make this choice so that if at least one $v|\bar{v}$ is in S_p^{st}, then $\tilde{v}$ is in S_p^{st}. Then for each $\bar{v} \in \overline{S}_p$ we consider the following three cases:

(cr-ord) $\tilde{v} \in S_p^{st}$ and $\tilde{v}^c \in S_p^{cr}$. Then $\iota_{\tilde{v}}(Q_{\bar{v}})$ is the standard parabolic given by the partition $(2, 1, 1)$.

(ord) $\tilde{v} \in S_p^{st}$ and $\tilde{v}^c \in S_p^{st}$. Then $\iota_{\bar{v}}(Q_{\bar{v}}) = \mathrm{B}_4$, the Borel subgroup.
(cr) $\tilde{v} \in S_p^{cr}$ and $\tilde{v}^c \in S_p^{cr}$. Then $Q_{\bar{v}} = P_{\bar{v}}$, the Siegel parabolic.

We have a Hecke algebra $\mathbb{T}_{\overline{G}}^{\mathcal{Q}_{\bar{S}_p}, \bar{S}_p-\mathrm{ord}}(\mathcal{C}^\bullet(K,\mathcal{O}))$, defined as in §4.2.1 by adding Hecke operators at places $v|p$ to $\mathbb{T}^T(\mathcal{C}^\bullet(K,\mathcal{O}))$.

Then we can find a coefficient field $E \subset \overline{\mathbb{Q}}_p$ and a maximal ideal $\mathfrak{m} \subset \mathbb{T}_{\overline{G}}^{\mathcal{Q}_{\bar{S}_p}, \bar{S}_p-\mathrm{ord}}(\mathcal{C}^\bullet(K,\mathcal{O}))$ such that $\overline{\rho}_{\mathfrak{m}} : G_{F,T} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ satisfies $\overline{\rho}_{\mathfrak{m}} \cong \overline{r}_\iota(\pi)$ (cf. [AKT19, Theorem 5.10]).

Moreover, for $v \in S_p^{st}$, since π is ι-ordinary at v, the Hecke operators $U_v := [K_v \iota_v^{-1} \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} K_v]$ are not in $\mathfrak{m}$.

Enlarging E if necessary, we assume that the residue field of $\mathfrak{m}$ is equal to k and that k contains all eigenvalues of the elements of $\overline{\rho}_{\mathfrak{m}}(G_F)$.

We now describe the global deformation problems we will be working with. They will depend on a choice of character $\chi = \prod_{v\in R} \chi_v : \prod_{v \in R} \mathcal{O}_{F_v}^\times \to \mathcal{O}^\times$ which is trivial modulo ϖ.

For each χ, we have the global deformation problem with fixed determinant

$$\mathcal{S}_\chi = (\overline{\rho}, \epsilon_p^{-1}, S, \{R_v^{\epsilon_p^{-1},\mathrm{BT}}\}_{v\in S_p^{cr}} \cup \{R_v^{\triangle}\}_{v \in S_p^{st}} \cup \{R_v^{\epsilon_p^{-1},\chi_v}\}_{v\in R} \cup \{R_v^{\epsilon_p^{-1}}\}_{v\in S-(S_p\cup R)}).$$

The character $\chi_v : \mathcal{O}_{F_v}^\times \to \mathcal{O}^\times$ (which we note factors through $k_v^\times$) determines a character

$$\chi_v : \mathrm{Iw}_v \to \mathcal{O}^\times$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \chi_v(a/d).$$

We have an $\mathcal{O}[K_S]$-module $\mathcal{O}(\chi^{-1})$, where K_S acts by the projection $K_S \to K_R = \prod_{v\in R} \mathrm{Iw}_v \xrightarrow{\prod_v \chi_v} \mathcal{O}^\times$.

For each χ, there is a canonical, surjective, $\mathcal{O}$-algebra map

$$\mathbb{T}_{\overline{G}}^{\mathcal{Q}_{\bar{S}_p}, \bar{S}_p-\mathrm{ord}}(\mathcal{C}^\bullet(K,\mathcal{O}(\chi^{-1}))) \to \mathbb{T}_{\overline{G}}^{\mathcal{Q}_{\bar{S}_p}, \bar{S}_p-\mathrm{ord}}(\mathcal{C}^\bullet(K,k))$$

inducing a bijection on maximal ideals. So the maximal ideal $\mathfrak{m}$ corresponds to a maximal ideal of $\mathbb{T}_{\overline{G}}^{\mathcal{Q}_{\bar{S}_p}, \bar{S}_p-\mathrm{ord}}(\mathcal{C}^\bullet(K,\mathcal{O}(\chi^{-1})))$ for each χ. We abusively denote all these ideals by $\mathfrak{m}$. The localisation $\mathcal{C}^\bullet(K,\mathcal{O}(\chi^{-1}))_{\mathfrak{m}}$ is a perfect complex of $\mathcal{O}$-modules (using Proposition 5.5.3 when K is not neat). We will also consider the Hecke algebra $\mathbb{T}_{\overline{G}}^{S,\mathcal{Q}_{\bar{S}_p}, \bar{S}_p-\mathrm{ord}}(\mathcal{C}^\bullet(K,\mathcal{O}(\chi^{-1}))_{\mathfrak{m}})$ obtained by adding in the spherical Hecke operators at places in $S^c - S$ to $\mathbb{T}_{\overline{G}}^{\mathcal{Q}_{\bar{S}_p}, \bar{S}_p-\mathrm{ord}}(\mathcal{C}^\bullet(K,\mathcal{O}(\chi^{-1}))_{\mathfrak{m}})$.

Proposition 5.6.2. *There exists an integer $N \geq 1$, depending only on $[F:\mathbb{Q}]$, an ideal $J \subset \mathbb{T}_{\overline{G}}^{S,\mathcal{Q}_{\bar{S}_p}, \bar{S}_p-\mathrm{ord}}(\mathcal{C}^\bullet(K,\mathcal{O}(\chi^{-1}))_{\mathfrak{m}})$ such that $J^N = 0$, and a continuous surjective homomorphism*

$$f_{\mathcal{S}_\chi} : R_{\mathcal{S}_\chi} \to \mathbb{T}_{\overline{G}}^{S,\mathcal{Q}_{\bar{S}_p}, \bar{S}_p-\mathrm{ord}}(\mathcal{C}^\bullet(K,\mathcal{O}(\chi^{-1}))_{\mathfrak{m}})/J$$

such that for each finite place $v \notin S$ of F, the characteristic polynomial of $f_{\mathcal{S}_\chi} \circ \rho_{\mathcal{S}_\chi}^{\mathrm{univ}}$ equals the image of $P_v(X)$ in $\mathbb{T}_{\overline{G}}^{S,\mathcal{Q}_{\bar{S}_p}-\mathrm{ord}}(\mathcal{C}^\bullet(K,\mathcal{O}(\chi^{-1}))_{\mathfrak{m}})/J$.

Proof. Proposition 5.5.2 already gives us a representation of $G_{F,S\cup S^c}$ with the right local properties at $v\notin S\cup S^c$, so it remains to check each prime $v\in S\cup S^c$. As in the proof of Proposition 5.5.2, we reduce to proving a similar local-global compatibility statement for a neat level in GL_2. For $v\in S_p$, we apply Theorem 4.2.15. For the remaining v, we proceed as in the proof of [ACC$^+$18, Proposition 6.5.3], using [ACC$^+$18, Theorem 3.1.1]. □

To complete the proof of Proposition 5.6.1 we need to show that the point of $\mathrm{Spec}(R_{\mathcal{S}_1})$ given by ρ is in the support of $H^*(C^\bullet(K,\mathcal{O})_{\mathfrak{m}})$. Then [AKT19, Theorem 5.10] implies that ρ is automorphic of weight 0.

We need a local–global compatibility statement allowing ramification at Taylor–Wiles primes. So we suppose we have a Taylor–Wiles datum $(Q,N,(\alpha_{v,1},\alpha_{v,2})_{v\in Q})$ for $\mathcal{S}_1$ (which is then also a Taylor–Wiles datum for every $\mathcal{S}_\chi$). We assume that each place of Q has residue characteristic split in an imaginary quadratic subfield of F. Now we define deformation problems

$$\mathcal{S}_{\chi,Q}=(\overline{\rho},\epsilon_p^{-1},S\cup Q,\{R_v^{\epsilon_p^{-1},\mathrm{BT}}\}_{v\in S_p^{cr}}\cup\{R_v^{\triangle}\}_{v\in S_p^{st}}\cup\{R_v^{\epsilon_p^{-1},\chi_v}\}_{v\in R}\cup\{R_v^{\epsilon_p^{-1}}\}_{v\in S\cup Q-(S_p\cup R)}).$$

For $v\in Q$, let $\Delta_v=k_v^\times(p)$, the maximal p-power quotient of $k_v^\times$. As in [AKT19, §A.1.4], the local lifting ring $R_v^{\epsilon_p^{-1}}$ is equipped with the structure of an $\mathcal{O}[\Delta_v]$-algebra. Setting $\Delta_Q=\prod_{v\in Q}\Delta_v$, we obtain an $\mathcal{O}[\Delta_Q]$-algebra structure on $R_{\mathcal{S}_{\chi,Q}}$.

We define subgroups $K_1(Q)\subset K_0(Q)\subset K$, with $K_0(Q)/K_1(Q)\cong\Delta_Q$ as in [ACC$^+$18, §6.5] (taking the image in PGL_2 of the subgroups defined there). From this point, we follow loc. cit. very closely, so we just explain the key points of the argument.

There is a direct summand $C^\bullet(K_0(Q)/K_1(Q),\mathcal{O}(\chi^{-1}))_{\mathfrak{n}_1^Q}$ of $C^\bullet(K_0(Q)/K_1(Q),\mathcal{O}(\chi^{-1}))$ in $\mathbf{D}(\mathcal{O}[\Delta_Q])$, defined using a maximal ideal in a Hecke algebra with operators $U_{v,i}$ at places $v\in Q$. It is a perfect complex, by [AKT19, Theorem 5.11].

We write $\mathbb{T}_{\chi,Q}$ for the image of the map

$$\mathbb{T}_{\overline{G}}^{S\cup Q,\mathcal{Q}_{\bar{S}_p},\bar{S}_p-\mathrm{ord}}\otimes_{\mathcal{O}}\mathcal{O}[\Delta_Q]\to\mathrm{End}_{\mathbf{D}(\mathcal{O}[\Delta_Q])}\left(C^\bullet(K_0(Q)/K_1(Q),\mathcal{O}(\chi^{-1}))_{\mathfrak{n}_1^Q}\right).$$

Proposition 5.6.3. *There exists an integer $N\geq 1$, depending only on $[F:\mathbb{Q}]$, an ideal $J\subset\mathbb{T}_{\chi,Q}$ such that $J^N=0$, and a continuous surjective $\mathcal{O}[\Delta_Q]$-algebra homomorphism*

$$f_{\mathcal{S}_{\chi,Q}}:R_{\mathcal{S}_{\chi,Q}}\to\mathbb{T}_{\chi,Q}/J$$

such that for each finite place $v\notin S\cup Q$ of F, the characteristic polynomial of $f_{\mathcal{S}_{\chi,Q}}\circ\rho^{\mathrm{univ}}_{\mathcal{S}_{\chi,Q}}$ equals the image of $P_v(X)$ in $\mathbb{T}_{\chi,Q}/J$.

