LONDON NUMBER THEORY STUDY GROUP: CYCLES ON SHIMURA VARIETIES VIA GEOMETRIC SATAKE, FOLLOWING LIANG XIAO AND XINWEN ZHU

ANA CARAIANI

Let (G, X) be a Shimura datum of Hodge type. For a neat compact open subgroup $K \subset G(\mathbb{A}_f)$, the corresponding Shimura variety

$$\mathrm{Sh}_K(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

has a canonical model, which is a smooth, quasi-projective variety over the reflex field E. Consider a prime $p > 2$ that is unramified for (G, X, K) and a prime $v \mid p$ of E. By work of Kisin and Vasiu, $\mathrm{Sh}_K(G, X)$ has a canonical model $\mathcal{S}$ over $\mathcal{O}_{E,v}$. Let

$$\mathrm{Sh}_\mu := \mathcal{S} \times_{\mathcal{O}_{E,v}} \mathbb{F}_v$$

denote the special fiber of the Shimura variety. The étale cohomology $H^i_{\text{ét}}(\mathrm{Sh}_{\mu,\overline{\mathbb{F}}_v}, \overline{\mathbb{Q}}_\ell)$ has natural commuting actions of $\mathrm{Gal}(\overline{\mathbb{F}}_v/\mathbb{F}_v)$ and of the Hecke algebra $\mathcal{H}_K := C_c^\infty\left(K\backslash G(\mathbb{A}_f)/K, \overline{\mathbb{Q}}_\ell\right)$. For an irreducible $\mathcal{H}_K$-module π_f, we consider the $\mathrm{Gal}(\overline{\mathbb{F}}_v/\mathbb{F}_v)$-module

$$W^i(\pi_f, \overline{\mathbb{Q}}_\ell) := \mathrm{Hom}_{\mathcal{H}_K}\left(\pi_f, H^i_{\text{ét}}(\mathrm{Sh}_{\mu,\overline{\mathbb{F}}_v}, \overline{\mathbb{Q}}_\ell)\right).$$

We let ϕ_v denote the geometric Frobenius element and we define the subspace of geometric Tate classes

$$T^i(\pi_f, \overline{\mathbb{Q}}_\ell) := \bigcup_{j\geq 1} W^{2i}(\pi_f, \overline{\mathbb{Q}}_\ell)(i)^{\phi_v^j}.$$

The Tate conjecture predicts that these Tate classes will come from codimension i "submotives" of $\mathrm{Sh}_{\mu,\overline{\mathbb{F}}_v}[\pi_f]$ via the cycle class map.

Let $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$ be the Langlands dual group of G equipped with a pinning. We define the lattice

$$\Lambda^{\mathrm{Tate}_p} := \left\{\lambda \in X^*(\hat{T}) \mid \sum_{i=1}^m \phi_p^i(\lambda) \in X_*(Z_G)\right\}$$

in $X^*(\hat{T})$. For an algebraic representation V of $\hat{G}$, we define the following subspace

$$V^{\mathrm{Tate}_p} := \bigoplus_{\lambda\in\Lambda^{\mathrm{Tate}_p}} V(\lambda)$$

of V. We let μ denote the (dominant) Hodge cocharacter determined by the Shimura datum and we set $\mu^* := -w_0(\mu) \in X^*(\hat{T}) = X_*(T)$. If $V_{\mu^*}^{\mathrm{Tate}_p} \neq 0$, there exists a unique inner form G' of G such that $G'(\mathbb{A}_f) \simeq G(\mathbb{A}_f)$ and $G'_{\mathbb{R}}$ is compact modulo center. The goal of this study group is to understand the proof of the following result, by Liang Xiao and Xinwen Zhu [XZ17].

Theorem 0.1. *Assume that (G, X) is of Hodge type and the center Z_G is connected. Let $K \subset G(\mathbb{A}_f)$ be a neat compact subgroup. Let $p > 2$ be a prime that is unramified for (G, X, K) and assume that $V_{\mu^*}^{\mathrm{Tate}_p} \neq 0$. Let G' be the unique inner form of G from above. Then:*

(1) *The basic Newton stratum $\mathrm{Sh}_{\mu,b}$ is pure of dimension $\frac{d}{2}$. In particular, d is always even. There is a $\mathcal{H}_K$-equivariant isomorphism*

$$H_d^{\mathrm{BM}}(\mathrm{Sh}_{\mu,b,\overline{\mathbb{F}}_v}) \simeq \mathcal{C}\left(G'(\mathbb{Q})\backslash G'(\mathbb{A}_f)/K, \overline{\mathbb{Q}}_\ell\right) \otimes V_{\mu^*}^{\mathrm{Tate}_p},$$

where we use the isomorphism $G'(\mathbb{A}_f) \simeq G(\mathbb{A}_f)$ to view K as a compact open subgroup of $G'(\mathbb{A}_f)$.

