Notes on modular forms of half-integral weight.

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Abstract

These are just notes on several aspects of modular forms of half-integral weight.

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1 Introduction

The basic idea is simple. Define $\theta = \sum_{n\in\mathbf{Z}} q^{n^2} = 1 + 2q + 2q^4 + \ldots$. The idea is that for $\kappa > 0$ an odd integer, we decree that θ^κ is a weight $\kappa/2$ modular form of level 4 and "with trivial character", and we basically define an arbitrary modular form of weight $\kappa/2$ to be a holomorphic function which transforms under (a congruence subgroup of) $\Gamma_0(4)$ in the same way as θ^κ does. This leads to a solid analytic theory, with Hecke operators and so on (although the Hecke operator story is rather different in the half-integral weight case). A very interesting observation of Nick Ramsey is that one can also build an algebraic theory: the idea is that θ^4 is classical, so defines a section of a sheaf, and hence a divisor D, and $D/4$ is a $\mathbf{Q}$-divisor but one can still talk about sections of $\kappa D/4$. One can go on to define p-adic half-integral weight forms (this was also done by Ramsey) and so on. Shimura and Shintani established an extraordinary connection between forms of weight $\kappa/2$ and forms of weight $\kappa-1$—the "Shimura lift" and the "Shintani lift" being essentially inverse operators between eigenforms on these spaces.

In these notes I'll build the theory from the ground up and then talk about more recent work.

2 Definitions (the analytic case).

A lot of the ideas here appear to be due to Shimura (Annals 97, 1973, pp 440–481).

Let $\mathcal{H}$ denote the upper half plane. Define $\theta : \mathcal{H} \to \mathbf{C}$ by $\theta(z) = \sum_{n\in\mathbf{Z}} q^{n^2}$, with $q = e^{2\pi i z}$ as usual. Note that θ^2 is a weight 1 modular form on $\Gamma_1(4)$ with character χ_4 (primitive of level 4), and hence for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\Gamma_0(4)$ and $z \in \mathcal{H}$ we have $\theta^2(\gamma z) = \chi_4(d)(cz+d)\theta^2(z)$. In particular, if we make the important definition (for $\gamma \in \Gamma_0(4)$ and $z \in \mathcal{H}$):

$$j(\gamma, z) := \theta(\gamma z)/\theta(z),$$

then we see that $j(\gamma, z)^2 = \chi_4(d)(cz+d)$. There is an explicit formula for $j(\gamma, z)$ in Serre-Stark [1] on page Se-St-3. It involves the quadratic residue symbol (c/d) (if $c \neq 0$), a 4th root of unity which depends on d mod 4, and an explicit branch of the square root function too. I guess the terrifying thing is not so much the 4th root of unity but the fact that it involves a quadratic residue symbol and hence is somehow not "controlled" in any way by a congruence subgroup.

Convention: *throughout these notes, κ will be a positive odd integer, and N will be a positive integer multiple of 4.* We will be considering modular forms of weight $\kappa/2$ and level N. The theory for weight in $\mathbf{Z} + \frac{1}{2}$ is sufficiently different from that of weight in $\mathbf{Z}$ that we specifically exclude weight $\mathbf{Z}$ here. And for weight in $\mathbf{Z} + \frac{1}{2}$ the level must, by definition, always be a multiple of 4.

