THE COHOMOLOGY OF SHIMURA VARIETIES WITH TORSION COEFFICIENTS

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Abstract. In this article, we survey recent work on some vanishing conjectures for the cohomology of Shimura varieties with torsion coefficients, under both local and global conditions. We discuss the p-adic geometry of Shimura varieties and of the associated Hodge–Tate period morphism, and explain how this can be used to make progress on these conjectures. Finally, we describe some applications of these results, in particular to the proof of the Sato–Tate conjecture for elliptic curves over CM fields.

1. Introduction

Shimura varieties are algebraic varieties defined over number fields and equipped with many symmetries, which often provide a geometric realisation of the Langlands correspondence. After base change to $\mathbb{C}$, they are closely related to certain locally symmetric spaces, but the beauty of Shimura varieties lies in their rich arithmetic.

To describe a Shimura variety, one needs to start with a Shimura datum (G, X). Here, G is a connected reductive group over $\mathbb{Q}$ and X is a conjugacy class of homomorphisms $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$ of algebraic groups over $\mathbb{R}$. Both G and X are required to satisfy certain highly restrictive axioms, cf. [Del79, §2.1]. In particular, this allows one to give the conjugacy class X a more geometric flavour, as a variation of polarisable Hodge structures. One can show that such an X is a disjoint union of finitely many copies of Hermitian symmetric domains.

Let $K \subset G(\mathbb{A}_f)$ be a sufficiently small compact open subgroup (the precise technical condition is called "neat"). The double quotient $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$, a priori a complex manifold, comes from an algebraic variety S_K defined over a number field E, called the reflex field of the Shimura datum. The varieties S_K are smooth and quasi-projective. Their étale cohomology groups (with or without compact support) $H^*_{(c)}(S_K \times_E \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$ are equipped with two kinds of symmetries. There is a Hecke symmetry coming from varying the level, i.e. the compact open subgroup K, and considering various transition morphisms between Shimura varieties at different levels. There is also a Galois symmetry, coming from the natural action of $\mathrm{Gal}(\overline{E}/E)$ on étale cohomology.

For this reason, Shimura varieties have played an important role in realising instances of the global Langlands correspondence over number fields. Indeed, a famous conjecture of Kottwitz predicts the relationship between the Galois representations occurring in the ℓ-adic étale cohomology of the Shimura varieties for G and those Galois representations associated with (regular, C-algebraic) cuspidal automorphic representations of G. See [XZ17, Remark 1.1.1] for a modern formulation of this conjecture.

There is a complete classification of groups that admit a Shimura datum. For example, if $G = \mathrm{GSp}_{2n}$, one can take X to be the Siegel double space

$$\left\{ Z \in \mathrm{M}_n(\mathbb{C}) \mid Z = Z^t, \mathrm{Im}(Z) \text{ positive or negative definite} \right\}. \tag{1.1}$$

The associated Shimura varieties are called Siegel modular varieties and they are moduli spaces of principally polarised abelian varieties. Many other Shimura varieties – those of so-called "abelian type" – can be studied using moduli-theoretic techniques, by relating them to Siegel modular varieties. See [Lan18] for an excellent introduction to the subject, which is focused on examples.

In this article, we will be primarily concerned with the geometry of the Shimura varieties S_K, after base change to a p-adic field, as well as with their étale cohomology groups $H^*_{(c)}(S_K \times_E \overline{\mathbb{Q}}, \mathbb{F}_\ell)$ with torsion coefficients. These groups are much less understood than their characteristic zero counterparts. We discuss certain conjectures about when these cohomology groups are expected to vanish, under both global and local conditions. Furthermore, we explain how the geometry of the Hodge–Tate period morphism, introduced in [Sch15] and refined in [CS17], can be used to make progress on these conjectures. Finally, we describe some applications of these results, in particular to the proof of the Sato–Tate conjecture for elliptic curves over CM fields [ACC+18].

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2. A vanishing conjecture for locally symmetric spaces

Let $G/\mathbb{Q}$ be a connected reductive group. We consider the symmetric space associated with the Lie group $G(\mathbb{R})$, which we define as $X = G(\mathbb{R})/K^\circ_\infty A^\circ_\infty$. Here, K°_∞ is the connected component of the identity in a maximal compact subgroup $K_\infty \subset G(\mathbb{R})$, and A°_∞ is the connected component of the identity inside the real points of the maximal $\mathbb{Q}$-split torus in the centre of G. Given a neat compact open subgroup $K \subset G(\mathbb{A}_f)$, we can form the double quotient $X_K = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$, which we call a locally symmetric space for G. This is a smooth Riemannian manifold, which does not have a complex structure in general.

Example 2.1. If $G = \mathrm{SL}_2/\mathbb{Q}$, we can identify $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ with the upper-half plane $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$ equipped with the hyperbolic metric, on which $\mathrm{SL}_2(\mathbb{R})$ acts transitively by the isometries

$$z \mapsto \frac{az+b}{cz+d} \text{ for } z \in \mathbb{H}^2 \text{ and } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R}).$$

Under this action, $\mathrm{SO}_2(\mathbb{R})$ is the stabiliser of the point i. By strong approximation for $\mathrm{SL}_2/\mathbb{Q}$, for any compact open subgroup $K \subseteq \mathrm{SL}_2(\widehat{\mathbb{Z}})$, there is only one double coset $\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A}_f)/K$. Write $\Gamma = \mathrm{SL}_2(\mathbb{Q}) \cap K$, which will be a congruence subgroup contained in $\mathrm{SL}_2(\mathbb{Z})$. The locally symmetric spaces X_K can be identified with quotients $\Gamma\backslash\mathbb{H}^2$. For Γ neat, these quotients inherit the complex structure on $\mathbb{H}^2$ and can be viewed as Riemann surfaces. Even more, these quotients arise from algebraic curves called *modular curves*, which are defined over finite extensions of $\mathbb{Q}$. Modular curves are examples of (connected) Shimura varieties. They represent moduli problems of elliptic curves endowed with additional structures. Even though they are some of the simplest Shimura varieties (the main complication being that they are non-compact), their geometry is already fascinating.

However, let $F/\mathbb{Q}$ be an imaginary quadratic field and take $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{SL}_2$. Then we can identify the symmetric space $X = \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2(\mathbb{R})$ with 3-dimensional hyperbolic space $\mathbb{H}^3$. Once again, we can identify the locally symmetric spaces X_K with quotients $\Gamma\backslash\mathbb{H}^3$, where $\Gamma = \mathrm{SL}_2(F) \cap K$ is a congruence subgroup. In this case, the locally symmetric spaces are arithmetic hyperbolic 3-manifolds and do not admit a complex structure. In particular, we cannot speak of Shimura varieties in this setting.

In general, Shimura varieties are closely related to locally symmetric spaces, as in the first example, though the latter are much more general objects. For example, the locally symmetric spaces for $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_n$ do not arise from Shimura varieties if $n \geq 3$, and, for $n = 2$, they can only be related to Shimura varieties if F is a totally real field. In some instances, such as for $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ with F a totally real field, one needs to replace $G(\mathbb{R})/K_\infty^\circ A_\infty^\circ$ by a slightly different quotient in order to obtain Shimura varieties[1]. We now define the invariants

$$l_0 = \mathrm{rk}(G(\mathbb{R})) - \mathrm{rk}(K_\infty) - \mathrm{rk}(A_\infty) \text{ and } q_0 = \frac{1}{2}(\dim_{\mathbb{R}}(X) - l_0).$$

These were first introduced by Borel–Wallach in [BW00]. There, they show up naturally in the computation of the $(\mathfrak{g}, K_\infty)$-cohomology of tempered representations of $G(\mathbb{R})$. In the Shimura variety setting, we consider the variants $l_0 = l_0(G^{\mathrm{ad}})$ and $q_0 = q_0(G^{\mathrm{ad}})$, because of the different quotient used. In this case, $l_0(G^{\mathrm{ad}})$ can be shown to be equal to 0 by the second axiom in the definition of a Shimura datum.

As K varies, we have a tower of locally symmetric spaces $(X_K)_K$, on which a spherical Hecke algebra $\mathbb{T}$ for G acts by correspondences. The systems of Hecke eigenvalues occurring in the cohomology groups $H^*_{(c)}(X_K, \mathbb{C})$ or equivalently, the maximal ideals of $\mathbb{T}$ in the support of these cohomology groups, can be related to automorphic representations of $G(\mathbb{A}_f)$ by work of Franke and Matsushima [Fra98]. The goal of this section is to state a conjecture on the cohomology of locally symmetric spaces with torsion coefficients $\mathbb{F}_\ell$, where ℓ is a prime number. This conjecture is formulated in [Eme14] (see the discussion around Conjecture 3.3 in *loc. cit.*) and in [CG18, Conjecture B]. Roughly, it says that the part of the cohomology outside the range of degrees $[q_0, q_0 + l_0]$ is somehow degenerate. Note that this range of degrees is symmetric about the middle $\frac{1}{2}\dim_{\mathbb{R}} X$ of the total range of cohomology and, in the Shimura variety case, it equals the middle degree of cohomology.