(2) *For an irreducible $\mathcal{H}_K$-module π_f, define*

$$H_d^{\mathrm{BM}}(\mathrm{Sh}_{\mu,b,\overline{\mathbb{F}}_v})[\pi_f] := \mathrm{Hom}_{\mathcal{H}_K}\left(\pi_f, H_d^{\mathrm{BM}}(\mathrm{Sh}_{\mu,b,\overline{\mathbb{F}}_v})\right) \otimes \pi_f.$$

Then the cycle class map

$$\mathrm{cl} : H_d^{\mathrm{BM}}(\mathrm{Sh}_{\mu,b,\overline{\mathbb{F}}_v}) \to H^d(\mathrm{Sh}_{\mu,\overline{\mathbb{F}}_v}, \overline{\mathbb{Q}}_\ell(\frac{d}{2})),$$

restricted to $H_d^{\mathrm{BM}}(\mathrm{Sh}_{\mu,b,\overline{\mathbb{F}}_v})[\pi_f]$ is injective if the Langlands parameter of $\pi_{f,p}$ is general with respect to V_{μ^*}.

(3) *Assume $\mathrm{Sh}_K(K, X)$ is a Kottwitz Shimura variety, or that G_{der} is anisotropic and simply-connected, and that there exists a prime $p' \neq p$ such that $\pi_{f,p'}$ is an unramified twist of Steinberg. Then the π_f^p-isotypical component of the cycle class map* cl *is surjective onto*

$$\sum_{\pi_p} T^{\frac{d}{2}}(\pi_f^p \pi_p, \overline{\mathbb{Q}}_\ell) \otimes \pi_f^p \pi_p$$

if the Langlands parameters of the π_p appearing in the sum are all strongly general with respect to V_{μ^*}. In particular, the Tate conjecture holds for these π_f^p.

Talks

Talks 1 & 2: Overview (180 mins). Oct 3 2018. Speaker: Ana Caraiani

Give some background and state the main theorem. Give an overview of the four different ingredients going into the proof of the main theorem, with a focus of the input from affine Deligne-Lusztig varieties and moduli of local shtukas. References: [XZ17].

Talk 3: The axioms of a Shimura variety (90 mins). Oct 10 2018. Speaker: Pol van Hoften

Introduce the notion of a Shimura datum, stating the axioms it must satisfy. Briefly review the necessary notions from Hodge theory, including the notion of a variation of Hodge structure, and use this to explain the axioms. Discuss the Hodge cocharacter, the reflex field and the existence of the canonical model of the Shimura variety.

Talk 4: Examples of Shimura varieties (90 mins). Oct 10 2018. Speaker: Andrew Graham

Discuss modular curves, Shimura curves and Siegel modular varieties in detail. Give examples of Shimura varieties of PEL type (such as those associated to unitary similitude groups) and Shimura varieties of Hodge type. Also discuss the notion of a Shimura set, state Corollary 2.1.6 and discuss Example 2.1.8 of [XZ17]. References: [Lan].

Talk 5: Integral models of Shimura varieties of Hodge type (90 mins). Oct 17 2018. Speaker: Ashwin Iyengar

Give an overview of Kisin's construction of canonical integral models of Shimura varieties of Hodge type. Emphasize the construction of the étale, de Rham and crystalline tensors over the integral model. References: [Kis10]

Talk 6: The Newton stratification on the special fiber (90 mins). Oct 17 2018. Speaker: Mafalda Santos

For a reductive group $G/\mathbb{Q}_p$ and a minuscule cocharacter μ, discuss the Kottwitz set $B(G,\mu)$. Recall the notion of a basic element and explain why $B(G,\mu)$ contains a unique basic element. Define the unramified elements of $B(G,\mu)$ and discuss some criteria for the basic element to be unramified, cf. §4.2 of [XZ17]. Describe how to construct the Newton stratification on the special fiber of a Shimura variety of Hodge type. Discuss dimension formulas for Newton strata. References: [RR96, Lov17].

Talk 7: The affine Grassmannian (90 mins). Oct 24 2018. Speaker: Carl Wang Erickson

Introduce the affine Grassmannian and its Schubert cells. Discuss both the equal characteristic case and the mixed-characteristic case. References: [Zhu17] and [BS17].

Talk 8: Perverse sheaves (90 mins). Oct 24 2018. Speaker: Pol van Hoften

Start with some motivation from intersection cohomology. Recall the notion of a t-structure on a triangulated category. Give as an example the usual t-structure on the bounded derived category of constructible l-adic sheaves $D_c^b(X,\overline{\mathbb{Q}}_\ell)$ (where X is a scheme of finite type defined over a finite or algebraically closed field). Discuss the six functor formalism and define the perverse t-structure on $D_c^b(X,\overline{\mathbb{Q}}_\ell)$. Define the middle extension. Explain why nearby cycles preserve perverse sheaves. References: [BBD82].