It is convenient to define the following extension $\mathcal{G}$ of $\mathrm{GL}_2^+(\mathbf{R})$ thus: an element of $\mathcal{G}$ is a pair (γ, ϕ) with $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbf{R})$ (that is, in $\mathrm{GL}_2(\mathbf{R})$ and with positive determinant) and with $\phi : \mathcal{H} \to \mathbf{C}$ holomorphic and such that $\phi(z)^2 = t.\det(\gamma)^{-1/2}(cz+d)$ with $t = t(\gamma, \phi) \in \mathbf{C}$ independent of z and satisfying $|t| = 1$. In fact let $\mathbf{T}$ denote $\{t \in \mathbf{C} : |t| = 1\}$. Note first that for any $\gamma \in \mathrm{GL}_2^+(\mathbf{R})$ the function $z \mapsto cz + d$ on the upper half plane will have a holomorphic square root, so the map $\mathcal{G} \to \mathrm{GL}_2^+(\mathbf{R})$ is surjective, and the fibre above $\gamma \in \mathrm{GL}_2^+(\mathbf{R})$ will be (typically non-canonically) isomorphic to $\mathbf{T}$ (because any holomorphic function whose square is constant will be constant). Moreover $\mathcal{G}$ can naturally be made into a group by $(\gamma, \phi)(\delta, \psi) = (\gamma\delta, z \mapsto \phi(\delta z).\psi(z))$. One checks that the latter is in $\mathcal{G}$ and that multiplication does indeed make $\mathcal{G}$ into a group. Let $P : \mathcal{G} \to \mathrm{GL}_2^+(\mathbf{R})$ denote the projection. Then the kernel of P is canonically $\mathbf{T}$, and this kernel is in the centre of $\mathcal{G}$.

My suspicion is that if we restrict furthermore to $t = \pm 1$ then we would get the metaplectic cover of $\mathrm{GL}_2^+(\mathbf{R})$.

So here's a nice definition: for f holomorphic on $\mathcal{H}$ and $\xi := (\gamma, \phi) \in \mathcal{G}$, define $f|_\kappa\xi(z) = f(\gamma z).\phi(z)^{-\kappa}$. One checks that this is an action of $\mathcal{G}$ on the set of all holomorphic functions on $\mathcal{H}$. It leads naturally to a definition of a modular form of level Γ for Γ a discrete subgroup of $\mathcal{G}$ (*not* of $\mathrm{SL}_2(\mathbf{R})$!). If $\mathcal{G}_0(4)$ is the pre-image of $\Gamma_0(4)$ in $\mathcal{G}$ then the map $\mathcal{G}_0(4) \to \Gamma_0(4)$ has a splitting, given by sending γ to $\gamma^* := (\gamma, j(\gamma, z))$. Hence we may talk about functions f such that $f|_\kappa\gamma^* = f$ for all $\gamma \in \Gamma$, assumed to be a congruence subgroup of $\Gamma_0(4)$; one checks easily that in this case the condition just says that

$$f(\gamma z) = f(z).j(\gamma, z)^{-\kappa},$$

where $j(\gamma, z) = \theta(\gamma z)/\theta(z)$ as before. For such a form to be a modular form, it must be "holomorphic at the cusps", which means the usual thing: for s in $\mathbf{Q} \cup \{\infty\}$ choose $\overline{\rho} \in \mathrm{SL}_2(\mathbf{Q})$ sending ∞ to s, lift to $\rho \in \mathcal{G}$, and then demand that $f|_\kappa\rho$ (which will be invariant under $z \mapsto z + h$ for some $h > 0$) has a q-expansion with no terms of negative degree. The same trick will enable one to define cusp forms.

So there's the definiton: for N a multiple of 4 and κ odd, a modular form of weight $\kappa/2$ and level $\Gamma_1(N)$ is a holomorphic function f on $\mathcal{H}$ with $f|_\kappa\gamma^* = f$ for all $\gamma \in \Gamma_1(N)$, and such that f is holomorphic at the cusps. Let $M_{\kappa/2}(\Gamma_1(N))$ denote such things and let $S_{\kappa/2}(\Gamma_1(N))$ denote the subspace of cusp forms. Note that $(\mathbf{Z}/N\mathbf{Z})^*/\pm 1$ acts naturally on these things (because $\Gamma_1(N)$ is normal in $\Gamma_0(N)$ and $\Gamma_0(N)$ also lifts to $\mathcal{G}$) and if, for χ an even Dirichlet character of level N, we define $M_{\kappa/2}(\Gamma_1(N), \chi)$ to be the $f \in M_{\kappa/2}(\Gamma_1(N))$ with $f|_\kappa\gamma^* = \chi(d)f$ then $M_{\kappa/2}(\Gamma_1(N))$ becomes the direct sum of the $M_{\kappa/2}(\Gamma_1(N), \chi)$ and similarly for the cusp forms.