To formulate this more precisely, we use the notion of a non-Eisenstein maximal ideal in the Hecke algebra, for which we need to pass to the Galois side of the global Langlands correspondence. For simplicity, we will restrict our formulation to the case of $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_n$ for some number field F, although the conjecture makes

[1] We make a small abuse of notation by using X to denote both the conjugacy class from the introduction, which is used in the definition of a Shimura datum, and the quotient $G(\mathbb{R})/K_\infty^\circ A_\infty^\circ$ considered in this section. See [Eme06, §2.4] for an extended discussion of the various quotients.

sense more generally. Let $\mathbb{T}$ be the abstract spherical Hecke algebra away from a finite set S of primes of F and let $\mathfrak{m} \subset \mathbb{T}$ be a maximal ideal in the support of $H^*_{(c)}(X_K, \mathbb{F}_\ell)$. Assume that there exists a continuous, semi-simple Galois representation $\bar{\rho}_\mathfrak{m} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$ *associated with* $\mathfrak{m}$: by this, we mean that $\bar{\rho}_\mathfrak{m}$ is unramified at all the primes of F away from the finite set S, and that, at any prime away from S, the Satake parameters of $\mathfrak{m}$ match the Frobenius eigenvalues of $\bar{\rho}_\mathfrak{m}$. (The precise condition is in terms of the characteristic polynomial of $\bar{\rho}_\mathfrak{m}$ applied to the Frobenius at such a prime and depends on various choices of normalisations. See, for example [ACC$^+$18, Theorem 2.3.5] for a precise formulation.) Since the Galois representation is assumed to be semi-simple and we are working with $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_n$, this property will characterise $\bar{\rho}_\mathfrak{m}$ by the Cebotarev density theorem and the Brauer–Nesbitt theorem. We say that $\mathfrak{m}$ is *non-Eisenstein* if such a $\bar{\rho}_\mathfrak{m}$ is absolutely irreducible.

The existence of $\bar{\rho}_\mathfrak{m}$ as above should be thought of as a mod ℓ version of the global Langlands correspondence, in the automorphic-to-Galois direction; in the case $F = \mathbb{Q}$, this existence was conjectured by Ash [Ash92]. The striking part of this conjecture is that it should apply to torsion classes in the cohomology of locally symmetric spaces, not just to those classes that lift to characteristic zero, and which can be related to automorphic representations of G. For general number fields, the existence of such Galois representations seems out of reach at the moment, even for classes in characteristic zero!

However, let F be a CM field: using non-standard terminology, we mean that F is either a totally real field or a totally complex quadratic extension thereof. In this case, Scholze constructed such Galois representations in the breakthrough paper [Sch15]. This strengthened previous work [HLTT16] that applied to cohomology with $\mathbb{Q}_\ell$-coefficients. Both these results relied, in turn, on the construction of Galois representations in the self-dual case, due to many people, including Clozel, Kottwitz, Harris–Taylor [HT01], Shin [Shi11] and Chenevier–Harris [CH13].

We can now state the promised vanishing conjecture for the cohomology of locally symmetric spaces with $\mathbb{F}_\ell$-coefficients.

Conjecture 2.2. *Assume that* $\mathfrak{m} \subset \mathbb{T}$ *is a non-Eisenstein maximal ideal in the support of* $H^*_{(c)}(X_K, \mathbb{F}_\ell)$. *Then* $H^i_{(c)}(X_K, \mathbb{F}_\ell)_\mathfrak{m} \neq 0$ *only if* $i \in [q_0, q_0 + l_0]$.

In the two examples discussed in 2.1, this conjecture can be verified "by hand", since one only needs to control cohomology in degree 0 (the top degree of cohomology can be controlled using Poincaré duality). In the case of $\mathrm{GL}_2 / \mathbb{Q}$, one can show that the systems of Hecke eigenvalues $\mathfrak{m}$ in the support of $H^0(X_K, \mathbb{F}_\ell)$ satisfy

$$\bar{\rho}_\mathfrak{m} \simeq \chi \oplus \chi_{\mathrm{cyclo}} \cdot \chi, \tag{2.1}$$

where χ is a suitable mod ℓ character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\chi_{\mathrm{cyclo}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{F}_\ell^\times$ is the mod ℓ cyclotomic character. Later, we will introduce a local *genericity* condition at an auxiliary prime $p \neq \ell$ and we will see that the $\bar{\rho}_\mathfrak{m}$ in (2.1) also fail to satisfy genericity everywhere. In addition to these and a few more low-dimensional examples, one can also consider the analogue of Conjecture 2.2 for $H^*_{(c)}(X_K, \mathbb{Q}_\ell)$. This analogue is related to Arthur's conjectures on the cohomology of locally symmetric spaces [Art96] and can be verified for GL_n over CM fields using work of Franke and Borel–Wallach (see [ACC$^+$18, Theorem 2.4.9]).

Conjecture 2.2 is motivated by the Calegari–Geraghty enhancement [CG18] of the classical Taylor–Wiles method for proving automorphy lifting theorems. The classical method works well in settings where the (co)homology of locally symmetric spaces is concentrated in one degree, for example for $\mathrm{GL}_2/\mathbb{Q}$ after localising at a non-Eisenstein maximal ideal, or for definite unitary groups over totally real fields. In general, however, a certain numerical coincidence that is used to compare the Galois and automorphic sides breaks down. Calegari and Geraghty had a significant insight: they reinterpret the failure of the numerical coincidence in terms of the invariant l_0. More precisely, l_0 arises naturally from a computation on the Galois side, and the commutative algebra underlying the method can be adjusted if one knows that the cohomology on the automorphic side, after localising at a non-Eisenstein maximal ideal, is concentrated in a range of degrees of length at most l_0. For an overview of the key ideas involved in the Calegari–Geraghty method, see [Cal21, §10].

In the case of Shimura varieties, Conjecture 2.2 predicts that the non-Eisenstein part of the cohomology with $\mathbb{F}_\ell$-coefficients is concentrated in the middle degree. The initial progress on this conjecture in the Shimura variety setting had rather strong additional assumptions: for example one needed ℓ to be an unramified prime for the Shimura datum and K_ℓ to be hyperspecial, as in the work of Dimitrov [Dim05] and Lan–Suh [LS12, LS13]. The theory of perfectoid Shimura varieties and their associated Hodge–Tate period morphism has been a game-changer in this area. For the rest of this article, we will discuss more recent progress on Conjecture 2.2 and related questions in the special case of Shimura varieties, as well as applications that go beyond the setting of Shimura varieties.

3. The Hodge–Tate period morphism

The Hodge–Tate period morphism was introduced by Scholze in his breakthrough paper [Sch15] and it was subsequently refined in [CS17]. It gives an entirely new way to think about the geometry and cohomology of Shimura varieties. In the past decade, it has had numerous striking applications to the Langlands programme: to Scholze's construction of Galois representations for torsion classes, to the vanishing theorems discussed in § 4 and § 5, to the construction of higher Coleman theory by Boxer and Pilloni [BP21], and to a radically new approach to the Fontaine–Mazur conjecture due to Pan [Pan20].

For simplicity, let us consider a Shimura datum (G, X) of *of Hodge type*. By this, we mean that (G, X) admits a closed embedding into a Siegel datum $(\widetilde{G}, \widetilde{X})$, where $\widetilde{G} = \mathrm{GSp}_{2n}$, for some $n \in \mathbb{Z}_{\geq 1}$, and $\widetilde{X}$ is as in (1.1). For example, (G, X) could be a Shimura datum of *PEL type* arising from a unitary similitude group: the corresponding Shimura varieties will represent a moduli problem of abelian varieties equipped with extra structures (polarisations, endomorphisms and level structures). This unitary case will be the main example to keep in mind, as this will also play a central role in § 4.

For some representative $h \in X$, we consider the *Hodge cocharacter*

$$\mu = h \times_{\mathbb{R}} \mathbb{C}|_{1\text{st } \mathbb{G}_m \text{ factor}} : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}.$$

The axioms in the definition of the Shimura datum imply that μ is minuscule. The reflex field E is the field of definition of the conjugacy class $\{\mu\}$; it is a finite extension of $\mathbb{Q}$ and the corresponding Shimura varieties admit canonical models over E. The cocharacter μ also determines two opposite parabolic subgroups P_μ^{std}

and P_μ, whose conjugacy classes are defined over E. These are given by

$$P_\mu^{\mathrm{std}} = \{g \in G \mid \lim_{t\to\infty} \mathrm{ad}(\mu(t))g \text{ exists}\}, \quad P^\mu = \{g \in G \mid \lim_{t\to 0} \mathrm{ad}(\mu(t))g \text{ exists}\}.$$

We let $\mathrm{Fl}^{\mathrm{std}}$ and Fl denote the associated flag varieties, which are also defined over E.