Talks 9 & 10: The geometric Satake equivalence (180 mins). Oct. 31 2018. Guest speaker: Timo Richarz

Explain the geometric Satake equivalence for function fields, then explain Zhu's adaptation to the Witt vector affine Grassmannian. Explain the relationship to classical Satake. References: [Ric16, Zhu17].

Talk 11: Affine Deligne–Lusztig varieties (90 mins). Nov. 7 2018, Speaker: Andrea Dotto

Define affine Deligne–Lusztig varieties and discuss their relationship to Rapoport–Zink spaces and their dimension. Also discuss unramified elements in $B(G)$ and give a criterion for the basic element of $B(G, \mu)$ to be unramified. References: §4.1 and 4.2 of [XZ17].

Talk 12: Mirkovic–Vilonen cycles (90 mins). Nov. 7 2018. Speaker: Dan Gulotta

Describe the irreducible components of affine Deligne–Lusztig varieties $X_\mu(b)$, for $b \in B(G, \mu)$ and unramified element, in terms of Mirkovic–Vilonen cycles. References: §3 of [XZ17] for the definition of Mirkovic–Vilonen cycles and §4.3 and 4.4 of [XZ17] for the application to affine Deligne–Lusztig varieties.

Talks 13 & 14: Moduli of local shtukas (180 mins). Nov. 14 2018. Speaker: James Newton

Introduce the local Hecke stack, the moduli of local shtukas, and describe the partial Frobenius maps and Hecke correspondences. Also do the same in the restricted setting. Define the category of perverse sheaves on the moduli of local shtukas. References: §5.1-5.3 of [XZ17].

Talk 15: Review of cohomological correspondences (90 mins). Nov. 21 2018. Speaker: Toby Gee

Discuss generalities about cohomological correspondences on perfect algebraic stacks. References: the appendix to [XZ17]

Talk 16: Cohomological correspondences on the moduli of local shtukas (90 mins). Nov 21 2018. Speaker: Rebecca Bellovin

Define the category $P^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}}_{\overline{\mathbb{F}}_q})$. Compute the endomorphisms of δ_1 in $P^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}}_{\overline{\mathbb{F}}_q})$. References: §5.4 of [XZ17].

Talks 17 & 18: The spectral action (180 mins). Nov. 28 2018. Speakers: Ashwin Iyengar and Ben Heuer

Discuss Theorem 6.0.1 of [XZ17] and its proof. References: §6 of [XZ17].

Talks 19 & 20: Correspondences between special fibers of Shimura varieties (180 mins). Dec. 5 2018. Speaker: David Helm

Relate the perfection of the special fiber $\mathrm{Sh}^{\mathrm{perf}}_\mu$ to the moduli of local shtukas. Discuss exotic Hecke correspondences between special fibers of different Shimura varieties. Prove the first part of the main theorem, using Rapoport–Zink uniformization. Describe the intersection matrix for cycle classes of the basic locus in terms of the spectral action by the ring of regular functions on the stack of Satake parameters. References: §7 of [XZ17].

Talk 21: The generalized Chevalley restriction map (90 mins). Dec 12 2018. Speaker: Eran Assaf

Discuss Theorem 1.4.1 of [XZ17] and its proof and use this to prove the second part of the main theorem. References: §7 of [XZ17] and [XZ18].

Talk 22: Application to Kottwitz's simple Shimura varieties (90 mins). Dec. 12 2018. Speaker: Mafalda Santos

Prove the third part of the main theorem by giving the application to Kottwitz's simple Shimura varieties, using the results in §2.2 of [XZ17].

References

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171.

[BS17] Bhargav Bhatt and Peter Scholze, *Projectivity of the Witt vector affine Grassmannian*, Invent. Math. **209** (2017), no. 2, 329–423.

[Kis10] Mark Kisin, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. **23** (2010), no. 4, 967–1012.

[Lan] Kai-Wen Lan, *An example-based introduction to Shimura varieties.*

[Lov17] T. Lovering, *Filtered F-crystals on Shimura varieties of abelian type*, ArXiv e-prints (2017).

[Ric16] Timo Richarz, *Affine Grassmannians and geometric Satake equivalences*, Int. Math. Res. Not. IMRN (2016), no. 12, 3717–3767.

[RR96] M. Rapoport and M. Richartz, *On the classification and specialization of F-isocrystals with additional structure*, Compositio Math. **103** (1996), no. 2, 153–181.

[XZ17] L. Xiao and X. Zhu, *Cycles on Shimura varieties via geometric Satake*, ArXiv e-prints (2017).

[XZ18] ———, *On vector-valued twisted conjugate invariant functions on a group*, ArXiv e-prints (2018).

[Zhu17] Xinwen Zhu, *Affine Grassmannians and the geometric Satake in mixed characteristic*, Ann. of Math. (2) **185** (2017), no. 2, 403–492.