Proof. This is proved in the same way as Proposition 5.6.2, using [ACC$^+$18, Theorem 3.1.1] to show that $f_{\mathcal{S}_{\chi,Q}}$ is an $\mathcal{O}[\Delta_Q]$-algebra homomorphism (cf. [ACC$^+$18, Proposition 6.5.11] and [AKT19, Proposition A.13]). □

It is convenient to patch complexes computing homology, so we define

$$C_{\chi,Q}:=R\,\mathrm{Hom}_{\mathcal{O}[\Delta_Q]}(C^\bullet(K_0(Q)/K_1(Q),\mathcal{O}(\chi^{-1}))_{\mathfrak{n}_1^Q},\mathcal{O}[\Delta_Q])$$

and

$$C_\chi:=R\,\mathrm{Hom}_{\mathcal{O}}(C^\bullet(K,\mathcal{O}(\chi^{-1}))_{\mathfrak{m}},\mathcal{O}).$$

Lemma 5.6.4. *$C_{\chi,Q}$ is a perfect complex of $\mathcal{O}[\Delta_Q]$-modules, with a canonical isomorphism*

$$C_{\chi,Q} \otimes^{\mathbb{L}}_{\mathcal{O}[\Delta_Q]} \mathcal{O} \cong C_\chi$$

in $\mathbf{D}(\mathcal{O})$.

Proof. This follows from the fact that we can identify $C_{\chi,Q}$ and C_χ with the duals of perfect complexes computing localisations (at $\mathfrak{n}_1^Q$ and $\mathfrak{m}$ respectively) of equivariant homology (see the second part of [AKT19, Theorem 5.11]). ☐

We now have everything we need to construct the objects required for §5.4, using [ACC⁺18, §6.4] and [AKT19, Proposition A.6] (existence of Taylor–Wiles primes) as in the proof of [AKT19, Theorem A.7]. In particular, we make use of two options for the tuple of characters χ: firstly, $\chi = 1$, and secondly a fixed tuple, denoted χ, given by a choice of character $\chi_v : \mathcal{O}_{F_v}^\times \to \mathcal{O}^\times$ with $\chi_v^2 \neq 1$ which is trivial mod ϖ for each $v \in R$.

The rings R_∞ and R'_∞ are power series rings over $R_{\mathrm{loc}} = R^{S,\mathrm{loc}}_{\mathcal{S}_1}$ and $R'_{\mathrm{loc}} = R^{S,\mathrm{loc}}_{\mathcal{S}_\chi}$ respectively, which come equipped with local R_{loc}-algebra (respectively R'_{loc}-algebra) surjections $R_\infty \to R_{\mathcal{S}_1}$ (respectively $R'_\infty \to R_{\mathcal{S}_\chi}$). We can assume, extending $\mathcal{O}$ if necessary, that all of the irreducible components of the local lifting rings appearing in the deformation problems $\mathcal{S}_1$ and $\mathcal{S}_\chi$ are geometrically irreducible.

It follows from formal smoothness of $R_v^{\square,\epsilon_p^{-1}}$ for $v \in S-(R\cup S_p)$, [AKT19, Lemma A.2], Proposition 5.3.2, Lemma 5.3.3 and [BLGHT11, Lemma 3.3] (which describes irreducible components of completed tensor products in terms of their factors) that R_∞ and R'_∞ satisfy Assumption 5.4.1. In particular, $\mathrm{Spec}(R'_\infty)$ has $2^{|S_p|}$ irreducible components, which biject with the irreducible components of $R^{S_p,\mathrm{loc}}_{\mathcal{S}_\chi} = R^{S_p,\mathrm{loc}}_{\mathcal{S}_1}$. We let C_a be an irreducible component of $\mathrm{Spec}(R_\infty)$ containing the point $r_{\pi,\iota}$ with $C_a \subset \mathrm{Spec}(T_\infty)$. It exists by Proposition 5.4.2. We let C be an irreducible component of $\mathrm{Spec}(R_\infty)$ containing the point given by ρ. It follows from conditions (3) and (4), Proposition 5.3.2, Lemma 5.3.4 and [BLGHT11, Lemma 3.3] that the generic points of $C \cap \mathrm{Spec}(R_\infty/\varpi)$ and $C_a \cap \mathrm{Spec}(R_\infty/\varpi)$ lie in the same irreducible component of $\mathrm{Spec}(R'_\infty)$. Note that condition (4) ensure that both C and C_a necessarily lie over the component X^{st} in $\mathrm{Spec}(R_v^{\triangle})$ for $v \in S_p^{st}$.

We will therefore be able to apply Corollary 5.4.3 to deduce automorphy of ρ.

This completes the proof of Proposition 5.6.1. ☐

The end of the proof of Theorem 5.2. We will use a variant of [DDT97, Lemma 4.11]:

Lemma 5.6.5. *Suppose that G is a finite group with a representatation $\overline{\rho} : G \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ for an odd prime p. Suppose that the character $\det \overline{\rho} : G \to \overline{\mathbb{F}}_p^\times$ has order $d > 1$, and for all g with $\det \overline{\rho}(g) \neq 1$ we have*

$$(\mathrm{tr}\overline{\rho}(g))^2 = (1 + \det \overline{\rho}(g))^2. \tag{5.6.1}$$

Then $\overline{\rho}|_{\ker(\det \overline{\rho})}$ is reducible.

Proof. This is an immediate consequence of [DDT97, Lemma 4.11], except when $d = 3$. So we assume $d = 3$. We write $Z \subset G$ for the subgroup $Z = \{g \in G : \overline{\rho}(g) \text{ scalar}\}$. As in loc. cit., (5.6.1) implies that $Z \subset \ker \det \overline{\rho}$, so $\det \overline{\rho}$ induces a surjective homomorphism $G' \twoheadrightarrow C_d$, where G' is the projective image of $\overline{\rho}$. Dickson's classification implies that $\overline{\rho}$ is reducible or $G' \cong A_4$. In the latter case, set $G_1 =$

$\ker(\det \overline{\rho})$. We have $\overline{\rho}(G_1)/\overline{\rho}(Z) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, and it follows that $\overline{\rho}|_{\ker(\det \overline{\rho})}$ is reducible. □

It suffices to prove that $\rho \cong r_\iota(\Pi)$ for a cuspidal automorphic representation Π of $\mathrm{GL}_2(\mathbb{A}_F)$; then the fact that $\det(\rho) = \epsilon_p^{-1}$ implies that Π has trivial central character. Let $L/F(\zeta_p)$ be the extension cut out by $\overline{\rho}|_{G_{F(\zeta_p)}}$. If F'/F is any finite solvable extension, we denote the base change of π to F' by $\pi_{F'}$.

We choose a finite set V of finite places of F exactly as in the proof of [AKT19, Theorem A.14], so that:

- For any proper extension L'/F contained in L, there is some $v \in V$ not splitting in L'.
- There is a rational prime $q \neq p$ such that $\overline{\rho}$ is decomposed generic for q and V contains all q-adic places of K.
- For each $v \in V$, $v \nmid 2p$ and both ρ and π are unramified at v.

This ensures that if F'/F is a finite Galois extension in which every place of V splits, then $\overline{\rho}|_{G_{F'}}$ remains decomposed generic and $\overline{\rho}(G_{F'(\zeta_p)}) = \overline{\rho}(G_{F(\zeta_p)})$.

Now we choose a solvable, Galois, CM extension F_0/F such that:

- Every place of V splits in F_0.
- For every finite place w of F_0, $\pi_{F_0}^{\mathrm{Iw}_w} \neq 0$.
- For every finite place $w \nmid p$ of F_0, either both $\pi_{F_{0,w}}$ and $\rho|_{G_{F_{0,w}}}$ are unramified, or $\rho|_{G_{F_{0,w}}}$ is unipotently ramified, $q_w \equiv 1 \bmod p$, and $\overline{\rho}|_{G_{F_{0,w}}}$ is trivial.
- For each $\bar{w}|p$ in F_0^+, $\bar{w}$ splits in F_0, $\overline{\rho}|_{G_{F_{0,w}}}$ is trivial for $w|\bar{w}$, the residue field k_w is strictly bigger than $\mathbb{F}_p$ and there exists a place $\overline{w}' \neq \overline{w}$ of F_0^+ such that $\overline{w}'|p$ and
$$\sum_{\overline{w}'' \neq \overline{w}, \overline{w}'} [F^+_{0,\overline{w}''} : \mathbf{Q}_p] > \frac{1}{2}[F_0^+ : \mathbf{Q}].$$
- If w lies over a place v of F with $\rho|_{G_{F_v}}$ potentially crystalline, then $\rho|_{G_{F_{0,w}}}$ is crystalline and $\pi_{F_{0,w}}$ is unramified. Moreover, $r_\iota(\pi)|_{G_{F_{0,w}}}$ is crystalline and it is ordinary if and only if $\rho|_{G_{F_{0,w}}}$ is ordinary.
- If w lies over a place v of F with $\rho|_{G_{F_v}}$ not potentially crystalline, $\rho|_{G_{F_{0,w}}}$ is a non-crystalline extension of ϵ_p^{-1} by the trivial character.

With respect to the penultimate item, we note that it follows from Theorem 4.3.1 that when $\pi_{F_{0,w}}$ is unramified, $r_\iota(\pi)|_{G_{F_{0,w}}}$ is automatically crystalline with all labelled Hodge–Tate weights equal to $(0, 1)$.

Making a further solvable extension F_1/F_0 by taking a composite with three imaginary quadratic fields, as in the proof of [AKT19, Theorem A.14], we furthermore satisfy:

- Let R be the set of finite places $w \nmid p$ of F_1 such that $\pi_{F_1,w}$ or $\rho|_{G_{F_{1,w}}}$ are ramified. Let S_p denote the p-adic places of F_1 and set $S' = S_p \cup R$. If l is a rational prime lying below an element of S', or which is ramified in F_1, then F_1 contains an imaginary quadratic field in which l splits.

We can now describe the data we need to apply Proposition 5.6.1 with $F = F_1$, $\rho = \rho|_{G_{F_1}}$ and $\pi = \pi_{F_1}$. We have already defined the set of places R. We let S_p^{cr} be the set of places $w|p$ where $\rho|_{G_{F_{0,w}}}$ is crystalline and let S_p^{st} be the set of places $w|p$ where $\rho|_{G_{F_{0,w}}}$ is non-crystalline. For $w \in S_p^{st}$, we know (by assumption) that $r_\iota(\pi)|_{G_{F_{0,w}}}$ is not crystalline, whilst it follows from Theorem 4.2.15 that it is ordinary.