Here is a more moduli-theoretic way to think about these the two parabolics. The chosen symplectic embedding $(G, X) \hookrightarrow (\widetilde{G}, \widetilde{X})$ gives rise to a faithful representation V of G. The embedding also gives rise to an abelian scheme A_K over the Shimura variety S_K at some level $K = G(\mathbb{A}_f) \cap \widetilde{K}$, obtained by restricting the universal abelian scheme over the Siegel modular variety at level $\widetilde{K} \subset \widetilde{G}(\mathbb{A}_f)$. The cocharacter μ induces a grading of $V_{\mathbb{C}}$, which in turn defines two filtrations on $V_{\mathbb{C}}$, a descending one $\mathrm{Fil}^\bullet$ and an ascending one $\mathrm{Fil}_\bullet$. The parabolic P_μ^{std} is the stabiliser of $\mathrm{Fil}^\bullet$, which is morally the Hodge–de Rham filtration on the Betti cohomology of A_K. There is a holomorphic, $G(\mathbb{R})$-equivariant embedding

$$\pi_{\mathrm{dR}} : X \hookrightarrow \mathrm{Fl}^{\mathrm{std}}(\mathbb{C}) = G(\mathbb{C})/P_\mu^{\mathrm{std}} \tag{3.1}$$

called the Borel embedding, defined by $h \mapsto \mathrm{Fil}^\bullet(\mu_h)$. The axioms of a Shimura datum imply that X is a variation of polarisable Hodge structures of abelian varieties. Moduli-theoretically, π_{dR} sends a Hodge structure, such as

$$H^1(A, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \simeq H^0(A, \Omega^1_A) \oplus H^1(A, \mathcal{O}_A),$$

to the associated Hodge–de Rham filtration, e.g. $H^0(A, \Omega^1_A) \subset H^1(A, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$. The embedding π_{dR} is an example of a *period morphism*. Historically, it has played an important role in the construction of canonical models of automorphic vector bundles over E (or even integrally), such as in work of Harris and Milne.

On the other hand, the parabolic subgroup P_μ is the stabiliser of the ascending filtration $\mathrm{Fil}_\bullet$. This gives rise to an anti-holomorphic embedding

$$X \hookrightarrow \mathrm{Fl}(\mathbb{C}) = G(\mathbb{C})/P_\mu \tag{3.2}$$

Morally, P_μ is the stabiliser of the Hodge–Tate filtration on the p-adic étale cohomology of A_K. The Hodge–Tate period morphism will be a p-adic analogue of the embedding (3.1) (or perhaps of the embedding (3.2), depending on one's perspective).

Let p be a rational prime, $\mathfrak{p} \mid p$ a prime of E, and let C be the completion of an algebraic closure of $E_{\mathfrak{p}}$. We consider the adic spaces $\mathcal{S}_K$ and $\mathscr{F}\ell$ over $\mathrm{Spa}(C, \mathcal{O}_C)$ corresponding to the algebraic varieties S_K and Fl over E. A striking result of Scholze shows that the tower of Shimura varieties $(\mathcal{S}_{K^pK_p})_{K_p}$ acquires the structure of a perfectoid space (in the sense of [Sch12]) as K_p varies over compact open subgroups of $G(\mathbb{Q}_p)$. More precisely, the following result was established in [Sch15, §3,4] and later refined in [CS17, §2], by correctly identifying the target of the Hodge–Tate period morphism.

Theorem 3.1. *There exists a unique perfectoid space $\mathcal{S}_{K^p}$ satisfying $\mathcal{S}_{K^p} \sim \varprojlim_{K_p} \mathcal{S}_{K^pK_p}$*[2]*, in the sense of* [SW13, Definition 2.4.1]*, and a $G(\mathbb{Q}_p)$-equivariant morphism of adic spaces*

$$\pi_{\mathrm{HT}} : \mathcal{S}_{K^p} \to \mathscr{F}\ell.$$

[2]It is enough to consider the Shimura varieties as adic spaces over $E_{\mathfrak{p}}$ and the tower still acquires a perfectoid structure in a non-canonical way. We work over C for simplicity and also because this gives rise to the étale cohomology groups we want to understand.

Moreover, π_{HT} is equivariant for the usual action of Hecke operators away from p on $\mathcal{S}_{K^p}$ and their trivial action on $\mathscr{F}\ell$.

In the Siegel case $G = \mathrm{GSp}_{2n}/\mathbb{Q}$, one can describe the Hodge–Tate period morphism π_{HT} from a moduli-theoretic perspective as follows. An abelian variety A/C, equipped with a trivialisation $T_pA \simeq \mathbb{Z}_p^{2n}$ will be sent to the first piece of the Hodge–Tate filtration

$$\mathrm{Lie}\ A \subset T_pA \otimes_{\mathbb{Z}_p} C \simeq C^{2n}.$$

Dually, one has the Hodge–Tate filtration on the p-adic étale cohomology of A:

$$0 \to H^1(A, \mathcal{O}_A) \to H^1_{\text{ét}}(A, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \to H^0(A, \Omega^1_{A/C})(-1) \to 0, \tag{3.3}$$

where (-1) denotes a Tate twist (which is important for keeping track of the Galois action). To show that the morphism defined this way on $\mathrm{Spa}(C, C^+)$-points comes from a morphism of adic spaces, it is important to know that the filtration (3.3) varies continuously. At the same time, to extend the result to Shimura varieties of Hodge type and to cut down the image to $\mathscr{F}\ell$, one needs to keep track of Hodge tensors carefully. Both problems are solved via relative p-adic Hodge theory for the morphism $\mathcal{A}_K \to \mathcal{S}_K$, where $\mathcal{A}_K$ is the restriction to $\mathcal{S}_K$ of a universal abelian scheme over an ambient Siegel modular variety. See [Car19, §3] for an overview.

Theorem 3.1 can be extended to minimal and toroidal compactifications of Siegel modular varieties, cf. [Sch15] and [PS16]. Moreover, there is a natural affinoid cover of $\mathscr{F}\ell$ such that the pre-image under π_{HT} of each affinoid in the cover is an affinoid perfectoid subspace of $\mathcal{S}^*_{K^p}$. The underlying reason for this is the fact that the partial minimal compactification of the ordinary locus is affine. The perfectoid structure on $\mathcal{S}^*_{K^p}$ and the affinoid nature of the Hodge–Tate period morphism play an important role in Scholze's p-adic interpolation argument, that is key for the construction of Galois representations associated with torsion classes. See also [Mor16] for an exposition of the main ideas.

Theorem 3.1 can also be extended to minimal and toroidal compactifications of Shimura varieties of Hodge type and even abelian type, cf. [She17, HJ20] and [BP21], although there are some technical issues at the boundary. For example, the cleanest formulation currently available in full generality is that the relationship $\mathcal{S}^*_{K^p} = \varprojlim_{K_p} \mathcal{S}^*_{K^pK_p}$, for a perfectoid space $\mathcal{S}^*_{K^p}$, holds in Scholze's category of diamonds [Sch17].

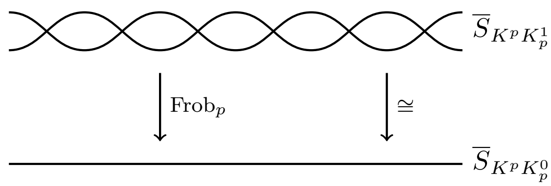
Example 3.2. To see where the perfectoid structure on $\mathcal{S}_{K^p}$ comes from, it is instructive to consider the case of modular curves and study the geometry of their special fibres: we are particularly interested in the geometry of the so-called Deligne–Rapoport model. Set $G = \mathrm{GL}_2/\mathbb{Q}$. Let $K_p^0 = \mathrm{GL}_2(\mathbb{Z}_p)$, the hyperspecial compact open subgroup and let $\overline{S}_{K^pK_p^0}/\mathbb{F}_p$ be the special fibre of the integral model over $\mathbb{Z}_{(p)}$ of the modular curve at this level. This is a smooth curve over $\mathbb{F}_p$ that represents a moduli problem (E, α) of elliptic curves equipped with prime-to-p level structures (determined by the prime-to-p level K^p). The isogeny class of the p-divisible group $E[p^\infty]$ induces the Newton stratification

$$\overline{S}_{K^pK_p^0} = \overline{S}^{\mathrm{ord}}_{K^pK_p^0} \sqcup \overline{S}^{\mathrm{ss}}_{K^pK_p^0} \tag{3.4}$$

into an open dense ordinary stratum $\overline{S}^{\mathrm{ord}}_{K^pK_p^0}$ (where $E[p^\infty]$ is isogenous to $\mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p$) and a closed supersingular stratum $\overline{S}^{\mathrm{ss}}_{K^pK_p^0}$ consisting of finitely many points (where $E[p^\infty]$ is connected).