3 Examples.

Visibly θ^κ is an example, but it's not cuspidal. Slightly terrifyingly, if $f \in M_{\kappa/2}(\Gamma_1(N))$ and $t \in \mathbf{Z}_{>0}$ then $z \mapsto f(tz)$ is in $M_{\kappa/2}(\Gamma_1(Nt))$ (unsurprising so far) but it will in general have a different character to f! This is because for $\gamma = \left(\begin{smallmatrix} a & b \\ Ntc & d \end{smallmatrix}\right) \in \Gamma_1(Nt)$ and $\gamma' = \left(\begin{smallmatrix} a & tb \\ Nc & d \end{smallmatrix}\right) \in \Gamma_1(N)$, it is not in general the case that $j(\gamma, z) = j(\gamma', tz)$; the explicit formula for j involves some quadratic residue symbol which changes, basically by the quadratic character associated to the positive integer t (note that this character is even and, if t is prime, will have conductor t for t congruent to 1 mod 4 but conductor 4t for other primes).

More general theta series are examples too: if ψ is even and primitive of conductor r, then $\theta_\psi := \sum_{n \in \mathbf{Z}} \psi(n)q^{n^2}$ is in $M_{1/2}(\Gamma_1(4r^2), \psi)$ (proposition 2.2 of Shimura's Annals paper). The key input is in some sense Poisson summation, which gets you from z to $-1/z$—this is all part of the general "yoga of theta series". Note that θ_ψ isn't the twist of θ by ψ! That would be $\sum \psi(n^2)q^{n^2}$. Note also that Serre and Stark proved that the weight 1/2 forms are spanned by these more general theta series and their associated oldforms $\theta_\psi(tz)$.

Which primitive even Dirichlet characters are the primitive character associated to the square of a Dirichlet character? One checks that the answer is the "totally even" Dirichlet characters, that is, those ψ whose prime-power components ψ_p are all even. If ψ is totally even then θ_ψ is (closely related to) a twist of θ, and hence one would not expect it to be cuspidal. However, if ψ is even but not totally even, then θ_ψ turns out to be a cusp form (the proof of this in Serre-Stark is analytic in nature, involving checking that the L-functions associated to these guys and their twists have no pole at $s = 1/2$).

The two cuspidal examples of weight 1/2 and smallest level are:

1) ψ of conductor 12, defined by $\psi(n) = 1$ for $n = \pm 1$ mod 12 and $\psi(n) = -1$ for $n = \pm 5$ mod 12, giving rise to

$$\begin{aligned}\theta_\psi &= \sum_{n\equiv\pm 1\ (12)} q^{n^2} - \sum_{n\equiv\pm 5\ (12)} q^{n^2} \\ &= 2(q - q^{25} - q^{49} + q^{121} + q^{169} - \ldots) \\ &= 2\eta(24z)\end{aligned}$$

with $\eta(z) = q^{1/24}\prod_n(1-q^n)$.

2) ψ of conductor 15, the product of odd characters of conductor 3 and 5, giving

$$\theta_\psi = 2(q - iq^4 - q^{16} + iq^{49} + iq^{64} + \ldots)$$

and its Galois conjugate (the above is associated to the character ψ with $\psi_5(2) = i$). This has level 900 and character ψ.

Non-cusp forms are much easier to stumble upon. For example θ itself has weight 1/2 and level 4, and $\theta(2z)$ has level 8 and character of conductor 8 associated to $\mathbf{Q}(\sqrt{2})$ (the one with kernel ± 1 mod 8), and there's an even character of conductor 5 giving a non-cusp form of level 100 (note that this form is a twist of θ).