If $\zeta_p \in F$, we set $S = S' = S_p \cup R$. If $\zeta_p \notin F$ (which entails $\zeta_p \notin F_1$), Lemma 5.6.5 shows that we can find an element $g \in \overline{\rho}(G_{F_1})$ such that $\det(g) \neq 1$, and the ratio of the eigenvalues of g does not equal $\det(g)^{\pm 1}$. Using Chebotarev density, we can find infinitely many finite places v_0 of F_1 of degree 1 over $\mathbb{Q}$ such that $v_0 \notin S' \cup R^c$, $q_{v_0} \not\equiv 1 \bmod p$, and the ratio of the eigenvalues of $\overline{\rho}(\mathrm{Frob}_{v_0})$ does not equal $q_{v_0}^{\pm 1}$. For such a place, $H^2(F_{1,v_0}, \mathrm{ad}^0\overline{\rho}) = 0$ and the rational prime below v_0 splits in any quadratic subfield of F. We choose two such places v_0, v_0' of distinct residue characteristic and set $S = S' \cup \{v_0, v_0'\}$. We are now in a situation where all the assumptions of Proposition 5.6.1 are satisfied. We deduce that $\rho|_{G_{F_1}}$ is automorphic, and solvable descent [ACC+18, Proposition 6.5.13] completes the job. □

6. Modularity of elliptic curves over CM fields

In this section, our goal is to combine Theorem 5.2 with the results of [AKT19] to prove the following:

Theorem 6.1. *Let F be an imaginary CM number field with $\zeta_5 \notin F$. Let E/F be an elliptic curve satisfying one of the following two conditions:*

1. *$\overline{r}_{E,3}$ is decomposed generic and $\overline{r}_{E,3}|_{G_{F(\zeta_3)}}$ is absolutely irreducible.*
2. *$\overline{r}_{E,5}$ is decomposed generic and $\overline{r}_{E,5}|_{G_{F(\zeta_5)}}$ is absolutely ireducible.*

Then E is modular.

By 'E is modular', we mean that either E has CM, or there is a cuspidal, regular algebraic automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ which is regular algebraic of weight 0, with $r_{\pi,\iota} \cong r_{E,p}^\vee$ for a prime p and an isomorphism $\iota : \overline{\mathbb{Q}}_p \to \mathbb{C}$.

Before proving the theorem, we give some corollaries. These will be improved further in the next section in the special case when F is imaginary quadratic.

Corollary 6.1.1. *Let F be an imaginary quadratic field. Let E/F be an elliptic curve satisfying one of the following two conditions:*

1. *$\overline{r}_{E,3}|_{G_{F(\zeta_3)}}$ is absolutely irreducible.*
2. *$\overline{r}_{E,5}|_{G_{F(\zeta_5)}}$ is absolutely ireducible.*

Them E is modular.

Proof. Combine Theorem 6.1 and Lemma 6.2.2. □

Corollary 6.1.2. *Let F be an imaginary CM field that is Galois over $\mathbb{Q}$ and such that $\zeta_5 \notin F$. Then 100% of Weierstrass equations over F, ordered by their height, define a modular elliptic curve.*

Proof. It follows from [AN20, Lemma 2.3] that, if $F/\mathbb{Q}$ is finite Galois and E/F is an elliptic curve such that the image of $\overline{r}_{E,5}$ contains $\mathrm{SL}_2(\mathbb{F}_5)$, then $\overline{r}_{E,5}$ is decomposed generic. Note than the decomposed generic condition used in *loc. cit.* is more restrictive than and therefore implies the one we are using. When the image of $\overline{r}_{E,5}$ contains $\mathrm{SL}_2(\mathbb{F}_5)$, we also have that $\overline{r}_{E,5}|_{G_{F(\zeta_5)}}$ is absolutely ireducible, so the hypotheses of the second part of Theorem 6.1 are satisfied.

To conclude, we observe that a quantitative version of Hilbert irreducibility, see for example [Zyw10, Prop. 5.2], implies that 100% of elliptic curves E over a fixed number field F have the property that the image of $\overline{r}_{E,5}$ contains $\mathrm{SL}_2(\mathbb{F}_5)$. □

We recall a useful lemma from [AKT19], which is proved using Varma's results on local-global compatibility [Var14].

Lemma 6.1.3. *Let F be a CM field and let E/F be a modular elliptic curve without CM, with $r_{\pi,\iota} \cong r_{E,p}^\vee$ for some choice of prime p and isomorphism $\iota : \overline{\mathbb{Q}}_p \to \mathbb{C}$. Then:*

1. *π has trivial central character and weight 0, and is uniquely determined by E.*
2. *For every prime p and isomorphism $\iota : \overline{\mathbb{Q}}_p \to \mathbb{C}$, there is an isomorphism $r_{\pi,\iota} \cong r_{E,p}^\vee$.*
3. *For every isomorphism $\iota : \overline{\mathbb{Q}}_p \to \mathbb{C}$ and finite place $v \nmid p$ of F, there is an isomorphism $WD(r_{E,p}^\vee|_{G_{F_v}})^{F-ss} \cong \mathrm{rec}_{F_v}^T(\pi_v)$.*
4. *Suppose $v|p$ is a place where E has potentially multiplicative reduction. Then π is ι-ordinary of weight 0 at v for any $\iota : \overline{\mathbb{Q}}_p \to \mathbb{C}$.*

Proof. The first three parts are contained in [AKT19, Lemma 9.1]. The final part is proved in the same way as [AKT19, Corollary 9.2]: applying the third part to $r_{E,l}^\vee$ for some $l \neq p$, we see that π_v is a twist of the Steinberg representation by a quadratic character (quadratic since the central character of π_v is trivial). Then the proof of [Ger19, Lemma 5.6] shows that π_v is ι-ordinary at v. □

Here is another useful lemma, taken from the proof of [AKT19, Corollary 9.14].

Lemma 6.1.4. *Let F be a CM field with $\zeta_5 \notin F$ and let E/F be an elliptic curve such that the projective image of $\overline{r}_{E,5}(G_{F(\zeta_5)})$ is conjugate to $\mathrm{PSL}_2(\mathbb{F}_5)$. Then the extension of F cut out by the projective image of $\overline{r}_{E,5}(G_F)$ does not contain ζ_5.*

Proof. The group $\overline{r}_{E,5}(G_{F(\zeta_5)})$ is a subgroup of $\mathrm{SL}_2(\mathbb{F}_5)$ surjecting onto $\mathrm{PSL}_2(\mathbb{F}_5)$, so it is equal to $\mathrm{SL}_2(\mathbb{F}_5)$. We let G be the kernel of the map from G_F to the projective image of $\overline{r}_{E,5}(G_F)$ and let $H \subset G$ be the kernel of the map from $G_{F(\zeta_5)}$ to the projective image of $\overline{r}_{E,5}(G_{F(\zeta_5)})$. The extension of F cut out by the projective image of $\overline{r}_{E,5}(G_F)$ contains ζ_5 if and only if $H = G$. Since $\zeta_5 \notin F$, $\det \overline{\rho}$ has order 2 or 4. In the first case, the projective image of $\overline{r}_{E,5}(G_F)$ is again $\mathrm{PSL}_2(\mathbb{F}_5)$ so $[G : H] = [F(\zeta_5) : F] > 1$. In the second case, $[F(\zeta_5) : F] > [\mathrm{PGL}_2(\mathbb{F}_5) : \mathrm{PSL}_2(\mathbb{F}_5)]$ which again means $H \neq G$. □

Now we state a variant of [AKT19, Proposition 9.13]:

Proposition 6.1.5. *Let F be an imaginary CM number field, and let*

$$\overline{\rho} : G_F \to \mathrm{GL}_2(\mathbb{F}_5)$$

be a continuous homomorphism with determinant $\bar{\epsilon}_5$. We assume $\overline{\rho}$ is decomposed generic.

Suppose we have a decomposition $S_5 = S_5^{st} \coprod S_5^{\mathrm{ord}} \coprod S_5^{\mathrm{ss}}$ of the set of places in F dividing 5.

Let F^{avoid}/F be a finite Galois extension. Then we can find a solvable CM extension L/F and an elliptic curve E/L satisfying the following conditions:

1. *E is modular.*

(2) *The extension L/F is linearly disjoint from F^{avoid}/F*
(3) *For each place $v|5$ in F and $w|v$ in L, E_{F_w} has good ordinary reduction if $v \in S_5^{\mathrm{ord}}$, good supersingular reduction if $v \in S_5^{\mathrm{ss}}$ and (split) multiplicative reduction if $v \in S_5^{st}$.*
(4) *There is an isomorphism $\overline{\rho}|_{G_L} \cong \overline{r}_{E,5}$.*
(5) *$\overline{\rho}|_{G_L}$ is decomposed generic.*

Proof. We choose L/F to be a solvable CM extension such that:

- For each place $w|2,3,5$ of L, $\overline{\rho}|_{G_{L_w}}$ is trivial and w is split over L^+.
- For $w|5$, there are elliptic curves E^{ord}/L_w, E^{ss}/L_w with good ordinary and good supersingular reduction respectively, and trivial action of G_{L_w} on their 5-torsion.
- For each place $w|2$ of L, the extension $L_w(\sqrt{-1})/L_w$ is unramified.
- L/F is linearly disjoint from F^{avoid}/F.
- There is a prime $q > 5$ which is decomposed generic for $\overline{\rho}$ and splits in L.

Now we apply [AKT19, Lemma 9.7] in the same way as in the proof of [AKT19, Proposition 9.13] to find an L-rational point of the modular curve $Y_{\overline{\rho}}$ corresponding to a modular elliptic curve E/L. The curve $Y_{\overline{\rho}}$ is isomorphic to an open subset of the projective line over F. Combining Hilbert irreducibility, in the form of [Ser08, Theorem 3.5.3], and weak approximation for the projective line, we can find points in $Y_{\overline{\rho}}(L)$ that avoid a thin subset ([Ser08, Definition 3.1.1]) and lie in specified non-empty w-adically open subsets $\Omega_w \subset Y_{\overline{\rho}}(L_w)$ for a finite set of places w. Compared to [AKT19, Proposition 9.13], we replace the condition that E_{L_w} is a Tate curve for each place $w|5$ with the condition that for w lying over a place in S_5^{ord}, E_{L_w} has good ordinary reduction, for w lying over a place in S_5^{ss}, E_{L_w} has good supersingular reduction and for w lying over a place in S_5^{st}, E_{L_w} is a Tate curve. Note that the modularity of E is then proved by applying [AKT19, Proposition 9.12], which is not sensitive to the 5-adic properties of E. □

We have a similar statement for mod 3 representations, which can be proved in the same way as [AKT19, Proposition 9.15]:

Proposition 6.1.6. *Let F be an imaginary CM number field with $\zeta_5 \notin F$, and let*

$$\overline{\rho} : G_F \to \mathrm{GL}_2(\mathbb{F}_3)$$

be a continuous homomorphism with determinant $\bar{\epsilon}_3$. We assume $\overline{\rho}$ is decomposed generic.

Suppose we have a decomposition $S_3 = S_3^{st} \coprod S_3^{\mathrm{ord}} \coprod S_3^{\mathrm{ss}}$ of the set of places in F dividing 3.