Now let $K_p^1 \subset \mathrm{GL}_2(\mathbb{Q}_p)$ be the Iwahori subgroup and $\overline{S}_{K^pK_p^1}/\mathbb{F}_p$ be the special fibre of the integral model of the modular curve at this level. This represents a moduli problem (E, α, D) of elliptic curves equipped with prime-to-p level structures and also with a level structure at p given by a finite flat subgroup scheme $D \subset E[p]$ of order p. Again, we have the pre-image of the Newton stratification $\overline{S}_{K^pK_p^1} = \overline{S}^{\mathrm{ord}}_{K^pK_p^1} \sqcup \overline{S}^{\mathrm{ss}}_{K^pK_p^1}$. The modular curve at this level is not smooth, but rather a union of irreducible components that intersect transversely at the finitely many supersingular points.

The open and dense ordinary locus $\overline{S}^{\mathrm{ord}}_{K^pK_p^1}$ is a disjoint union of two Kottwitz–Rapoport strata: the one where $D \simeq \mu_p$ and the one where $D \simeq \mathbb{Z}/p\mathbb{Z}$. Both of these Kottwitz–Rapoport strata can be shown to be abstractly isomorphic to the ordinary stratum at hyperspecial level. If we restrict the natural forgetful map $\overline{S}^{\mathrm{ord}}_{K^pK_p^1} \to \overline{S}^{\mathrm{ord}}_{K^pK_p^0}$ to the Kottwitz–Rapoport stratum where $D \simeq \mathbb{Z}/p\mathbb{Z}$, the map can be identified (up to an isomorphism) with the geometric Frobenius. (The restriction of the map to the Kottwitz–Rapoport stratum where $D \simeq \mu_p$ is an isomorphism.)



(3.5)

On the adic generic fibre, one can extend this picture to an *anticanonical* ordinary tower, where the transition morphisms reduce modulo p to (powers of) the geometric Frobenius, giving a perfectoid space in the limit. To extend beyond the ordinary locus, Scholze uses the theory of the canonical subgroup, the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ at infinite level, and a rudimentary form of the Hodge–Tate period morphism that is just defined on the underlying topological spaces.

The above strategy generalises relatively cleanly to higher-dimensional Siegel modular varieties, modulo subtleties at the boundary. To extend Theorem 3.1 to general Shimura varieties of Hodge type, Scholze considers an embedding at infinite level into a Siegel modular variety. It is surprisingly subtle to understand directly the perfectoid structure on a general Shimura variety of Hodge type (especially in the case when $G_{\mathbb{Q}_p}$ is non-split) and this is related to the discussion in § 5. This is also related to the fact that the geometry of the EKOR stratification is more intricate when $G_{\mathbb{Q}_p}$ is non-split.

For simplicity, let us now assume that (G, X) is a Shimura datum of PEL type and that p is an unramified prime for this Shimura datum. Recall the Kottwitz set $B(G)$ classifying isocrystals with $G_{\mathbb{Q}_p}$-structure. The Hodge cocharacter μ defines a subset $B(G, \mu^{-1}) \subset B(G)$ of μ^{-1}-admissible elements. The special fibre of the Shimura variety with hyperspecial level at p admits a *Newton stratification*

$$\overline{S}_{K^pK_p^0} = \bigsqcup_{b \in B(G,\mu^{-1})} \overline{S}^b_{K^pK_p^0}$$

into locally closed strata indexed by this subset. This stratification is in terms of isogeny classes of p-divisible groups with $G_{\mathbb{Q}_p}$-structure and generalises the stratification (3.4) from the modular curve case.

For each $b \in B(G, \mu^{-1})$, one can choose a (completely slope divisible) p-divisible group with $G_{\mathbb{Q}_p}$-structure $\mathbb{X}_b/\overline{\mathbb{F}}_p$ and define the corresponding *Oort central leaf.* This is a smooth closed subscheme $\mathscr{C}^{\mathbb{X}_b}$ of the Newton stratum $\overline{S}^b_{K^pK_p^0}$, such that the isomorphism class of the p-divisible group with $G_{\mathbb{Q}_p}$-structure over each geometric point of the leaf is constant and equal to that of $\mathbb{X}_b$:

$$\mathscr{C}^{\mathbb{X}_b} = \left\{ x \in \overline{S}^b_{K^pK_p^0} \mid \overline{A}_{K^pK_p^0}[p^\infty] \times \kappa(\bar{x}) \simeq \mathbb{X}_b \times \kappa(\bar{x}) \right\}.$$

In general, there can be infinitely many leaves inside a given Newton stratum. Over each central leaf, one has the *perfect Igusa variety* $\mathrm{Ig}^b/\overline{\mathbb{F}}_p$, a pro-finite cover of $\mathscr{C}^{\mathbb{X}^b}$ which parametrises trivialisations of the universal p-divisible group with $G_{\mathbb{Q}_p}$-structure.

Variants of Igusa varieties were introduced in [HT01] in the special case of Shimura varieties of Harris–Taylor type. They were defined more generally for Shimura varieties of PEL type by Mantovan [Man05] and their ℓ-adic cohomology was computed in many cases by Shin using a counting point formula [Shi09, Shi10, Shi11]. All these authors consider Igusa varieties as pro-finite étale covers of central leaves, which trivialise the graded pieces of the slope filtration on the universal p-divisible group. Taking perfection gives a more elegant moduli-theoretic interpretation, while preserving ℓ-adic cohomology. However, the coherent cohomology of Igusa varieties is also important for defining and studying p-adic families of automorphic forms on G, as pioneered by Katz and Hida. Taking perfection is too crude for this purpose.

While the central leaf $\mathscr{C}^{\mathbb{X}_b}$ depends on the choice of $\mathbb{X}_b$ in its isogeny class, one can show that the perfect Igusa variety Ig^b only depends on the isogeny class: this follows from the equivalent moduli-theoretic description in [CS17, Lemma 4.3.4] (see also [CT21, Lemma 4.2.2], which keeps track of the extra structures more carefully). In particular, the pair (G, μ) is not determined by the Igusa variety Ig^b – it can happen that Igusa varieties that are a priori obtained from different Shimura varieties are isomorphic. See [CT21, Theorem 4.2.4] for an example and [Sem21] for a systematic analysis of this phenomenon in the function field setting.

Because $\mathrm{Ig}^b/\overline{\mathbb{F}}_p$ is perfect, the base change $\mathrm{Ig}^b \times_{\overline{\mathbb{F}}_p} \mathcal{O}_C/p$ admits a canonical lift to a flat formal scheme over Spf $\mathcal{O}_C$. We let $\mathfrak{Ig}^b$ denote the adic generic fibre of this lift, which is a perfectoid space over $\mathrm{Spa}(C, \mathcal{O}_C)$. The spaces Ig^b and $\mathfrak{Ig}^b$ have naturally isomorphic ℓ-adic cohomology groups and they both have an action of a locally profinite group $G_b(\mathbb{Q}_p)$, where G_b is an inner form of a Levi subgroup of G.

For each $b \in B(G, \mu^{-1})$, one can also consider the associated *Rapoport–Zink space*, a moduli space of p-divisible groups with $G_{\mathbb{Q}_p}$-structure that is a local analogue of a Shimura variety. Concretely in the PEL case, one considers a moduli problem of p-divisible groups equipped with $G_{\mathbb{Q}_p}$-structure, satisfying the Kottwitz determinant condition with respect to μ, and with a modulo p quasi-isogeny to the fixed p-divisible group $\mathbb{X}_b$. This moduli problem was shown by Rapoport–Zink [RZ96] to be representable by a formal scheme over Spf $\mathcal{O}_{\breve{E}_\mathfrak{p}}$, where $\breve{E}_\mathfrak{p}$ is the completion of the maximal unramified extension of $E_\mathfrak{p}$. We let $\mathcal{M}^b$ denote the adic generic fibre of this formal scheme[3], base changed to $\mathrm{Spa}(C, \mathcal{O}_C)$, and let $\mathcal{M}^b_\infty$ denote the corresponding infinite-level Rapoport–Zink space. The latter object can be shown to be a perfectoid space using the techniques of [SW13]. By *loc. cit.*, the infinite-level Rapoport–Zink space admits a local analogue of the Hodge–Tate period morphism

$$\pi^b_{\mathrm{HT}} : \mathcal{M}^b_\infty \to \mathscr{F}\ell.$$

It turns out that the geometry of π_{HT} is intricately tied up with the geometry of its local analogues π^b_{HT}. The following result is a conceptually cleaner, infinite-level version of the Mantovan product formula established in [Man05], which describes Newton strata inside Shimura varieties in terms of a product of Igusa varieties and Rapoport–Zink spaces.