4 Interlude: zeros of θ

This is just some classical mathematics. Note that the sheaf ω exists on $Y_1(4)$ because it's the solution to a moduli problem, but $Y_1(4)$ has an irregular cusp (the middle one) and hence only ω^2 is guaranteed to exist on $X_1(4)$, and ω^2 will have degree 1 and θ^4 will be a section of it. The fact that it has degree 1 means that it will have a unique zero somewhere. Where is this zero?? It's at a cusp! I know this because I just computed Δ/θ^4, which is either a level 4 weight 10 modular form (if the zero is at a cusp), or has a pole on $\mathcal{H}$ (if the zero is on $\mathcal{H}$), and lo and behold I can find a level 4 weight 10 modular form whose q-expansion agrees with Δ/θ^4 up to $O(q^{100})$, which is proof enough for me (use the Sturm bound).

Indeed, it seems to me that if q is a real number with $|q| < 1$ and q tends down to -1 then $1 + 2(q + q^4 + q^9 + q^{16} + \ldots)$ is tending to zero, which is evidence for θ having a zero at the middle cusp. Indeed, θ^2 is a section of a sheaf which has degree 1/2 and only one "magic point" (the middle cusp), so if θ^4 has a simple zero then θ^2 had better have a zero at the magic point.

Note that $X_1(4p)$, for p any prime (including $p = 2$), has no magic points and hence ω exists on $X_1(4p)$. Where are the zeros of θ^2 on $X_1(4p)$? And what are their orders of vanishing? Hmm, a back-of-an-envelope calculation seems to suggest that $X_1(8)$ has two "middle cusps" but (under the natural degeneracy map) only one of them maps to the middle cusp of $X_1(4)$. So probably the ramification degree is 4 at this cusp and θ^2 has a zero of order 2 at one of the middle cusps of $X_1(8)$. Indeed, the stabiliser of 1/2 in $\mathrm{SL}_2(\mathbf{Z})$ is $\left(\begin{smallmatrix} u-2h & -4h \\ h & 2h+u\end{smallmatrix}\right)$ with $h \in \mathbf{Z}$ and $u = \pm 1$, so the stabilisers in $\Gamma_1(4)$ and $\Gamma_1(8)$ are cyclic, but for $\Gamma_1(4)$ the generator has trace -2 and $u = -1$, and for $\Gamma_1(8)$ it's the 4th power of this.

But where is this going? I was hoping to be able to construct half-integral weight modular forms by dividing cusp forms by θ—but the problem is that, when considered as a form of high level, θ will have zeros of big degrees at random middle cusps, so life isn't so easy. So perhaps this isn't going anywhere.

5 Hecke operators.

One might hope that the general yoga of Hecke operators applies. One has to be a little careful though. We're thinking of levels as discrete subgroups of $\mathcal{G}$, and so the kind of calculation we need to do is: what is the commensurator of $\Gamma_0(4)$ in $\mathcal{G}$? And things like: if $\gamma \in \mathrm{GL}_2^+(\mathbf{Q})$, does a random lift of γ to $\mathcal{G}$ give us a legal (i.e. no problems with commensurability) and interesting (i.e. not trivially zero) Hecke operator?

Of course the general yoga is: if $f \in M_{\kappa/2}(\Gamma_1(N))$ and $\xi \in \mathcal{G}$ such that $\xi\Gamma_1(N)^*\xi^{-1}$ is commensurable with $\Gamma_1(N)^*$ then we'll get a Hecke operator.