Let F^{avoid}/F be a finite Galois extension. Then we can find a solvable CM extension L/F and an elliptic curve E/L satisfying the following conditions:

(1) *E is modular.*
(2) *The extension L/F is linearly disjoint from F^{avoid}/F*
(3) *For each place $v|3$ in F and $w|v$ in L, E_{F_w} has good ordinary reduction if $v \in S_3^{\mathrm{ord}}$, good supersingular reduction if $v \in S_3^{\mathrm{ss}}$ and and (split) multiplicative reduction if $v \in S_3^{st}$.*
(4) *There is an isomorphism $\overline{\rho}|_{G_L} \cong \overline{r}_{E,3}$.*
(5) *$\overline{\rho}|_{G_L}$ is decomposed generic.*

Lemma 6.1.7. *Let E be the modular elliptic curve produced by Proposition 6.1.5 or Proposition 6.1.6 with $r_{E,p}^\vee \cong r_\iota(\pi)$ for $p = 3$ or 5 respectively. Let $w|p$ be a place of L, lying over the place v of F. Then π is ι-ordinary at w if $v \in S_p^{st}$. The local factor π_w is unramified if $v \in S_p^{\mathrm{ord}} \coprod S_p^{\mathrm{ss}}$.*

Proof. This follows from Lemma 6.1.3. □

Proof of Theorem 6.1. Choose $p \in \{3, 5\}$ so that $\overline{r}_{E,p}$ is decomposed generic and $\overline{r}_{E,p}|_{G_{F(\zeta_p)}}$ is absolutely irreducible. Now we apply Proposition 6.1.5 or 6.1.6 with $\overline{\rho} = \overline{r}_{E,p}$, S_p^{ord} the set of places above p where E has potentially good ordinary reduction, S_p^{ss} the set of places where E has potentially good supersingular reduction and S_p^{st} the set of places where E has potentially multiplicative reduction. The appropriate proposition gives us a solvable extension L/F and a modular elliptic curve A/L with $\overline{r}_{E,p}|_{G_L} \cong \overline{r}_{A,p}$, such that the hypotheses of Theorem 5.2 apply to $\rho = r_{E,p}^\vee|_{G_L}$ (we use Lemma 6.1.7 here). The assumption that $\zeta_5 \notin F$ is sufficient to check the condition on the projective image of $\overline{\rho}$ in the theorem, by Lemma 6.1.4. We deduce that E_L is modular, and the modularity of E follows by solvable descent. □

6.2. **Group theory.** We now do a little bit of group theory to optimise the statement of Theorem 6.1 when F is quadratic (cf. Corollary 6.1.1). Our main tool will be the following well-known lemma:

Lemma 6.2.1 (Goursat's lemma)**.** *Let G_1, G_2 be finite groups and suppose $H \subset G_1 \times G_2$ is a subgroup with the projection map $p_i : H \to G_i$ surjective for $i = 1$ and 2. Then we have normal subgroups $N_i = H \cap G_i \subset G_i$ and an isomorphism $\phi : G_1/N_1 \cong G_2/N_2$ such that*

$$H = \{(g_1, g_2) : \phi(g_1 N_1) = g_2 N_2\}.$$

Lemma 6.2.2. *Let $F/\mathbb{Q}$ be a quadratic field, let p be an odd prime, and let $\overline{\rho} : G_F \to \mathrm{GL}_2(\mathbb{F}_p)$ be a homomorphism. Suppose that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible. Then $\overline{\rho}$ is decomposed generic.*

Proof. We consider the homomorphism $P = \mathrm{Proj}(\overline{\rho}) : G_F \to \mathrm{PGL}_2(\mathbb{F}_p)$. Set $L = \overline{F}^{\ker P}$, so P factors through an embedding $P : \mathrm{Gal}(L/F) \hookrightarrow \mathrm{PGL}_2(\mathbb{F}_p)$. We also set $L_1 = \overline{F}^{\ker P \cap G_{F(\zeta_p)}} = L(\zeta_p)$.

Let $\widetilde{L}$ be the Galois closure of L over $\mathbb{Q}$ in $\overline{F}$. Fixing a lift $c \in \mathrm{Gal}(\widetilde{L}/\mathbb{Q})$ of the non-trivial element in $\mathrm{Gal}(F/\mathbb{Q})$, we have an injective map:

$$\mathrm{Gal}(\widetilde{L}/F) \hookrightarrow \mathrm{Gal}(L/F) \times \mathrm{Gal}(L/F)$$
$$\sigma \mapsto (\sigma|_L, (c^{-1}\sigma c)|_L)$$

whose composition with each of the two projection maps to $\mathrm{Gal}(L/F)$ is surjective. Injectivity follows from the fact that $\widetilde{L}$ is the composite of L and $c(L)$.

The Galois closure $\widetilde{L_1}$ of L_1 over $\mathbb{Q}$ is $\widetilde{L}(\zeta_p)$, and restricting the above map to the subgroup $\mathrm{Gal}(\widetilde{L_1}/F(\zeta_p)) = \mathrm{Gal}(\widetilde{L}/\widetilde{L} \cap F(\zeta_p))$ gives an injective map

$$\mathrm{Gal}(\widetilde{L_1}/F(\zeta_p)) \hookrightarrow \mathrm{Gal}(L_1/F(\zeta_p)) \times \mathrm{Gal}(L_1/F(\zeta_p))$$

whose composition with each of the two projection maps to $\mathrm{Gal}(L_1/F(\zeta_p))$ is surjective.

We are going to show that there is an element $\tau \in \mathrm{Gal}(\widetilde{L_1}/F(\zeta_p))$ whose image under each projection map to $\mathrm{Gal}(L_1/F(\zeta_p))$ is a non-identity element of order prime to p. First we need to show that $\mathrm{Gal}(L_1/F(\zeta_p))$ itself contains a non-identity element of order prime to p. This group contains the image of $\overline{\rho}(G_F(\zeta_p))$ in $\mathrm{PGL}_2(\mathbb{F}_p)$. The irreducibility of $\overline{\rho}|_{G_{F(\zeta_p)}}$ implies that $\overline{\rho}(G_F(\zeta_p))$ either has order prime to p or contains $\mathrm{SL}_2(\mathbb{F}_p)$ (by Dickson's classification, or, more simply, [Ser72, Proposition 15]). It follows that we can find a non-identity element $T \in \mathrm{Gal}(L_1/F(\zeta_p))$ of order prime to p.

We now denote $\mathrm{Gal}(\widetilde{L_1}/F(\zeta_p))$ by H and $\mathrm{Gal}(L_1/F(\zeta_p))$ by G. Goursat's lemma tells us that there are normal subgroups $N_1, N_2 \lhd G$ and an isomorphism $\phi : G/N_1 \cong G/N_2$ such that $H = \{(g_1, g_2) : \phi(g_1N_1) = g_2N_2\}$. Note that N_1 and N_2 are necessarily of the same order. We separate into two cases:

- The N_i are p-groups (we include the possibility that the N_i are trivial). In this case, we fix a lift $T' \in G$ of $\phi(TN_1) \in G/N_2$. The order of $\phi(TN_1)$ is equal to the order of T, and is equal to the order of T' up to a p-power factor. So, replacing T and T' by a sufficiently large pth power if necessary, we have $(T, T') \in H$ with T and T' non-identity elements with order prime to p.
- The N_i are not p-groups. Then we can let T_1 be a non-identity element of N_1 of order prime to p and T_2 a non-identity element of N_2 of order prime to p. The element (T_1, T_2) is contained in H.

We have now constructed the desired element $\tau \in \mathrm{Gal}(\widetilde{L_1}/F(\zeta_p))$. By Chebotarev density, we can choose a rational prime l, unramified in $\widetilde{L_1}$, such that Frob_l is the conjugacy class of τ in $\mathrm{Gal}(\widetilde{L_1}/\mathbb{Q})$. Since τ fixes $F(\zeta_p)$, we have $l \equiv 1 \mod p$ and l splits completely in F. The Frobenius elements Frob_v for $v|l$ in F are given by the $\mathrm{Gal}(\widetilde{L_1}/F)$-conjugacy classes contained in Frob_l. These are the $\mathrm{Gal}(\widetilde{L_1}/F)$-conjugacy classes of τ and $c^{-1}\tau c$ (which could coincide).

By construction, $\overline{\rho}(\tau)$ and $\overline{\rho}(c^{-1}\tau c)$ have image in $\mathrm{PGL}_2(\mathbb{F}_p)$ a non-identity element of order prime to p. It follows that they are both regular semisimple elements of $\mathrm{GL}_2(\mathbb{F}_p)$. Since $l \equiv 1 \mod p$, we have shown that l is a decomposed generic prime for $\overline{\rho}$. □

7. Quadratic points on modular curves and modularity over quadratic fields

In this section, inspired by the proof of modularity of elliptic curves over real quadratic fields [FLHS15], our goal is to extend Corollary 6.1.1 to cover many of the excluded cases where $r_{E,p}|_{G_{F(\zeta_p)}}$ is absolutely reducible for $p = 3$ and 5. We first state the main theorem of this section.

Theorem 7.1. *Let F be an imaginary quadratic field, and let E/F be an elliptic curve such that one of the following conditions holds:*

(1) *The action of G_F on $E[5]$ is irreducible (not necessarily absolutely irreducible).*
(2) *The action of G_F on $E[3]$ is irreducible and the image of G_F in $\mathrm{Aut}(E[3])$ is not the normalizer of a split Cartan subgroup.*

Then E is modular.

The following lemma helps explicate the condition that $r_{E,p}|_{G_{F(\zeta_p)}}$ is absolutely reducible for $p = 3$ and 5.

Lemma 7.1.1. *Let F be a number field and $\overline{\rho} : G_F \to \mathrm{GL}_2(\mathbb{F}_p)$ an irreducible representation with determinant $\overline{\epsilon}_p$ and $\overline{\rho}|_{G_{F(\zeta_p)}}$ absolutely reducible.*

1. *$\overline{\rho}(G_F)$ is a subgroup of the normalizer of a Cartan subgroup.*
2. *If $p = 3$, $\overline{\rho}(G_F)$ is conjugate to $C_{\mathrm{s}}^+(3)$ (the normalizer of a split Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_3)$) or to a subgroup of $C_{\mathrm{ns}}(3)$ (a non-split Cartan).*
3. *If $p = 5$ and $[F(\zeta_5) : F] = 4$, then $\overline{\rho}(G_F)$ is conjugate to a subgroup of $C_{\mathrm{ns}}^+(5)$ (the normalizer of a non-split Cartan).*

Proof. Let $G = \overline{\rho}(G_F)$. If G acts absolutely reducibly on $\mathbb{F}_p^2$ then it is a subgroup of a Cartan subgroup (G acts semisimply, because it acts irreducibly). Suppose G acts absolutely irreducibly. We show that G is a subgroup of the normalizer of a Cartan subgroup. We have $G' = G \cap \mathrm{SL}_2(\mathbb{F}_p) = \overline{\rho}(G_{F(\zeta_p)})$, and we apply [FLHS15, Lemma 2.2] to conclude the proof of the first part.

Now assume that G is absolutely irreducible and $\det(G) = \mathbb{F}_p^\times$. It follows from [FLHS15, Lemma 2.2] that, if $\mathrm{GL}_2^+(\mathbb{F}_p)$ is the subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$ consisting of matrices with square determinant, then $G^+ = G \cap \mathrm{GL}_2^+(\mathbb{F}_p)$ is contained in a Cartan subgroup C with G contained in the normalizer of C.