Theorem 3.3. *There exists a Newton stratification*

$$\mathscr{F}\ell = \bigsqcup_{b \in B(G,\mu^{-1})} \mathscr{F}\ell^b$$

into locally closed strata.

For each $b \in B(G, \mu^{-1})$, one can consider the Newton stratum $\mathcal{S}^{\circ b}_{K^p}$ as a locally closed subspace of the good reduction locus $\mathcal{S}^\circ_{K^p}$. There exists a Cartesian diagram of diamonds over $\mathrm{Spd}(C, \mathcal{O}_C)$

$$\begin{array}{ccc} \mathcal{M}^b_\infty \times_{\mathrm{Spd}(C,\mathcal{O}_C)} \mathfrak{Ig}^b & \longrightarrow & \mathcal{M}^b_\infty \\ \big\downarrow & & \big\downarrow {\scriptstyle \pi^b_{\mathrm{HT}}} \\ \mathcal{S}^{\circ b}_{K^p} & \xrightarrow{\ \pi_{\mathrm{HT}}\ } & \mathscr{F}\ell^b \end{array} .$$

Moreover, each vertical map is a pro-étale torsor for the group diamond $\widetilde{G}_b$ (identified with $\mathrm{Aut}_G(\widetilde{\mathbb{X}}_b)$, in the notation of [CS17, §4]*).*

The decomposition into Newton strata is defined in [CS17, §3]. Morally, one first constructs a map of v-stacks $\mathscr{F}\ell \to \mathrm{Bun}_G$, where the latter is the v-stack of G-bundles on the Fargues–Fontaine curve. To construct this map of v-stacks, it is convenient to notice that one can identify the diamond associated to $\mathscr{F}\ell$ with the minuscule Schubert cell defined by μ inside the B^+_{dR}-Grassmannian for G. Once the map to Bun_G is in the picture, one uses Fargues's result that the points of Bun_G are in bijection with the Kottwitz set $B(G)$, cf. [Far20] (see also [Ans19] for an alternative proof that also works in equal characteristic). Moreover, the Newton decomposition is a *stratification*, in the sense that, for $b \in B(G, \mu)$, we have

$$\overline{\mathscr{F}\ell^b} = \bigsqcup_{b' \geq b} \mathscr{F}\ell^{b'},$$

where $\geq$ denotes the Bruhat order. The latter fact follows from a recent result of Viehmann, see [Vie21, Theorem 1.1].

On rank one points, π_{HT} is compatible with the two different ways of defining the Newton stratification: via pullback from $\overline{S}_{K^pK^\circ_p}$ on $\mathcal{S}_{K^p}$ and via pullback from Bun_G on $\mathscr{F}\ell$. The behaviour is more subtle on higher rank points. This is related to the fact that the closure relations are reversed in the two settings: the basic

[3] As a consequence of the comparison with moduli spaces of local shtukas in [SW20], one obtains a group-theoretic characterisation of Rapoport–Zink spaces as local Shimura varieties determined by the tuple (G, b, μ). We suppress (G, μ) from the notation for simplicity.

locus inside $\overline{S}_{K^pK_p^0}$ is the unique closed stratum, whereas each basic stratum inside Bun_G is open. On the other hand, the (μ)-ordinary locus is open and dense inside $\overline{S}_{K^pK_p^0}$, whereas it is a zero-dimensional closed stratum inside $\mathscr{F}\ell$. The infinite-level product formula is established in [CS17, §4], although it is formulated in terms of functors on $\mathrm{Perf}_{\breve{E}_{\mathfrak{p}}}$ [4]. This was extended to Shimura varieties of Hodge type by Hamacher [Ham19].

Assume that the Shimura varieties S_K are compact. We have the following consequence for the fibres of π_{HT}: let $\bar{x} : \mathrm{Spa}(C, C^+) \to \mathscr{F}\ell^b$ be a geometric point. Then there is an inclusion of $\mathfrak{Ig}^b$ into $\pi_{\mathrm{HT}}^{-1}(\bar{x})$, which identifies the target with the canonical compactification of the source, in the sense of [Sch17, Proposition 18.6]. In [CS19, Theorem 1.10], we extend the computation of the fibres to minimal and toroidal compactifications of (non-compact) Shimura varieties attached to quasi-split unitary groups. In this case, the fibres can be obtained from partial minimal and toroidal compactifications of Igusa varieties. It would be interesting to extend the whole infinite-level product formula to compactifications.

Example 3.4. We make the geometry of π_{HT} explicit in the case of the modular curve, i.e. for $G = \mathrm{GL}_2/\mathbb{Q}$. In this case, we identify $\mathscr{F}\ell = \mathbb{P}^{1,\mathrm{ad}}$ and we have the decomposition into Newton strata (at least on points of rank one):

$$\begin{array}{ccccccc} \mathcal{S}^*_{K^p} & = & \mathcal{S}^{*,\mathrm{ord}}_{K^p} & \sqcup & \mathcal{S}^{\mathrm{ss}}_{K^p} \\ \downarrow{\scriptstyle \pi_{\mathrm{HT}}} & & \downarrow & & \downarrow \\ \mathbb{P}^{1,\mathrm{ad}} & = & \mathbb{P}^{1,\mathrm{ad}}(\mathbb{Q}_p) & \sqcup & \Omega. \end{array}$$

The ordinary locus inside $\mathbb{P}^{1,\mathrm{ad}}$ consists of the set of points defined over $\mathbb{Q}_p$ and the basic / supersingular locus is its complement Ω, the Drinfeld upper half-plane.

The fibres of π_{HT} over the ordinary locus are "perfectoid versions" of Igusa curves. The infinite-level version of the product formula reduces, in this case, to the statement that the ordinary locus is parabolically induced from $\mathfrak{Ig}^{\mathrm{ord}}$, as in [CT21, §6]. The fibres of π_{HT} over the supersingular locus are profinite sets: the corresponding Igusa varieties can be identified with double cosets $D^\times\backslash D^\times(\mathbb{A}_f)/K^p$, where $D/\mathbb{Q}$ is the quaternion algebra ramified precisely at ∞ and p. This precise result is established in [How18], although the idea goes back to Deuring–Serre. One should be able to give an analogous description for basic Igusa varieties in much greater generality – this is closely related to Rapoport–Zink uniformisation.

4. Cohomology with mod ℓ coefficients

In this section, we outline some recent strategies for computing the cohomology of Shimura varieties with modulo ℓ coefficients using the p-adic Hodge–Tate period morphism, where ℓ and p are two *distinct* primes. We emphasise the strategies developed in [CS17], [CS19], [CT21], and [Kos21].

We will assume throughout that (G, X) is a Shimura datum of abelian type and, in practice, we will focus on two examples: the case of Shimura varieties

[4]The result precedes the notion of diamonds and, in order to ensure that $\mathcal{S}^{\circ b}_{K^p}$ is a diamond, one needs to take care in defining it. At hyperspecial level, one should consider the adic generic fibre of the formal completion of the integral model of the Shimura variety along the Newton stratum indexed by b in its special fibre.

associated with unitary similitude groups and the case of Hilbert modular varieties. Let $\mathfrak{m} \subset \mathbb{T}$ be a maximal ideal in the support of $H^*_{(c)}(S_K(\mathbb{C}), \mathbb{F}_\ell)$. By work of Scholze, cf. [Sch15, Theorem 4.3.1] and by the construction of Galois representations in the essentially self-dual case, we know in many cases how to associate a global modulo ℓ Galois representation $\bar{\rho}_\mathfrak{m}$ to the maximal ideal $\mathfrak{m}$. Therefore, the non-Eisenstein condition makes sense and one can at least formulate Conjecture 2.2. In order to make progress on this conjecture, we impose a local representation-theoretic condition at the prime p, which we treat as an auxiliary prime.