Shimura shows the following. Say $\alpha \in \mathrm{GL}_2^+(\mathbf{Q})$ and Γ is a congruence subgroup of $\Gamma_0(4)$. Let ξ be an arbitrary lifting of α to $\mathcal{G}$. Then for $\gamma \in \Gamma \cap \alpha^{-1}\Gamma\alpha$ we can compare γ^* (a lifting of γ to $\mathcal{G}$) with $(\alpha\gamma\alpha^{-1})^*$. Indeed, we get a natural map $\Gamma \cap \alpha^{-1}\Gamma\alpha \to \mathbf{T}$ defined by sending γ to $(\alpha\gamma\alpha^{-1})^*.[\xi.\gamma^*.\xi^{-1}]^{-1}$; this is easily checked to be in $\mathbf{T}$. Indeed this natural map is a homomorphism of groups. Shimura proves that if the image is finite then $\xi\Gamma^*\xi^{-1}$ is commensurable with Γ^*, but also that if the image is not within the κth roots of unity then the resulting Hecke operator is zero! This latter is just mucking about with group theory.

One deduces that for $\alpha = \left(\begin{smallmatrix} m & 0 \\ 0 & n \end{smallmatrix}\right)$ with m and n positive integers, the resulting Hecke operators exist (strictly speaking we must lift to $\mathcal{G}$, and we do this by setting $\phi(z) = (n/m)^{1/4}$ with (n/m) the ratio, not the quadratic character) but the image of the funny map above mentions quadratic residue symbols (mn/d) so (because κ is assumed odd) will often give Hecke operators which *trivially act as zero*. If mn is a square then one gets something interesting though, and if m and n divide N then sometimes one gets interesting things too.

Before I start on Hecke operators, let me mention that the matrix $\left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ also gives something non-zero on $M_{\kappa/2}(\Gamma_1(N))$ and changes a character to its complex conjuage as in the usual story.

OK so what about the analogues of T_p? Here's the story at primes dividing the level. If m is a positive integer such that (a) all primes dividing m also divide N, and (b) the conductor of $\mathbf{Q}(\sqrt{m})$ divides N [so for $m = p$ prime we are demanding $p|N$ and furthermore if $p = 2$ we want $8|N$] then the Hecke operator associated to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & m \end{smallmatrix}\right)$ acts on $M_{\kappa/2}(\Gamma_1(N))$. Warning: it doesn't preserve characters! It sends a form of character χ to one of character $\chi.(m/.)$. On q-expansions it sends $\sum a_n q^n$ to $\sum a_{nm} q^n$.

Now what about away from N? Let n be a positive integer, and let T_{n^2} denote the Hecke operator associated with $\left(\begin{smallmatrix} 1 & 0 \\ 0 & n^2 \end{smallmatrix}\right)$. Shimura does the group theory to verify that T_{m^2} and T_{n^2} commute if either m and n are coprime, or if all prime factors of m also divide N.

Shimura now computes what these do to q-expansions: if $f = \sum_{n\geq 0} a_n q^n$ is in $M_{\kappa/2}(\Gamma_1(N), \chi)$ and if p is prime then $T_{p^2} f$ is also in $M_{\kappa/2}(\Gamma_1(N), \chi)$ (so T_{p^2} is preserving characters), and $T_{p^2} f = \sum_{n\geq 0} b_n q^n$ with, writing $\lambda = (\kappa - 1)/2 \in \mathbf{Z}$,

$$b_n = a_{np^2} + \chi_1(p)(n/p)p^{\lambda-1}a_n + \chi(p^2)p^{\kappa-2}a_{n/p^2}$$

where $a(n/p^2) := 0$ if p^2 doesn't divide n, and with the middle term vanishing by definition if $p = 2$ and, for odd p, we have $\chi_1(p) = \chi(p)(-1/p)^\lambda$.

The proof is of course a gory double coset computation. There are $p^2 + p$ single cosets in the double coset if p doesn't divide N. If p does divide N (for example if $p = 2$), then note that T_{p^2} is just "U_p^2" on q-expansions, so it's $(T_p)^2$).