For the second part, if $F = \mathbb{Q}(\zeta_3)$, then $G = G'$ is absolutely reducible and is therefore a subgroup of a non-split Cartan. We can now assume G is absolutely irreducible and $\det(G) = \mathbb{F}_3^\times$. Suppose G' is contained in a non-split Cartan C_{ns} and G is contained in its normalizer C_{ns}^+. After conjugation, we can assume that

$$C_{\mathrm{ns}} = \{\begin{pmatrix} x & -y \\ y & x \end{pmatrix} : (x, y) \in \mathbb{F}_3^2 - \{(0, 0)\}.$$

Note that $C_{\mathrm{ns}} \cap \mathrm{SL}_2(\mathbb{F}_3) = \langle \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$ is cyclic of order 4 and is also contained in the normalizer of the diagonal split Cartan $C_{\mathrm{s}}(3)$. If G' is contained in the scalars, then G is the pre-image of a single order 2 element in $\mathrm{PGL}_2(\mathbb{F}_p)$ and is therefore absolutely reducible. So $G' = C_{\mathrm{ns}} \cap \mathrm{SL}_2(\mathbb{F}_3)$. Considering the possibilities for $G \subset C_{\mathrm{ns}}^+$ with $G \subsetneq C_{\mathrm{ns}}$ which contain G' with index 2, we can see that G must also be contained in the normalizer $C_{\mathrm{s}}^+(3)$ of $C_{\mathrm{s}}(3)$. The irreducible proper subgroups of $C_{\mathrm{s}}^+(3)$ are also contained in $C_{\mathrm{ns}}(3)$.

Finally, suppose $p = 5$ and that G^+ is contained in the diagonal split Cartan $C_s(5)$ with G contained in $C_s^+(5)$. We assume that G is irreducible (not necessarily absolutely irreducible). If G^+ contains an element with eigenvalues 1 and -1, then we are in the situation of (the proof of) [FLHS15, Proposition 4.1(b)], which shows that G is a subgroup (of index 3) in the normalizer of a non-split Cartan. So we assume that G^+ does not contain such an element. It follows that G^+ is equal to the subgroup of scalar matrices in $\mathrm{GL}_2(\mathbb{F}_p)$, so G is the pre-image of an order 2 element in $\mathrm{PGL}_2(\mathbb{F}_p)$. We conclude that G is absolutely reducible, hence a subgroup of a non-split Cartan. □

At this point we need to introduce modular curves with special level structures at 3 and 5. We follow the notation of [FLHS15, §2.2], so for a subgroup $H \subset \mathrm{GL}_2(\mathbb{F}_p)$ containing $-I$ and with $\det(H) = \mathbb{F}_p^\times$, we have a modular curve $X(H)/\mathbb{Q}$ equipped with its j-invariant map $X(H) \to X(1)$. If $H_1 \subset \mathrm{GL}_2(\mathbb{F}_{p_1})$ and $H_2 \subset \mathrm{GL}_2(\mathbb{F}_{p_2})$ with $p_1 \neq p_2$, $X(H_1, H_2)$ is the modular curve given by the normalization of the fibre product $X(H_1) \times_{X(1)} X(H_2)$. The cuspidal points of $X(H_1, H_2)$ are those lying over $\infty \in X(1)$.

If $F \subset \overline{\mathbb{Q}}$ is a number field, the non-cuspidal F-rational points of $X(H_1, H_2)$ correspond to $\overline{\mathbb{Q}}$-isomorphism classes of pairs $(E, [\eta])$, where E is an elliptic curve over F and $[\eta]$ is an $(H_1 \times H_2)$-orbit of isomorphisms

$$\eta : \prod_{i=1}^{2} \mathbb{F}_{p_i}^2 \cong \prod_{i=1}^{2} E[p_i](\overline{\mathbb{Q}})$$

such that $\eta^{-1}\overline{r}_{E,p}(G_F)\eta$ is contained in H_i for $i = 1, 2$.

We will be concerned with the subgroups bp, sp, ns$p$ which are respectively the upper triangular Borel, $C_{\mathrm{s}}^+(p)$ and $C_{\mathrm{ns}}^+(p)$ in $\mathrm{GL}_2(\mathbb{F}_p)$. We will also need the Cartan subgroups themselves: s$p^\circ := C_{\mathrm{s}}(p)$ and ns$p^\circ := C_{\mathrm{ns}}(p)$.

Before proving Theorem 7.1, we give a corollary. Recall that $X_0(15) = X(\mathrm{b}3, \mathrm{b}5)$ is an elliptic curve of rank zero over $\mathbb{Q}$. It is the curve with Cremona label 15A1 (see [FLHS15, Lemma 5.6]).

Corollary 7.1.2. *Let F be an imaginary quadratic field such that $X_0(15)(F)$ is finite. Then every elliptic curve E/F is modular.*

Proof. By Theorem 7.1, we only need to consider E/F giving rise to an F-rational point P of $X_0(15) = X(\mathrm{b}3, \mathrm{b}5)$ or $X(\mathrm{s}3, \mathrm{b}5)$. We can assume that E does not have CM (otherwise it would be modular), so if $(E', [\eta])$ also gives the point P then E' is isomorphic to E or a quadratic twist of E. Now it suffices to show that each of the j-invariants coming from points of $X_0(15)(F)$ and $X(\mathrm{s}3, \mathrm{b}5)(F)$ are modular. The curve $X(\mathrm{s}3, \mathrm{b}5)$ is an ellipic curve, with Cremona label 15A3 ([FLHS15, Lemma 5.7]), and is isogenous to $X_0(15)$. We are assuming that $X_0(15)(F)$ is finite, so $X(\mathrm{s}3, \mathrm{b}5)(F)$ is also finite. It remains to check modularity for the non-cuspidal torsion points defined over imaginary quadratic fields.

We have $X_0(15)(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. A Legendre form for $X_0(15)/\mathbb{Q}$ is $y^2 = x(x+16)(x+25)$. It follows from [Kwo97, Theorem 1] that the only quadratic fields F with $X_0(15)(\mathbb{Q}) \subsetneq X_0(15)(F)^{\mathrm{tors}}$ are $F = \mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{5})$. This information is also contained in the LMFDB. It remains to show that elliptic curves giving rise to the 8 points in $X_0(15)(\mathbb{Q}(\sqrt{-1}))\backslash X_0(15)(\mathbb{Q})$ are modular. This reduces to checking modularity of a single elliptic curve (isogenous and conjugate curves give the other points), and its modularity can be verified using the Faltings–Serre method [DGP10] which has been carried out as part of the LMFDB project [LMF22, Elliptic curve 4050.1-c3 over number field $\mathbb{Q}(\sqrt{-1})$].

We also have $X(\mathrm{s}3, \mathrm{b}5)(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. A Legendre form for $X(\mathrm{s}3, \mathrm{b}5)/\mathbb{Q}$ is $y^2 = x(x+1)(x+16)$, and it follows from [Kwo97, Theorem 1] that the only quadratic field F with $X_0(15)(\mathbb{Q}) \subsetneq X_0(15)(F)^{\mathrm{tors}}$ is $F = \mathbb{Q}(\sqrt{5})$ (again, this information is contained in the LMFDB). □

Combining Theorem 6.1 with Lemma 7.1.1, to prove Theorem 7.1 we need to show modularity of the elliptic curves defined over imaginary quadratic fields giving rise to points of the following modular curves:

(1) $X(\mathrm{ns}3^\circ, \mathrm{b}5)$
(2) $X(\mathrm{b}3, \mathrm{ns}5)$
(3) $X(\mathrm{ns}3^\circ, \mathrm{ns}5)$
(4) $X(\mathrm{s}3, \mathrm{ns}5)$

We will do this is the following subsections. We used Magma to do the computations [BCP97], and there are associated Magma files available at `https://github.com/jjmnewton/modularity-iqf`.

To help us find equations for these curves, we use the following well-known facts about 'small' modular curves:

Proposition 7.1.3.
(1) *The curve* $X(\mathrm{b3})$ *is isomorphic to* $\mathbb{P}^1_{\mathbb{Q}}$ *with co-ordinate* x *and* j*-invariant* $\frac{(x+27)(x+3)^3}{x}$.
(2) *The curve* $X(\mathrm{b5})$ *is isomorphic to* $\mathbb{P}^1_{\mathbb{Q}}$ *with co-ordinate* x *and* j*-invariant* $\frac{(x^2+250x+5^5)^3}{x^5}$. *The Fricke involution* w_5 *is given by* $x \mapsto 5^3/x$.
(3) *The curve* $X(\mathrm{ns3})$ *is isomorphic* $\mathbb{P}^1_{\mathbb{Q}}$ *with co-ordinate* x *and* j*-invariant* x^3.
(4) *The curve* $X(\mathrm{ns5})$ *is isomorphic to* $\mathbb{P}^1_{\mathbb{Q}}$ *with co-ordinate* x *and* j*-invariant* $\frac{5^3x(2x+1)^3(2x^2+7x+8)^3}{(x^2+x-1)^5}$.
(5) *The curve* $X(\mathrm{ns3}, \mathrm{ns5})$ *is an elliptic curve over* $\mathbb{Q}$*, isomorphic to the curve with Cremona label* $225A1$*, with Mordell–Weil group* $X(\mathrm{ns3}, \mathrm{ns5})(\mathbb{Q}) \cong \mathbb{Z}$.
(6) *The curve* $X(\mathrm{s3})$ *is isomorphic to* $\mathbb{P}^1_{\mathbb{Q}}$ *with co-ordinate* x *and* j*-invariant* $\frac{3^3(x+1)^3(x-3)^3}{x^3}$

Proof. For the first and last part, see [SZ17, Table 1]. For the second part, see [McM04, Table 3]. For the third and fourth parts, see [Che99, Proposition 5.1, Corollary 6.3]. For the fifth part, it follows from the third and fourth parts that a singular model for $X(\mathrm{ns3}, \mathrm{ns5})$ is given by the equation $x^3 = \frac{5^3y(2y+1)^3(2y^2+7y+8)^3}{(y^2+y-1)^5}$. Some simple manipulations show that this is birational to the elliptic curve $y^2 - y = x^3 + 1$, which has Cremona label $225A1$ as claimed. This is also checked in the Magma file `ns3ns5-elliptic.m`. ☐

7.2. **Quadratic points on** $X(\mathrm{ns3}^\circ, \mathrm{b5})$**.** We begin by specifying an equation for the modular curve $X(\mathrm{ns3}^\circ, \mathrm{b5})$, together with the properties we will need to show modularity of quadratic points.

Proposition 7.2.1.
(1) *A model for* $X(\mathrm{ns3}^\circ, \mathrm{b5})$ *is given by the genus one curve*
$$C : y^2 = -3(x^4 + 2x^3 - x^2 + 10x + 25),$$
where (the homogeneization of) this equation defines a smooth curve in the weighted projective space $\mathbb{P}(1, 2, 1)$.
(2) *With the above equation,* w_5 *transforms* x*-co-ordinates by* $x \mapsto 5/x$.
(3) *The Jacobian* Jac_C *is isomorphic to the elliptic curve with Cremona label* $45A2$*, and in particular has* $\mathrm{Jac}_C(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. For the first part, we proceed in a similar way to the proof of [Zyw15, Lemma 4.4]. It follows from Proposition 7.1.3 that $X(\mathrm{ns3}, \mathrm{b5})$ is isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$, with j-invariant given by $j(x) = \frac{(x^6+250x^3+5^5)^3}{x^{15}}$ and w_5 given by $x \mapsto 5/x$. The map $X(\mathrm{ns3}^\circ, \mathrm{b5}) \to X(\mathrm{ns3}, \mathrm{b5})$ has degree two, and is ramified at 4 points with j-invariant 1728. The fibre of the j-invariant map on $X(\mathrm{ns3}, \mathrm{b5})$ at 1728 is cut out by the vanishing of the polynomial
$$\begin{aligned} &(x^6 + 250x^3 + 5^5)^3 - 1728x^{15} \\ = &(x^2-5x-25)^2(x^2-2x+5)(x^4+2x^3-x^2+10x+25)(x^4+5x^3+50x^2-125x+625)^2. \end{aligned}$$

It follows that the ramification locus is cut out by either $(x^2-5x-25)^2$ or $x^4+2x^3-x^2+10x+25$. We claim that it is the latter. If we are in the former case, then we have an elliptic curve $E/\mathbb{Q}(\sqrt{5})$ with j-invariant 1728 and image of $G_{\mathbb{Q}(\sqrt{5})}$ on $\mathrm{Aut}(E[3])$ contained in a non-split Cartan subgroup. Since $\mathbb{Q}(\sqrt{5})$ is totally real, this image also contains an element conjugate to $\mathrm{diag}(1,-1)$ but non-split Cartans mod 3 contain no such elements.