Definition 4.1. *Let $\mathbb{F}$ be a finite field of characteristic ℓ.*

1. *Let $p \neq \ell$ be a prime, $K/\mathbb{Q}_p$ be a finite extension, and $\bar{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{F})$ be a continuous representation. We say that $\bar{\rho}$ is* generic *if it is unramified and the eigenvalues (with multiplicity) $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{F}}_\ell$ of $\bar{\rho}(\mathrm{Frob}_K)$ satisfy $\alpha_i/\alpha_j \neq |\mathcal{O}_K/\mathfrak{m}_K|$ for $i \neq j$.*
2. *Let F be a number field and $\bar{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{F})$ be a continuous representation. We say that a prime $p \neq \ell$ is* decomposed generic *for $\bar{\rho}$ if p splits completely in F and, for every prime $\mathfrak{p} \mid p$ of F, $\bar{\rho}|_{\mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})}$ is generic. We say that $\bar{\rho}$ is* decomposed generic *if there exists a prime $p \neq \ell$ which is decomposed generic for $\bar{\rho}$. (If one such prime exists, then infinitely many do.)*

Remark 4.2. The condition for the local representation $\bar{\rho}$ of $\mathrm{Gal}(\overline{K}/K)$ to be generic implies that any lift to characteristic 0 of $\bar{\rho}$ corresponds under the local Langlands correspondence to a generic principal series representation of $\mathrm{GL}_n(K)$. Such a representation can never arise from a non-split inner form of GL_n/K via the Jacquet–Langlands correspondence. For this reason, a generic $\bar{\rho}$ cannot be the modulo ℓ reduction of the L-parameter of a smooth representation of a non-split inner form of GL_n/K.

A semi-simple 2-dimensional representation $\bar{\rho}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is either decomposed generic or it satisfies (2.1): the case where $\bar{\rho}$ is a direct sum of two characters can be analysed by hand, and the case where $\bar{\rho}$ is absolutely irreducible follows from the paragraph after Theorem 3.1 in [Kos20]. More generally, the condition for a global representation $\bar{\rho}$ of $\mathrm{Gal}(\overline{F}/F)$ to be decomposed generic can be ensured when $\bar{\rho}$ has large image. For example, if $\ell > 2$, F is a totally real field, and $\bar{\rho}$ is a totally odd 2-dimensional representation with non-solvable image, then $\bar{\rho}$ is decomposed generic (cf. [CT21, Lemma 7.1.8]).

Let F be an imaginary CM field. Let $(B, *, V, \langle\ ,\ \rangle)$ be a PEL datum of type A, where B is a central simple algebra with centre F. We let (G, X) be the associated Shimura datum. For a neat compact open subgroup $K \subset G(\mathbb{A}_f)$, we let S_K/E be the associated Shimura variety, of dimension d. The following conjecture is a slightly different formulation of [Kos21, Conjecture 1.2], with essentially the same content.

Conjecture 4.3. *Let $\mathfrak{m} \subset \mathbb{T}$ be a maximal ideal in the support of $H^i_{(c)}(S_K(\mathbb{C}), \mathbb{F}_\ell)$. Assume that $\bar{\rho}_\mathfrak{m}$ is decomposed generic. Then the following statements hold true:*

1. *if $H^i_c(S_K(\mathbb{C}), \mathbb{F}_\ell)_\mathfrak{m} \neq 0$, then $i \leq d$;*
2. *if $H^i(S_K(\mathbb{C}), \mathbb{F}_\ell)_\mathfrak{m} \neq 0$, then $i \geq d$.*

If the Shimura varieties S_K are compact, or if we additionally assume $\mathfrak{m}$ to be non-Eisenstein, Conjecture 4.3 implies a significant part of Conjecture 2.2 for Shimura varieties of PEL type A. Analogues of Conjecture 4.3 can be formulated (and are perhaps within reach) for other Shimura varieties, such as Siegel modular varieties.

Theorem 4.4 ([CS17] strengthened in [Kos21])**.** *Assume that G is anisotropic modulo centre, so that the Shimura varieties S_K are compact. Then Conjecture 4.3 holds true.*

Theorem 4.5 ([CS19] strengthened in [Kos21])**.** *Assume that $B = F$, $V = F^{2n}$ and G is a quasi-split group of unitary similitudes. Then Conjecture 4.3 holds true.*

Remark 4.6. The more recent results of [Kos21] have significantly fewer technical assumptions than the earlier ones of [CS17] and [CS19]. For example, Koshikawa's version of Theorem 4.5 allows F to be an imaginary quadratic field. It seems non-trivial to obtain this case with the methods of [CS19]. In the non-compact case, his results rely on the geometric constructions in [CS19], in particular on the semi-perversity result for Shimura varieties attached to quasi-split unitary groups that is established there. As he notes, a generalisation of this semi-perversity result should lead to a full proof of Conjecture 4.3 for Shimura varieties of PEL type A. The more general semi-perversity result will be obtained in the upcoming PhD thesis of Mafalda Santos.

In the case of Harris–Taylor Shimura varieties, Theorem 4.4 was first proved by Boyer [Boy19]. Boyer's argument uses the integral models of Shimura varieties of Harris–Taylor type, but it is close in spirit to the argument carried out in [CS17] on the generic fibre. What is really interesting about Boyer's results is that he goes *beyond genericity*, in the following sense. Given the eigenvalues (with multiplicity) $\alpha_1, \ldots, \alpha_n$ of $\bar{\rho}_{\mathfrak{m}}(\mathrm{Frob}_{\mathfrak{p}})$, with $\mathfrak{p} \mid p$ the relevant prime of F[5], one can define a "defect" that measures how far $\bar{\rho}_{\mathfrak{m}}$ is from being generic at $\mathfrak{p}$. Concretely, set $\delta_{\mathfrak{p}}(\mathfrak{m})$ to be equal to the length of the maximal chain of eigenvalues where the successive terms have ratio equal to $|\mathcal{O}_{F_{\mathfrak{p}}}/\mathfrak{m}_{F_{\mathfrak{p}}}|$. Boyer shows that the cohomology groups $H^i_{(c)}(S_K(\mathbb{C}), \mathbb{F}_\ell)_{\mathfrak{m}}$ are non-zero at most in the range $[d - \delta_{\mathfrak{p}}(\mathfrak{m}), d + \delta_{\mathfrak{p}}(\mathfrak{m})]$. As noted by both Emerton and Koshikawa, such a result is consistent with Arthur's conjectures on the cohomology of Shimura varieties with $\mathbb{C}$-coefficients and points towards a modulo ℓ analogue of these conjectures.

Let us also discuss the analogous vanishing result in the Hilbert case. Let F be a totally real field of degree g and let $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$. For a neat compact open subgroup $K \subset G(\mathbb{A}_f)$, we let $S_K/\mathbb{Q}$ be the corresponding Hilbert modular variety, of dimension g.

Theorem 4.7. [CT21, Theorem A] *Let $\ell > 2$ and $\mathfrak{m} \subset \mathbb{T}$ be a maximal ideal in the support of $H^i_{(c)}(S_K(\mathbb{C}), \mathbb{F}_\ell)$. Assume that the image of $\bar{\rho}_{\mathfrak{m}}$ is not solvable, which implies that $\bar{\rho}_{\mathfrak{m}}$ is absolutely irreducible and decomposed generic. Then $H^i_c(S_K(\mathbb{C}), \mathbb{F}_\ell)_{\mathfrak{m}} = H^i(S_K(\mathbb{C}), \mathbb{F}_\ell)_{\mathfrak{m}}$ is non-zero only for $i = g$.*

The same result holds for all quaternionic Shimura varieties and we can even prove the analogue of Boyer's result that goes beyond genericity in all these settings. As an application, we deduce (under some technical assumptions) that the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ occurs in the completed cohomology of Hilbert modular varieties, when p is a prime that splits completely in F. This uses

[5]In this special case, one does not have to impose the condition that p splits completely in F, and it suffices to have genericity at one prime $\mathfrak{p} \mid p$.

the axiomatic approach via patching introduced in [CEG$^+$16] and further developed in [CEG$^+$18, GN16].

We now outline the original strategy for proving Theorem 4.4, which was introduced in [CS17]. Let p be a prime and $K = K^p K_p \subset G(\mathbb{A}_f)$ be a neat compact open subgroup. The Hodge–Tate period morphism gives rise to a $\mathbb{T}$-equivariant diagram

$$\begin{array}{ccc} & \mathcal{S}_{K^p} & \\ \swarrow & & \searrow^{\pi_{\mathrm{HT}}} \\ \mathcal{S}_{K^p K_p} & & \mathscr{F}\ell. \end{array} \tag{4.1}$$

The standard comparison theorems between various cohomology theories allow us to identify $H^*_{(c)}(S_K(\mathbb{C}), \mathbb{F}_\ell)_{\mathfrak{m}}$ with $H^*_{(c)}(\mathcal{S}_K, \mathbb{F}_\ell)_{\mathfrak{m}}$. The arrow on the left hand side of (4.1) is a K_p-torsor, so the Hochschild–Serre spectral sequence allows us to recover $H^*_{(c)}(\mathcal{S}_K, \mathbb{F}_\ell)_{\mathfrak{m}}$ from $H^*_{(c)}(\mathcal{S}_{K^p}, \mathbb{F}_\ell)_{\mathfrak{m}}$. The idea is now to compute $H^*_{(c)}(\mathcal{S}_{K^p}, \mathbb{F}_\ell)_{\mathfrak{m}}$ in two stages: first understand the complex of sheaves $(R\pi_{\mathrm{HT}*}\mathbb{F}_\ell)_{\mathfrak{m}}$ on $\mathscr{F}\ell$, then compute the total cohomology using the Leray–Serre spectral sequence.