Here are some slightly simpler formulas for b_n above: if n is prime to p then

$$b_n = a_{np^2} + \chi(p)(-1/p)^\lambda(n/p)p^{\lambda-1}a_n,$$

if p divides n once then

$$b_n = a_{np^2}$$

and if p divides n at least twice then

$$b_n = a_{np^2} + \chi(p^2)p^{\kappa-2}a_{n/p^2}.$$

For example, on weight 1/2 modular forms of level 4 and trivial character and p odd we see that $b_n = a_{np^2} + (n/p)p^{-1}a_n + p^{-3/2}a_{n/p^2}$ and we check that T_{p^2} has eigenvalue $1 + p^{-1}$ on θ. However T_2 has eigenvalue 1.

More generally, let's compute the coefficient of q^1 in $T_{p^2}(\frac{1}{2}\theta_\psi)$ (the half to make $a_1 = 1$). By the above formulae, for $p > 2$ prime to $N = 4r^2$, it's $a_{p^2} + \psi(p)p^{-1} = \psi(p)(1 + p^{-1})$. So it's going to be an eigenform with eigenvalue $\psi(p)(1+p^{-1})$. Note that if ψ is totally even then θ_ψ is cuspidal but this eigenvalue looks quite Eisensteiny!

In particular, for $p >= 5$ the eigenvalue of T_{p^2} on the interesting-looking cusp form $\eta(24z)$ is going to be quite dull! In fact this isn't a surprise: see the next section.

6 Interlude: old and new.

This whole notion is much more messy in the half-integral weight case. Serre and Stark do some calculations; I'll summarise them and then do some more. If p divides $N/4$ and χ is a Dirichlet character of level N then the "oldforms of the first kind" in $M_{\kappa/2}(\Gamma_1(N),\chi)$ only exist if χ is definable mod N/p, in which case they're the image of the elements in $M_{\kappa/2}(\Gamma_1(N/p),\chi)$ which are eigenforms for all but finitely many of the T_{ℓ^2}, and the oldforms of the second kind only exist if $\chi.chi_p$ is definable mod N/p, in which case they're the $f(pz)$ at level N/p and character $\chi.chi_p$, with f an eigenform for all but finitely many of the T_{ℓ^2}. The oldforms are the space spanned by all the oldforms of all kinds as p varies. The newforms are the space spanned by eigenforms for all but finitely many of the T_{p^2} that aren't in the oldforms. What a palaver!

I need to figure out twisting. OK I figured it out. If f has level N and character χ, and ψ has level r with r prime to N, then $g := f \otimes \psi$ has level Nr^2, character $\chi.\psi^2$, and if $T_{p^2} f = \lambda_p f$ then $T_{p^2} g = \lambda_p.\psi(p^2)g$.

Note in particular the following rather scary thing: say f has level prime to ℓ and ℓ is odd. Then I can think of three forms of level $N\ell^2$ with the same T_{p^2}-eigenvalue for all p away from $N\ell^2$, namely $f(z)$, $f(\ell^2 z)$ and $f \otimes (./\ell)$ (the twist by the Dirichlet character of conductor ℓ). Actually I guess this isn't so scary—we get three forms in the integral weight setting too.

So now the natural question is as follows: say f has level N and is an eigenvector for T_{p^2} for all primes p. Now consider the space spanned by $A := f(z)$, $C := f(\ell^2 z)$ and $B := f \otimes \chi_\ell$.

The first question is: what is the dimension of this space?? One would imagine that in general it's 3. For example, for the weight 5/2 form below ($\eta(24z)^5$) the dimension is indeed 3. But in fact I just found an example when it's only 2: if $f = \theta_\psi = \sum_n \psi(n)q^{n^2}$ then $A = f$, $B = \sum_{\ell \nmid n} \psi(n)q^{n^2}$ and hence $A = B + \psi(\ell)C$. Note that θ_ψ could be cuspidal in this calculation.

Let's assume the space is 3-dimensional. If $T_{p^2} f = \lambda_p f$ then this 3-dimensional space will have T_{p^2} acting on it via λ_p for all p away from $p = \ell$. But how does U_{ℓ^2} act?