We conclude that $X(\mathrm{ns}3^\circ, \mathrm{b}5)$ is defined by an equation

$$y^2 = c(x^4 + 2x^3 - x^2 + 10x + 25)$$

for some squarefree $c \in \mathbb{Z}$. We claim that $c = -3$. We see from the equation (since the constant term 25 is a square) that $\mathbb{Q}(\sqrt{c})$ is the field of definition of the cusps of $X(\mathrm{ns}3^\circ, \mathrm{b}5)$, or equivalently the field of definition of the cusps of $X(\mathrm{ns}3^\circ)$. It is well-known that this field is $\mathbb{Q}(\sqrt{-3})$. Here is one justification: by ramification considerations as above, the double cover $X(\mathrm{ns}3^\circ) \to X(\mathrm{ns}3)$ is given by $y^2 = d(x^2+12x+144)$ for a squarefree $d \in \mathbb{Z}$. The unique rational point of $X(\mathrm{ns}3)$ with j-invariant 1728 is $x = 12$. The fibre over this point in $X(\mathrm{ns}3^\circ)$ is given by $y^2 = 432d$, so $\mathbb{Q}(\sqrt{3d})$ is determined by the image of $G_{\mathbb{Q}(\sqrt{3d})}$ in $\mathrm{Aut}(E[3])$ being contained in a non-split Cartan, where $E/\mathbb{Q}$ has j-invariant 1728 (hence CM by $\mathbb{Z}[\sqrt{-1}]$). It follows that $\mathbb{Q}(\sqrt{3d}) = \mathbb{Q}(\sqrt{-1})$ and we deduce that $d = -3$.

Having computed a model for our genus one curve, the Jacobian can be identified using classical invariant theory and the Mordell–Weil group determined with the help of a two-descent (see [AKM$^+$01] for equations for the Jacobian, we did the calculations using Magma as documented in the file `ns3ob5.m`). □

We will make use of the following points in $C(\mathbb{Q}(\sqrt{-3}))$:

$$\begin{aligned}
\infty^+ &= (1 : \sqrt{-3} : 0)\\
\infty^- &= (1 : -\sqrt{-3} : 0)\\
0^+ &= (0 : 5\sqrt{-3} : 1)\\
P_1 &= (-2 : -\sqrt{-3} : 1)\\
P_2 &= (-5/2 : 5\sqrt{-3}/4 : 1).
\end{aligned}$$

We have $C(\mathbb{Q}) = \emptyset$, as can be checked 3-adically. Using the isomorphism between $C_{\mathbb{Q}(\sqrt{-3})}$ and $\mathrm{Jac}_{C,\mathbb{Q}(\sqrt{-3})}$ given by $P \mapsto [P] - [\infty^-]$, we can match up the order 2 rational points of Jac_C with the equivalence classes of the divisors:

$$\begin{aligned}
D_0 &= 0^+ - \infty^-\\
D_1 &= P_1 - \infty^-\\
D_2 &= P_2 - \infty^-.
\end{aligned}$$

At this point, we want to compute which points of $\mathrm{Jac}_C(\mathbb{Q})$ are representented by rational divisors, not just rational equivalence classes. We denote the group of degree 0 rational divisors modulo linear equivalence by $\mathrm{Pic}^0(C)$.

Proposition 7.2.2. $\mathrm{Pic}^0(C) \cong \mathbb{Z}/2\mathbb{Z}$*, with generator* $[0^+ - \infty^-]$.

Proof. Our computations here are again documented in the file `ns3ob5.m`. There is an injective map $\mathrm{Pic}^0(C) \hookrightarrow \mathrm{Jac}_C(\mathbb{Q})$, so we need to determine which of the divisors D_i (for $i = 0, 1, 2$) are linearly equivalent to a rational divisor. For this, we essentially follow the procedure outlined in [BF04]. Translating by the rational divisor $\infty^+ + \infty^-$, we consider the degree 2 divisors $E_i = D_i + \infty^+ + \infty^-$. For each divisor, we compute a basis $\{1, f_i\}$ for each of the two-dimensional Riemann–Roch spaces $L(E_i)$ defined over $\mathbb{Q}(\sqrt{-3})$. We get:

$$f_0 = \frac{y + (\sqrt{-3}x^2 + 5\sqrt{-3})y}{x}$$
$$f_1 = \frac{y + (\sqrt{-3}x^2 - 5\sqrt{-3})y}{x+2}$$
$$f_2 = \frac{y + (\sqrt{-3}x^2 - 5\sqrt{-3})y}{x+5/2}.$$

A divisor class $[E_i]$ contains a rational divisor if and only if there is an effective degree 2 rational divisor $D_i' = P + \sigma(P)$ linearly equivalent to E_i, with $P \in C(F)$ for a quadratic field F and σ the non-trivial element of $\mathrm{Gal}(F/\mathbb{Q})$. If this is the case, then D_i' is the divisor of zeroes of a meromorphic function $f_i - \alpha$ for some $\alpha \in \mathbb{Q}(\sqrt{-3})$. For $i = 0, 1, 2$ respectively this gives us equations:

$$y = \alpha x - (\sqrt{-3}x^2 + 5\sqrt{-3})$$
$$y = \alpha(x+2) - (\sqrt{-3}x^2 - 5\sqrt{-3})$$
$$y = \alpha(x+5/2) - (\sqrt{-3}x^2 - 5\sqrt{-3}).$$

Squaring both sides, using the equation for C, and dividing out by the linear factor in x (which doesn't correspond to a zero of $f_i - \alpha$) gives us equations:

$$(6 - 2\sqrt{-3}\alpha)x^2 + (\alpha^2 - 33)x + (30 - 10\sqrt{-3}\alpha) = 0 \tag{7.2.1}$$

$$(6 - 2\sqrt{-3}\alpha)x^2 + (\alpha^2 + 15)x + (2\alpha^2 + 10\sqrt{-3}\alpha) = 0 \tag{7.2.2}$$

$$(6 - 2\sqrt{-3}\alpha)x^2 + (\alpha^2 + 12)x + (5\alpha^2/2 + 10\sqrt{-3}\alpha) = 0. \tag{7.2.3}$$

Equation (7.2.1) has the solution $x = 1 + 2i$, $\alpha = 3 + 2\sqrt{-3}$, and E_0 is linearly equivalent to $P + \sigma(P)$ for $P = (1 + 2i : 3 + 6i : 1)$. Note that, since (7.2.1) rescales to $x^2 + \cdots + 5 = 0$, we have $x(P)x(\sigma(P)) = 5$ for any P defined over a quadratic field with $E_0 \sim P + \sigma(P)$.

For the zeroes of Equation (7.2.2) to come from a rational divisor, we must have $\frac{\alpha^2+15}{6-2\sqrt{-3}\alpha} \in \mathbb{Q}$. Writing $\alpha = \alpha_0 + \alpha_1\sqrt{-3}$, with $\alpha_i \in \mathbb{Q}$, we deduce that $\alpha_0 = 0$ or $\alpha_0^2 + 3\alpha_1^2 + 6\alpha_1 + 15 = 0$. The case $\alpha_0 = 0$ can be excluded by considering the equation for y in terms of x. The equation $\alpha_0^2 + 3\alpha_1^2 + 6\alpha_1 + 15 = 0$ has no rational solutions (check 3-adically). The associated conic is the Brauer–Severi variety associated to the divisor class $[E_1]$, as described in [BF04].

Similarly, from Equation (7.2.3) we get the conic equation $\alpha_0^2 + 3\alpha_1^2 + 6\alpha_1 + 12 = 0$, which again has no rational points (check 3-adically). □

Remark 7.2.3. The conic associated to the divisor E_0 has equation $\alpha_0^2 + 3\alpha_1^2 + 6\alpha_1 - 33 = 0$, which does have rational points (for example $(3, 2)$, which corresponds to $\alpha = 3 + 2\sqrt{-3}$).

Proposition 7.2.4. *Suppose $P \in C(F)$ for a quadratic field F. We assume P is not one of the two points at infinity. Then either $x(P) \in \mathbb{Q}$ or $x(P)x(\sigma(P)) = 5$, where σ is the non-trivial element of* $\mathrm{Gal}(F/\mathbb{Q})$.

Proof. We have $[P+\sigma(P)-\infty^+-\infty^-]=0$ or $[0^+-\infty^-]$ in $\mathrm{Pic}^0(C)$, so we deduce that $[P+\sigma(P)]=[\infty^++\infty^-]$ or $[0^++\infty^+]$. In the second case, the proof of the previous proposition shows that $x(P)x(\sigma(P))=5$. In the first case, we have $P+\sigma(P)=\infty^++\infty^-+\mathrm{div}(f)$, where $f\in L(\infty^++\infty^-)$ is a $\mathbb{Q}$-linear combination of 1 and x. In particular, P is a zero of $x-\alpha$ for some $\alpha\in\mathbb{Q}$, so $x(P)\in\mathbb{Q}$. □

Corollary 7.2.5. *Let F be a quadratic field and E/F elliptic curve such that E is the elliptic curve underlying an F-point $P\in X((\mathrm{ns}3)^\circ,\mathrm{b}5)(F)$. Then E is modular.*

Proof. According to Proposition 7.2.4 we have two cases. In the first case, P maps to a point in $X(\mathrm{ns}3,\mathrm{b}5)(\mathbb{Q})$. In particular, E has rational j-invariant and is therefore modular.

In the second case, P maps to a $\mathbb{Q}$-point of the Atkin–Lehner quotient $X(\mathrm{ns}3,\mathrm{b}5)/w_5$. This shows that E and E^σ are 5-isogenous over $\overline{\mathbb{Q}}$, where σ is the non-trivial element of $\mathrm{Gal}(F/\mathbb{Q})$. In particular, E is a $\mathbb{Q}$-curve, and is therefore modular (cf. [FLHS15, §12]). □

7.3. **Quadratic points on** $X(\mathrm{b}3,\mathrm{ns}5)$**.** For this curve, our approach is similar to that of [FLHS15, Lemma 5.1]. Our computations are documented in the file `b3ns5.m`.