Two miraculous things happen that greatly simplify the structure of $(R\pi_{\mathrm{HT}*}\mathbb{F}_\ell)_{\mathfrak{m}}$. The first is that $(R\pi_{\mathrm{HT}*}\mathbb{F}_\ell)_{\mathfrak{m}}$ behaves like a perverse sheaf on $\mathscr{F}\ell$. This is because π_{HT} is simultaneously *affinoid*, as discussed after Theorem 3.1, and *partially proper*, because the Shimura varieties were assumed to be compact. In particular, the restriction of $(R\pi_{\mathrm{HT}*}\mathbb{F}_\ell)_{\mathfrak{m}}$ to a highest-dimensional stratum in its support is concentrated in one degree. By the computation of the fibres of π_{HT}, this implies that the cohomology groups $R\Gamma(\mathfrak{Ig}^b, \mathbb{Z}_\ell)_{\mathfrak{m}}$ are concentrated in one degree and torsion-free. The second miracle is that, whenever the group $G_b(\mathbb{Q}_p)$ acting on $\mathfrak{Ig}^b$ comes from a non-quasi-split inner form, the localisation $R\Gamma(\mathfrak{Ig}^b, \mathbb{Q}_\ell)_{\mathfrak{m}}$ vanishes. This uses the genericity of $\bar{\rho}_{\mathfrak{m}}$ at each $\mathfrak{p} \mid p$ and suggests that the cohomology of Igusa varieties satisfies some form of local-global compatibility. Finally, the condition that p splits completely in F guarantees that the only Newton stratum for which G_b is quasi-split is the ordinary one. Therefore, the hypotheses of Theorem 4.4 guarantee that $(R\pi_{\mathrm{HT}*}\mathbb{F}_\ell)_{\mathfrak{m}}$ is as simple as possible - it is supported in one degree on a zero-dimensional stratum!

The computation of $R\Gamma(\mathrm{Ig}^b, \mathbb{Q}_\ell)_{\mathfrak{m}}$, at least at the level of the Grothendieck group, can be done using the trace formula method pioneered by Shin [Shi10]. This is the method used for Shimura varieties of PEL type A in [CS17] and [CS19]. For inner forms of $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$, with F a totally real field, one can avoid these difficult computations, cf. [CT21]. In this setting, one can reinterpret results of Tian–Xiao [TX16] on geometric instances of the Jacquet–Langlands corresponcence as giving rise to exotic isomorphisms between Igusa varieties arising from different Shimura varieties. This is what happens for the basic stratum in Example 3.4. Then one can conclude by applying the classical Jacquet–Langlands correspondence.

In [Kos21], Koshikawa introduces a novel and complementary strategy for proving these kinds of vanishing theorems. He shows that, under the same genericity assumption in Definition 4.1, only the restriction of $(R\pi_{\mathrm{HT}*}\mathbb{F}_\ell)_{\mathfrak{m}}$ to the ordinary locus contributes to the total cohomology of the Shimura variety. To achieve this, he proves the analogous generic vanishing theorem for the cohomology $R\Gamma_c(\mathcal{M}^b, \mathbb{Z}_\ell)_{\mathfrak{m}_p}$ of the Rapoport–Zink space, where $\mathfrak{m}_p$ is a maximal ideal of the local spherical

Hecke algebra at p. This relies on the recent work of Fargues–Scholze on the geometrisation of the local Langlands conjecture [FS21].

Koshikawa's strategy is more flexible, allowing him to handle with ease the case where F is an imaginary quadratic field. On the other hand, the original approach also gives information about the complexes of sheaves $(R\pi_{\mathrm{HT}*}\mathbb{F}_\ell)_{\mathfrak{m}}$, rather than just about the cohomology groups $H^*_{(c)}(S_K(\mathbb{C}), \mathbb{F}_\ell)_{\mathfrak{m}}$. These complexes should play an important role for questions of local-global compatibility in Fargues's geometrisation conjecture, cf. [Far16, §7].

5. Cohomology with mod p and p-adic coefficients

The most general method for constructing p-adic families of automorphic forms from the cohomology of locally symmetric spaces is via *completed cohomology*. First introduced by Emerton in [Eme06], this has the following definition

$$\widetilde{H}^*(K^p, \mathbb{Z}_p) = \varprojlim_n \left(\varinjlim_{K_p} H^*\left(X_{K^pK_p}, \mathbb{Z}/p^n\right)\right),$$

where $K^p \subset G(\mathbb{A}_f)$ is a sufficiently small, fixed tame level, and $K_p \subset G(\mathbb{Q}_p)$ runs over compact open subgroups. This space has an action of the spherical Hecke algebra $\mathbb{T}$, built from Hecke operators away from p, as well as an action of the group $G(\mathbb{Q}_p)$. One can make the analogous definition for completed cohomology with compact support, and a variant gives completed homology and completed Borel–Moore homology. See [Eme14] for an excellent survey that gives motivation, examples, and sketches the basic properties of these spaces.

Motivated by heuristics from the p-adic Langlands programme, Calegari and Emerton made several conjectures about the range of degrees in which one can have non-zero completed (co)homology and about the codimension of completed homology over the completed group rings $\mathbb{Z}_p[\![K_p]\!]$. See [CE12, Conjecture 1.5] for the original formulation and [HJ20, Conjecture 1.3] for a slightly different formulation, which emphasises the natural implications between the various conjectures. In particular, Calegari–Emerton conjectured that

$$\widetilde{H}^i_c(K^p, \mathbb{Z}_p) = \widetilde{H}^i(K^p, \mathbb{Z}_p) = 0 \text{ for } i > q_0.$$

For Shimura varieties of pre-abelian type, the Calegari–Emerton conjectures were proved by Hansen–Johansson in [HJ20], building on work of Scholze who proved the vanishing of completed cohomology with compact support for Shimura varieties of Hodge type [Sch15].

We sketch Scholze's argument, which illustrates the role of p-adic geometry in this result. It is enough to show that

$$\widetilde{H}^i_c(K^p, \mathbb{F}_p) = \varinjlim_{K_p} H^i_c(S_{K^pK_p}(\mathbb{C}), \mathbb{F}_p)$$

vanishes for $i > d = \dim_E S_K$. Since (G, X) is a Shimura datum of Hodge type, we are in the setting of Theorem 3.1 – in fact, we know that the minimal compactification $\mathcal{S}^*_{K^p}$ is perfectoid. The primitive comparison theorem of [Sch13] gives an almost isomorphism between $\widetilde{H}^i_c(K^p, \mathbb{F}_p) \otimes \mathcal{O}_C/p$ and $H^i_{\text{ét}}(\mathcal{S}^*_{K^p}, \mathcal{I}^+/p)$, where $\mathcal{I}^+ \subseteq \mathcal{O}^+$ is the subsheaf of sections that vanish along the boundary. On an affinoid perfectoid space, Scholze proved the almost vanishing of the étale cohomology of $\mathcal{O}^+/p$ in degree $i > 0$. With some care at the boundary, one deduces that it is enough to prove that the analytic cohomology groups $H^i_{\mathrm{an}}(\mathcal{S}^*_{K^p}, \mathcal{I}^+/p)$ are almost 0 in degree $i > d$. This final step follows from a theorem of Scheiderer on the cohomological dimension of spectral spaces.

In [CGH+20] and [CGJ19], we study Shimura varieties with unipotent level at p. More precisely, assume that (G, X) is a Shimura datum of Hodge type and that $G_{\mathbb{Q}_p}$ is split. Choose a split model of G and a Borel subgroup B over $\mathbb{Z}_p$, and let $U \subset B$ be the unipotent radical.