Well, set $A = f$, $B = f \otimes (./\ell)$ and $C = f(\ell^2 z)$. Then the formula for how T_{ℓ^2} acts on q-expansions gives immediately that

$$\lambda_\ell f = T_{\ell^2} f = U_{\ell^2} f + \chi(\ell)(-1/\ell)^\lambda \ell^{\lambda-1} B + \chi(\ell^2)\ell^{\kappa-2} C$$

and hence the matrix representating U_{ℓ^2} on the 3-d space with basis A, B, C is

$$\begin{pmatrix} \lambda_\ell & 0 & 1 \\ -\chi(\ell)(-1/\ell)^\lambda \ell^{\lambda-1} & 0 & 0 \\ -\chi(\ell)^2 \ell^{\kappa-2} & 0 & 0 \end{pmatrix}$$

This matrix has one eigenvalue zero (eigenvector B) and other eigenvectors given by the roots of $X^2 - \lambda_\ell X + \chi(\ell)^2 \ell^{\kappa-2}$—so the eigenforms have U_{ℓ^2}-eigenvalue given precisely by the U_ℓ-eigenvalues of the three oldforms at level $N\ell^2/2$ in the Shimura lift.

Now let's do the case of θ_ψ. We see via an easy calculation that we lose the $\ell^{-1}\psi(\ell)$ eigenvalue.

While we're here, let's go back to the general case but do level $N\ell^4$. Then A, B and C above, plus $B(q^{\ell^2})$ and $C(q^{\ell^2})$ are all in, and if A, B, C are linearly independent then thinking about coefficients of q^n with n a multiple of ℓ^2 convinces me that these five forms span a 5-dimensional space at level ℓ^4. However we can now see a 2-dimensional space in the kernel of U_{p^2}, because the matrix representing U_{p^2} at level p^4 is

$$\begin{pmatrix} \lambda_\ell & 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 1 & 0 \\ \beta & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and both B and $A - \lambda C - \alpha B(q^{\ell^2}) - \beta C(q^{\ell^2})$ are in the kernel! This makes me wonder whether Nick Ramsey's eigencurve isn't actually quite isomorphic to the Coleman-Mazur eigencurve.

7 What Serre and Stark do.

In their paper, Serre and Stark prove that the weight 1/2 modular forms are spanned over $\mathbf{C}$ by theta series $\sum_{n\in\mathbf{Z}} \psi(n)q^{tn^2}$ with ψ even and primitive. They do this by showing that formulae of Shimura regarding how Hecke operators act on q-expansions implies that if f is an eigenform for T_{p^2} for all p not in a finite set S (containing all the divisors of N) then there's some Dirichlet character ψ such that the eigenvalue of T_{p^2} will be $\psi(p)(1+p^{-1})$ for all p not in S. They go on to show that this is enough via a straightforward argument (and Deligne gives a second approach to get the result in an appendix).

In fact Serre and Stark really find bases for all spaces of weight 1/2 modular and cusp forms, so one can get explicit formulae for dimensions.

Here are some examples of conclusions one can draw from their work:

1) You'll not find any interesting eigenforms in weight 1/2; the eigenvalue of T_{p^2} will be $\psi(p)(1+p^{-1})$ for some Dirichlet character ψ.

2) (an example in Serre-Stark). If s is squarefree (and it could be even), and the product of h primes, and $N = 4s$, then a basis for $M_{1/2}(\Gamma_1(N))$ is given by the $\theta_(tz) = \sum_n q^{tn^2}$ for t running through the 2^h divisors of N. Each $\theta(tz)$ is an eigenform, with character χ_t, the primitive Dirichlet character associated to $\mathbf{Q}(\sqrt{t})/\mathbf{Q}$. So, for example, if $t = 1$ then $\chi_t = 1$, if $t = 2$ then χ_t is even and has conductor 8, if t is a prime congruent to 1 mod 4 then χ_t has conductor t and is even and of order 2. and if t is a prime congruent to 3 mod 4 then χ_t has conductor $4t$. Note that distinct χ_t give distinct characters, so $M_{1/2}(\Gamma_1(N), \chi_t)$ is 1-dimensional for each of the 2^h χ_t and is zero for all the other Dirichlet characters of level N. Note also that $S_{1/2}(\Gamma_1(N)) = 0$: no cusp forms of level N.