Proposition 7.3.1. (1) *$X(\mathrm{b}3,\mathrm{ns}5)$ is a curve of genus 2, with hyperelliptic equation*

$$C: y^2=9x^6-6x^5-35x^4+40x^2+12x-8.$$

(2) *The hyperelliptic involution on $X(\mathrm{b}3,\mathrm{ns}5)$ is equal to the Atkin–Lehner involution w_3.*
(3) *The Mordell–Weil group $\mathrm{Jac}(C)(\mathbb{Q})$ of the Jacobian of C is isomorphic to $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/10\mathbb{Z}$.*

Proof. We used our description for $X(\mathrm{b}3)$ and $X(\mathrm{ns}5)$ to write down (an affine patch of) the fibre product $X(\mathrm{b}3)\times_{X(1)}X(\mathrm{ns}5)$ and then used the Magma routines `IsHyperelliptic` and `SimplifiedModel` to find the displayed equation for $X(\mathrm{b}3,\mathrm{ns}5)$, together with a birational map from $X(\mathrm{b}3)\times_{X(1)}X(\mathrm{ns}5)$ to C.

To show that w_3 is the hyperelliptic involution, it suffices to show that it has (at least) 6 fixed points. Alternatively, we can compute the automorphism group of $X(\mathrm{b}3,\mathrm{ns}5)$ [LSR21] and note that it has order two, so the only non-trivial automorphism is the hyperelliptic involution. The fixed points of w_3 come from the elliptic curves $E_0=\mathbb{C}/\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$ and $E_1=\mathbb{C}/\mathbb{Z}[\sqrt{-3}]$ with b3 level structure coming from the kernel of the multiplication by $\sqrt{-3}$ map; using the moduli description of [KP16] we can check that there are indeed 6 fixed points for w_3, 2 from E_0 and 4 from E_1.

We computed the Mordell–Weil group using the Magma routine `MordellWeilGroupGenus2`. For our curve, a two-descent suffices to prove that the Jacobian has rank 0. We understand that this routine was created by Michael Stoll, see also [Sto01]. To double check that the torsion subgroup has size at most 20, we can compute that $\mathrm{Jac}_C(\mathbb{F}_7)\cong\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/20\mathbb{Z}$ and $\mathrm{Jac}(C)(\mathbb{Q})[2]$ has size 4 ($\mathrm{Jac}(C)(\overline{\mathbb{Q}})[2]$ is given by $(x_1,0)-(x_2,0)$ where the x_i are roots of $9x^6-6x^5-35x^4+40x^2+12x-8$; this polynomial has 3 rational roots and an irreducible cubic factor over $\mathbb{Q}$, whence it follows that there are 4 rational 2-torsion points). □

We write D_∞ for the degree two (hyperelliptic) divisor at ∞ for C.

Lemma 7.3.2. *Let P be a quadratic point of C, with conjugate $\sigma(P)$. Then we are in one of the two following cases:*

(1) *$[P + \sigma(P)] = [D_\infty]$ in $\mathrm{Pic}(C)$. Then $P = (x, \pm\sqrt{f(x)})$ where $f(x)$ is the polynomial defining C, with $x \in \mathbb{Q}$.*
(2) *$[P + \sigma(P) - D_\infty]$ is non-zero in $\mathrm{Pic}(C)$. The divisor $P + \sigma(P)$ is the unique degree 2 effective divisor representing its divisor class.*

Proof. Note that $[D_\infty]$ is the canonical divisor class. In the first case we have $L(D_\infty) = \langle 1, x\rangle$ so P and $\sigma(P)$ are zeroes of $x - \alpha$ for $\alpha \in \mathbb{Q}$. In the second case, by Riemann–Roch we have $\dim L(P + \sigma(P)) = 1$. □

Proposition 7.3.3. *Suppose $P \in C(F)$ for an imaginary quadratic field F. We assume P is not one of the two points at infinity. Then $x(P) \in \mathbb{Q}$ or $F = \mathbb{Q}(\sqrt{-11})$ and, up to complex conjugation, $P = (\frac{-5+\sqrt{-11}}{6}, \pm\frac{17-\sqrt{-11}}{6})$.*

Proof. By the previous lemma, it suffices to determine the non-zero elements $A \in \mathrm{Jac}(C)(\mathbb{Q})$ with $A + [D_\infty]$ represented by an effective divisor with points in its support defined over an imaginary quadratic field. Magma returns elements of $\mathrm{Jac}(C)(\mathbb{Q})$ using the Mumford representation, which in particular gives a minimal polynomial for the x co-ordinate. So it is easy to see which points may be imaginary quadratic. There are 9 non-zero divisor classes supported on points with rational x co-ordinate, 2 supported at ∞, 2 supported on the imaginary quadratic points specified in the statement of this proposition (and their conjugates); the remaining 6 are supported on real quadratic points. □

Corollary 7.3.4. *Let F be a quadratic field and E/F an elliptic curve such that E is the elliptic curve underlying an F-point $P \in X(\mathrm{b3}, \mathrm{ns5})(F)$. Then E is modular.*

Proof. The points $P \in C(F)$ with $x(P) \in \mathbb{Q}$ (and the points at infinity) have $\sigma(P) = w_3(P)$. So in this case, E is a $\mathbb{Q}$-curve. We are now only concerned with the $\mathbb{Q}(\sqrt{-11})$ points identified in Proposition 7.3.3. They are related by conjugation and w_3, so it suffices to show modularity of one of them. We conside the elliptic curve $E_0/\mathbb{Q}(\sqrt{-11})$ described by [LMF22, Elliptic curve 8100.2-a2 over number field $\mathbb{Q}(\sqrt{-11})$]. On the one hand, it is modular, because the data in loc. cit. shows that the image of $G_{\mathbb{Q}(\sqrt{-11})}$ on $\mathrm{Aut}(E_0[5])$ is the full normalizer of a non-split Cartan, and therefore E_0 satisfies the hypotheses of Corollary 6.1.1 (modularity was also checked explicitly by LMFDB using the Faltings–Serre method). On the other hand, we can write down a $\mathbb{Q}(\sqrt{-11})$-rational point of $X(\mathrm{b3}) \times_{X(1)} X(\mathrm{ns5})$ with the same j-invariant as E_0, map it to C, and verify that we obtain one of our points of interest. We deduce that these points are also modular. □

7.4. **Quadratic points on $X(\mathrm{s3}, \mathrm{ns5})$ and $X(\mathrm{ns3}^\circ, \mathrm{ns5})$.** Both curves $X(\mathrm{s3}, \mathrm{ns5})$ and $X(\mathrm{ns3}^\circ, \mathrm{ns5})$ are bi-elliptic, admitting degree two maps to $X(\mathrm{ns3}, \mathrm{ns5})$. The quadratic points with image a rational point in $X(\mathrm{ns3}, \mathrm{ns5})$ are modular, so we need to understand quadratic points which are not pulled back from rational points. To do this, we use Siksek's relative symmetric power Chabauty method [Sik09]. We can closely follow the implementation of this method by Box for some modular curves $X_0(N)$ [Box21]. Our computations here are documented in the Magma files `ns3ons5.m` and `s3ns5.m`.

We outline Box's method, following [Box21, §2.4]. We consider a smooth geometrically irreducible projective curve $X/\mathbb{Q}$ with Jacobian J, equipped with a degree two map $\pi : X \to C$ to another smooth curve $C/\mathbb{Q}$. Our goal is to describe the rational points of the symmetric square $X^{(2)}$, which include pairs of conjugate quadratic points $(P, \overline{P})$ of X.

We will use the following input:

(1) Primes $p_1, \dots, p_r$ of good reduction for C and X.
(2) Divisors $D_1, \dots, D_n$ generating a subgroup G of $J(\mathbb{Q})$ of finite index. We have an associated surjective homomorphism $\phi : \mathbb{Z}^n \to G$.
(3) A positive integer I such that $I \cdot J(\mathbb{Q}) \subset G$.
(4) A finite non-empty set $\mathcal{L} \subset X^{(2)}(\mathbb{Q})$, with a fixed element $\infty \in \mathcal{L}$. We have a partition $\mathcal{L} = \mathcal{L}_{pb} \coprod \mathcal{L}_{npb}$, with $x \in \mathcal{L}_{pb}$ if and only if it is pulled back from a point in $C(\mathbb{Q})$.

For each $i = 1, \dots, r$ we have a subset $\mathcal{L}_i^{\text{good}} \subset \mathcal{L}$ of known points where the (relative) Chabauty p_i-adic criterion applies. So $x \in \mathcal{L}_i^{\text{good}}$ if [Box21, Theorem 2.1] applies to x (which entails that x is the unique point of $X^{(2)}(\mathbb{Q})$ in its residue class mod p_i) or $x = \pi^*(P)$ for $P \in C(\mathbb{Q})$ and [Box21, Theorem 2.4] applies to x (which entails that every point in the residue class of x mod p_i is pulled back from a point in $C(\mathbb{Q})$). The following proposition is immediate from the definition.

Proposition 7.4.1. *Suppose* $x \in X^{(2)}(\mathbb{Q})$ *and* $\text{red}_{p_i}(x) \in \text{red}_{p_i}(\mathcal{L}_i^{\text{good}})$ *for at least one* $i = 1, \dots, r$. *Then* $x \in \mathcal{L} \cup \pi^* C(\mathbb{Q})$.

In practice, we will use a version of this proposition which lends itself to explicit computation. Consider the commutative diagram

$$\begin{array}{ccc} X^{(2)}(\mathbb{Q}) & \xrightarrow{\ \iota\ } & G \\ {\scriptstyle \text{red}_p}\downarrow & & \downarrow{\scriptstyle \text{red}_p} \\ X^{(2)}(\mathbb{F}_p) & \xrightarrow{\ \iota_p\ } & J(\mathbb{F}_p) \end{array}$$

where $\iota(x) = I([x] - \infty)$ and ι_p is the same map on the reduction mod p. For $i = 1, \dots, r$, let $\mathcal{M}_i^{\text{bad}}$ be the subset

$$\mathcal{M}_i^{\text{bad}} := \iota_{p_i}^{-1}(\text{red}_{p_i}(G)) - \text{red}_{p_i}(\mathcal{L}_i^{\text{good}}) \subset X^{(2)}(\mathbb{F}_{p_i}).$$

It follows from Proposition 7.4.1 that if x is a rational point in $X^{(2)}(\mathbb{Q})$ which is not in $\mathcal{L} \cup \pi^* C(\mathbb{Q})$, then $\text{red}_{p_i}(x) \in \mathcal{M}_i^{\text{bad}}$ for each $i = 1 \dots r$.

Theorem 7.4.2. [Box21, Theorem 2.6] *If the set*

$$\bigcap_{i=1}^{r} \text{red}_{p_i}^{-1}(\iota_{p_i}(\mathcal{M}_i^{\text{bad}})) \subset G$$

is empty, then $X^{(2)}(\mathbb{Q}) = \mathcal{L} \cup \pi^* C(\mathbb{Q})$.

Proof. Suppose $x \in X^{(2)}(\mathbb{Q}) - (\mathcal{L} \cup \pi^* C(\mathbb{Q}))$. We have already observed that $\text{red}_{p_i}(x) \in \mathcal{M}_i^{\text{bad}}$ for each i. Then $\iota(x) \in G$ and we have $\text{red}_{p_i}(\iota(x)) \in \iota_{p_i}(\mathcal{M}_i^{\text{bad}})$ for each i. $\square$

For each i, $\text{red}_{p_i}^{-1}(\iota_{p_i}(\mathcal{M}_i^{\text{bad}}))$ will give us a union of cosets of $\ker(\text{red}_{p_i})$ in G (possibly an empty union, if $\mathcal{M}_i^{\text{bad}} = \emptyset$).