Theorem 5.1. [CGJ19, Theorem 1.1] *Let $H \subseteq U(\mathbb{Z}_p)$ be a closed subgroup. We have*

$$\varinjlim_{K_p \supseteq H} H^i_c(S_{K^pK_p}(\mathbb{C}), \mathbb{F}_p) = 0 \text{ for } i > d.$$

This result is stronger than the Calegari–Emerton conjecture for completed cohomology with compact support, since we can take $H = \{1\}$ and recover Scholze's result discussed above. In addition to the argument sketched above, the key new idea needed for Theorem 5.1 is that the *Bruhat decomposition* on the Hodge–Tate period domain $\mathscr{F}\ell$ remembers how far different subspaces of $\mathcal{S}^*_{K^pU(\mathbb{Z}_p)}$ are from being perfectoid.

Example 5.2. Assume that $G = \mathrm{GL}_2/\mathbb{Q}$, so that we are working in the modular curve case. The Bruhat decomposition is given by $\mathbb{P}^{1,\mathrm{ad}} = \mathbb{A}^{1,\mathrm{ad}} \sqcup \{\infty\}$, with the two Bruhat cells in natural bijection with the two components of the ordinary locus in (3.5). We have a morphism of sites

$$\pi_{\mathrm{HT}/U(\mathbb{Z}_p)} : \left(\mathcal{S}^*_{K^pU(\mathbb{Z}_p)}\right)_{\text{ét}} \to |\mathbb{P}^{1,\mathrm{ad}}|/U(\mathbb{Z}_p),$$

where we take the quotient $|\mathbb{P}^{1,\mathrm{ad}}|/U(\mathbb{Z}_p)$ only as a topological space. The preimage of $|\mathbb{A}^{1,\mathrm{ad}}|/U(\mathbb{Z}_p)$ in $\mathcal{S}_{K^pU(\mathbb{Z}_p)}$ is a perfectoid space, as proved by Ludwig in [Lud17]. The preimage of $|\infty|/U(\mathbb{Z}_p)$ has a $\mathbb{Z}_p$-cover that is an affinoid perfectoid space. This allows us to bound the support of each $R^i\pi_{\mathrm{HT}*/U(\mathbb{Z}_p)}(\mathcal{I}^+/p)$ and we conclude by the Leray spectral sequence.

More generally, the Bruhat decomposition $G = \sqcup_{w \in W^{P_\mu}} BwP_\mu$ gives a decomposition $\mathscr{F}\ell = \sqcup_{w \in W^{P_\mu}} \mathscr{F}\ell^w$ into locally closed Schubert cells indexed by certain Weyl group elements. For each $\mathscr{F}\ell^w/U(\mathbb{Z}_p)$, we can quantify how far its preimage in $\mathcal{S}^*_{K^pU(\mathbb{Z}_p)}$ is from being a perfectoid space, which depends on the length of the Weyl group element w. The assumption that $G_{\mathbb{Q}_p}$ is split guarantees that all the Weyl group elements lie in the ordinary locus inside $\mathscr{F}\ell$, which greatly simplifies the analysis. However, the analogue of Theorem 5.1 may hold even without the assumption that $G_{\mathbb{Q}_p}$ is split, and even when the ordinary locus is empty. There is some evidence in this direction, e.g. by using embeddings into higher-dimensional Shimura varieties attached to split groups.

The Bruhat decomposition on $\mathscr{F}\ell$ has more recently been used by Boxer and Pilloni to define a version of higher Coleman theory indexed by each $w \in W^{P_\mu}$ in [BP21]. The development of higher Coleman and higher Hida theories shows that the geometric theory of p-adic automorphic forms on Shimura varieties is much richer than previously expected. Furthermore, the Bruhat decomposition indicates the form a p-adic Eichler–Shimura isomorphism should take, relating completed cohomology to these more geometric theories. In joint work in progress with Mantovan and Newton, we use the geometry described in Example 5.2 to give a new proof of the ordinary Eichler–Shimura isomorphism due to Ohta [Oht95, Oht00]. Our result decomposes the ordinary completed cohomology of the modular curve in terms of Hida theory and higher Hida theory, the latter recently developed by Boxer and Pilloni in [BP20].

Theorem 5.1 seems far away from Conjecture 2.2, because it concerns Shimura varieties with "infinite level" at p. However, one could ask whether a version of Theorem 5.1 holds already at level $B(\mathbb{Z}_p)$, at least after applying an ordinary idempotent, as in Hida theory. If that were the case, the control theorems in Hida theory (specifically the result known as independence of level) and a careful application of Poincaré duality would imply that an $\ell = p$ analogue of Conjecture 4.3 holds, with *generic* replaced by *ordinary*. More precisely, in this case, the "auxiliary prime" p where we impose a representation-theoretic condition is no longer auxiliary but rather equal to ℓ.

6. Applications beyond Shimura varieties

While the focus of this article has been the cohomology of Shimura varieties, Theorems 4.5 and 5.1 have surprising applications to understanding the cohomology of more general locally symmetric spaces. For example, let F be an imaginary CM field and $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_n$. Then G can be realised as the Levi quotient of the Siegel maximal parabolic of a quasi-split unitary group $\widetilde{G}$. The Borel–Serre compactification $\widetilde{X}^{\mathrm{BS}}_{\widetilde{K}}$ for the locally symmetric spaces associated with the unitary group $\widetilde{G}$ gives rise to a Hecke-equivariant long exact sequence of the form

$$\cdots \to H^i_c(\widetilde{X}_{\widetilde{K}}, \mathbb{Z}/\ell^n\mathbb{Z}) \to H^i(\widetilde{X}_{\widetilde{K}}, \mathbb{Z}/\ell^n\mathbb{Z}) \to H^i(\partial\widetilde{X}_{\widetilde{K}}, \mathbb{Z}/\ell^n\mathbb{Z}) \to H^{i+1}_c(\widetilde{X}_{\widetilde{K}}, \mathbb{Z}/\ell^n\mathbb{Z}) \to \dots, \tag{6.1}$$

where $\partial\widetilde{X}_{\widetilde{K}} = \widetilde{X}^{\mathrm{BS}}_{\widetilde{K}} \setminus \widetilde{X}_{\widetilde{K}}$ is the boundary of the Borel–Serre compactification. The usual and compactly supported cohomology of $\widetilde{X}_{\widetilde{K}}$ can be simplified to some extent by applying either of the two vanishing theorems. On the other hand, the cohomology of X_K can be shown to contribute to the cohomology of $\partial\widetilde{X}_{\widetilde{K}}$, in some more or less controlled fashion.

Let $\mathfrak{m} \subset \mathbb{T}$ be a non-Eisenstein maximal ideal in the support of $R\Gamma(X_K, \mathbb{Z}_\ell)$ and let $\mathbb{T}(K)_\mathfrak{m}$ denote the quotient of $\mathbb{T}$ that acts faithfully on $R\Gamma(X_K, \mathbb{Z}_\ell)_\mathfrak{m}$. In addition to the residual Galois representation $\bar{\rho}_\mathfrak{m}$, Scholze associates to $\mathfrak{m}$ a deformation $\rho_\mathfrak{m}$ valued in $\mathbb{T}(K)_\mathfrak{m}/I$, for some nilpotent ideal I. This was subsequently shown by Newton and Thorne in [NT16] to satisfy $I^4 = 0$. In [CGH+20], we used a variant of Theorem 5.1 together with the excision sequence (6.1) to eliminate this nilpotent ideal entirely, under the assumption that ℓ splits completely in the CM field F. This leads to a more natural statement on the existence of Galois representations in this setting.

The Galois representations $\rho_\mathfrak{m}$ are expected to satisfy a certain property known as *local-global compatibility*, which is particularly subtle to state and prove at primes above ℓ. For example, after inverting ℓ, the $\rho_\mathfrak{m}$ are expected to be de Rham, in the sense of Fontaine, but it is less clear what the right condition should be for torsion Galois representations. In another application, Theorem 4.5 is crucially used in [ACC+18] together with the excision sequence (6.1) to prove that $\rho_\mathfrak{m}$ satisfies the expected local-global compatibility at primes above ℓ in two restricted families of cases: the ordinary case and the Fontaine–Laffaille case[6]. In joint work in progress with Newton, we should be able to extend these methods to cover significantly more.

The local-global compatibility results established in [ACC+18] are already extremely useful: they help us implement the Calegari–Geraghty method unconditionally for the first time in arbitrary dimension. A striking application is the following result.

Theorem 6.1. [ACC+18, Theorem 1.0.1] *Let F be a CM field and E/F be an elliptic curve that does not have complex multiplication. Then E is potentially automorphic and satisfies the Sato–Tate conjecture.*

The potential automorphy of E was established at the same time in work of Boxer–Calegari–Gee–Pilloni [BCGP18], who also showed the potential automorphy of abelian surfaces over totally real fields. Their work relies on the Calegari–Geraghty method for the *coherent cohomology* of Shimura varieties and uses a preliminary version of higher Hida theory, due to Pilloni, as a key ingredient.

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