To get started applying this method in practice, we need equations for the curves and a formula for the bi-elliptic involution with quotient $X = X(\text{ns}3, \text{ns}5)$. It turns out that both curves have automorphism groups of order two, so the non-trivial automorphism must be the bi-elliptic involution.

Proposition 7.4.3. (1) *$X(\mathrm{ns}3^\circ, \mathrm{ns}5)$ is isomorphic to the plane quartic C_1 with equation*

$$9x^4+19x^2y^2+y^4+9x^3+19x^2y+22xy^2+2y^3+10x^2+22xy+13y^2+7x+12y+11=0.$$

It has a unique automorphism w_1 of order 2 defined over $\mathbb{Q}$, given by $(x,y)\mapsto(x,-y-1)$.

(2) *$X(\mathrm{s}3, \mathrm{ns}5)$ is isomorphic to the plane quartic C_2 with equation*

$$-x^4+2x^3y+x^2y^2+8x^3+2x^2y-2xy^2-y^3-3x^2-3xy+3y^2+2x-3y+1.$$

It has a unique automorphism w_2 of order 2 defined over $\mathbb{Q}$, given by $(x,y)\mapsto(\frac{3x+y+2}{4x-2y+1}, \frac{8x+y-8}{4x-2y+1})$.

(3) *The $\mathbb{Q}(\sqrt{-55})$ points*

$$P_1=(\frac{1+\sqrt{-55}}{28}, \frac{27-\sqrt{-55}}{56}), P_2=(\frac{3-\sqrt{-55}}{4}, \frac{3+3\sqrt{-55}}{4})$$

of C_2 have j-invariant -32768.

Proof. We determined a model $y^2=-3(x^2+12x+144)$ for the conic $X(\mathrm{ns}3^\circ)$ in Proposition 7.2.1. Together with our model for $X(\mathrm{ns}5)$ this gives us a singular model for $X(\mathrm{ns}3^\circ, \mathrm{ns}5)$. Using Magma, we determined a model for the desingularization and checked it was isomorphic to the plane quartic C_1 with the given equation. The automorphisms of C_1 can be computed using the function `AutomorphismGroupOfPlaneQuartic` written by Lercier, Sijsling and Ritzenthaler [LSR21]. The same procedure was used for $X(\mathrm{s}3, \mathrm{ns}5)$. We also wrote down the points in the singular model (where the j-invariant is easy to compute) which map to P_1 and P_2. □

We write $\pi_i : C_i \to X := X(\mathrm{ns}3, \mathrm{ns}5)$ for the bi-elliptic quotient map.

Proposition 7.4.4. (1) *We have $\mathrm{rk}(\mathrm{Jac}_{C_i}(\mathbb{Q}))=\mathrm{rk}(\mathrm{Jac}_X(\mathbb{Q}))=1$ for $i=1$ and 2.*

(2) *The torsion subgroups $\mathrm{Jac}_{C_i}(\mathbb{Q})^{\mathrm{tors}}$ satisfy*
- *$\mathrm{Jac}_{C_1}(\mathbb{Q})^{\mathrm{tors}}$ is isomorphic to a subgroup of $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$*
- *$\mathrm{Jac}_{C_2}(\mathbb{Q})^{\mathrm{tors}}\cong\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/10\mathbb{Z}$ (explicit generators will be identified in the proof).*

(3) *Pick a generator $D\in\mathrm{Jac}_X(\mathbb{Q})\cong\mathbb{Z}$. Set $G_i=\langle\pi_i^*D\rangle\subset\mathrm{Jac}_{C_i}(\mathbb{Q})$. Then $4(\mathrm{Jac}_{C_1}(\mathbb{Q}))\subset 2G_1$ and $10(\mathrm{Jac}_{C_2}(\mathbb{Q}))\subset\langle 5G_1, \mathrm{Jac}_{C_2}(\mathbb{Q})[2]\rangle$.*

Proof. Using Chen's isogeny (in the form of [dSE00, Théoréme 2]) and the fact that $X(\mathrm{ns}3^\circ)$ and $X(\mathrm{ns}5)$ have genus 0, we see that Jac_{C_1} is isogenous to the new part of the Jacobian of $X_0(225)/w_{25}$. The relevant space of cuspforms $S_2(\Gamma_0(225))^{w_{25}=1,\mathrm{new}}$ has dimension 3. It has a basis of Hecke eigenforms with rational q-expansions, with LMFDB labels 225.2.*a*.*b*, 225.2.*a*.*c* and 225.2.*a*.*e*. The associated isogeny classes of elliptic curves are those with Cremona labels 225*c*, 225*a* and 225*d* respectively. They have Mordell–Weil rank 0, 1 and 0 respectively, so we deduce that $\mathrm{rk}(\mathrm{Jac}_{C_1}(\mathbb{Q}))=1$.

Similarly, Jac_{C_2} is isogenous to the 5-new part of the Jacobian of $X_0(225)/\langle w_9, w_{25}\rangle$. The relevant space of cuspforms is now $S_2(\Gamma_0(225))^{w_9=1,w_{25}=1,5-\mathrm{new}}$. The newform 225.2.*a*.*c*, with associated rank 1 elliptic curve 225*a*, still contributes (as it should, since C_2 maps to $X(\mathrm{ns}3, \mathrm{ns}5)$). We also get contributions from the 2 rational eigenforms in $S_2(\Gamma_0(75))^{w_{25}=1,5-\mathrm{new}}$, labels 75.2.*a*.*a* and 75.2.*a*.*b* in the LMFDB (w_9 has characteristic polynomial $X^2 - 1$ on the oldspaces generated by each eigenform). The associated isogeny classes of elliptic curves (Cremona labels 75*c* and 75*a* respectively) both have rank 0.

It now follows from [Box21, Proposition 3.1] that $2(\mathrm{Jac}_{C_i}(\mathbb{Q})) \subset \langle G_i, \mathrm{Jac}_{C_i}(\mathbb{Q})^{\mathrm{tors}}\rangle$ for $i = 1, 2$. We claim that

(1) $2(\mathrm{Jac}_{C_1}(\mathbb{Q})^{\mathrm{tors}}) = 0$
(2) $10(\mathrm{Jac}_{C_2}(\mathbb{Q})^{\mathrm{tors}}) = 0$.

In the first case, the orders of $\mathrm{Jac}_{C_1}(\mathbb{F}_p)$ for $p = 7, 11$ and 13 have gcd 4, so $\mathrm{Jac}_{C_1}(\mathbb{Q})^{\mathrm{tors}}$ has order dividing 4. Moreover, $\mathrm{Jac}_{C_1}(\mathbb{F}_{13}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/1710\mathbb{Z}$, so $\mathrm{Jac}_{C_1}(\mathbb{Q})^{\mathrm{tors}}$ must be isomorphic to a subgroup of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

In the second case, the orders of $\mathrm{Jac}_{C_2}(\mathbb{F}_p)$ for $p = 7, 11$ and 13 have gcd 20, so $\mathrm{Jac}_{C_2}(\mathbb{Q})^{\mathrm{tors}}$ has order dividing 20. On the other hand, we can write down a subgroup of $\mathrm{Jac}_{C_2}(\mathbb{Q})^{\mathrm{tors}}$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$, so we deduce that this is the full torsion subgroup. Generators for the torsion subgroup are given by D_1 of order 10 and D_2 of order 2, for

$$D_1 = [(0,1,1)] - [(-3,7,1)], D_2 \quad = 5([(0,1,0)] + [(-1/2,-1/2,1)] + [P_2] + [\overline{P}_2] - 2[P_1] - 2[\overline{P}_1]),$$

where $P_1 = (\frac{5+\sqrt{21}}{2}, -\sqrt{21} - 4)$, $P_2 = (\frac{-1-\sqrt{5}}{2}, \frac{3\sqrt{5}+9}{2})$ and $\overline{P}_i$ denotes the Galois conjugate of P_i.

We deduce from this that $4(\mathrm{Jac}_{C_1}(\mathbb{Q})) \subset \langle 2G_1\rangle$ and $10(\mathrm{Jac}_{C_2}(\mathbb{Q})) \subset \langle 5G_1, \mathrm{Jac}_{C_2}(\mathbb{Q})[2]\rangle$. □

This gives us all the inputs we need to apply the relative symmetric power Chabauty method. The description of vanishing differentials and their reduction goes through exactly as in [Box21, §3.4], replacing the Atkin–Lehner involution of $X_0(N)$ which appears there with our bi-elliptic involutions. We used Box's code, available at `https://github.com/joshabox/quadraticpoints/`, to carry out the computations. This also includes code written by Ozman and Siksek to search for rational points in $X^{(2)}(\mathbb{Q})$.

Proposition 7.4.5. (1) *Let* $X = X(\mathrm{ns}3^\circ, \mathrm{ns}5)$*,* $C = X(\mathrm{ns}3, \mathrm{ns}5)$ *and* $\pi : X \to C$ *the natural quotient map. We have* $X^{(2)}(\mathbb{Q}) = \pi^* C(\mathbb{Q})$*.*

(2) *Let* $X = X(\mathrm{s}3, \mathrm{ns}5)$*,* $C = X(\mathrm{ns}3, \mathrm{ns}5)$ *and* $\pi : X \to C$ *the natural quotient map. There are eight conjugate pairs of quadratic points of* X *which do not have image in* $C(\mathbb{Q})$*. Two are defined over an imaginary quadratic field, corresponding to the points* P_1, P_2 *identified in Proposition 7.4.3.*

Proof. For the first part, it turns out that we just need to apply the relative Chabauty criterion of [Box21, Theorem 2.4] for $p = 43$ and with $\mathcal{L}$ consisting of 8 degree two divisors (all pulled back from $C(\mathbb{Q})$). Digging in to what's happening, it turns out that $\mathrm{red}_{43}(G)$ is cyclic of order 7 and $\iota_{43}^{-1}(\mathrm{red}_{43}(G))$ also has size 7. The divisions in $\mathcal{L}$ cover all these possibilities, and they all satisfy the hypotheses of [Box21, Theorem 2.4].

For the second part, we run Box's Mordell–Weil sieve for the primes $11, 43$, $G = \langle 5G_1, \mathrm{Jac}_{C_2}(\mathbb{Q})[2]\rangle$ and $I = 10$. The set $\mathcal{L}$ includes 16 rational degree two divisors which are not pulled back from $C(\mathbb{Q})$, 8 of which are sums of two points in $X(\mathbb{Q})$ (not interchanged by the bi-elliptic involution). □

This proposition has the immediate corollary:

Corollary 7.4.6. (1) *Let F be a quadratic field and E/F an elliptic curve such that E is the elliptic curve underlying an F-point $P \in X(\mathrm{ns}3^\circ, \mathrm{ns}5)(F)$. Then E is modular.*
(2) *Let F be an imaginary quadratic field and E/F an elliptic curve such that E is the elliptic curve underlying an F-point $P \in X(\mathrm{s}3, \mathrm{ns}5)(F)$. Then E is modular.*

Proof. We have shown that all relevant quadratic points map to rational points of $X(\mathrm{ns}3, \mathrm{ns}5)$, with the exception of the points $P_1, P_2 \in X(\mathrm{s}3, \mathrm{ns}5)(\mathbb{Q}(\sqrt{-55}))$ and their conjugates. The latter points have rational j-invariant so we are done in all cases. □

We have now completed the proof of Theorem 7.1!

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