8 Interlude: what Cohen and Oesterlé do in the next paper in Antwerp 6.

Cohen and Oesterlé use Riemann-Roch to compute dimensions of spaces of modular forms. Their formulae work for half-integral weights. Hence they give explicit formulae for

$$\dim(S_{1/2}(\Gamma_1(N), \chi)) - \dim(M_{3/2}(\Gamma_1(N)), \chi))$$

and

$$\dim(S_{3/2}(\Gamma_1(N), \chi)) - \dim(M_{1/2}(\Gamma_1(N)), \chi))$$

Because Serre and Stark compute the dimensions of $M_{1/2}(\Gamma_1(N)), \chi)$ and $S_{1/2}(\Gamma_1(N), \chi)$ it's possible to compute the dimensions of $M_{3/2}(\Gamma_1(N)), \chi)$ and $S_{3/2}(\Gamma_1(N), \chi)$.

In fact there seems to be a typo in Cohen-Oesterlé: their table after Theorem 2, the one giving the value of some constant ζ involved in the formula—in the "non (C)" case I think that "$k - 1/2 \in \mathbf{Z}$" should say "$k - 1/2 \in 2\mathbf{Z}$ and similarly $k - 3/2 \in \mathbf{Z}$ should be replaced by "$k - 3/2 \in 2\mathbf{Z}$".

Here are examples of conclusions. If p is an odd prime, and $N = 4p$, then what are the dimensions of the weight 3/2 modular forms of level N and character χ?

Well, as we saw above, $S_{1/2}(\Gamma_1(4p), \chi)$ is always zero, and $M_{1/2}(\Gamma_1(4p), \chi)$ is also zero unless χ is either trivial, or is equal to χ_p (the character associated to $\mathbf{Q}(\sqrt{p})$, which is even, has order 2, and conductor p (resp. $4p$) for $p \equiv 1$ (resp. $p \equiv 3$) mod 4.

Now Cohen-Oesterlé in this case says that for any even Dirichlet character, we have

$$\dim(S_k(\Gamma_1(4p), \chi)) - \dim(M_{2-k}(\Gamma_1(4p), \chi)) = (k-1)(p+1)/2 - \zeta$$

where ζ is the following thing:

if $p \equiv 3$ mod 4 then $\zeta = 2$.

If $p \equiv 1$ mod 4 then:

*) $\zeta = 3/2$ if ($k \in 1/2 + 2\mathbf{Z}$ and χ_2 is trivial), or if ($k \in 3/2 + 2\mathbf{Z}$ and χ_2 is non-trivial), and

*) and $\zeta = 5/2$ in the other cases ($k \in 1/2 + 2\mathbf{Z}$ and χ_2 non-trivial, or $k \in 3/2 + 2\mathbf{Z}$ and χ_2 trivial).

So, for example (setting $k = 1/2$), if χ is even we have

$$\dim(M_{3/2}(\Gamma_1(4p), \chi)) = (p+1)/4 + \zeta$$

with $\zeta = 2$ for $p = 3$ mod 4, and for $p = 1$ mod 4 we have $\zeta = 3/2$ if χ_2 is trivial (i.e. if χ is "totally even") and $\zeta = 5/2$ if χ_2 is non-trivial.

And (setting $k = 3/2$) we have

$$\dim(S_{3/2}(\Gamma_1(4p), \chi)) - \dim(M_{1/2}(\Gamma_1(4p), \chi)) = (p+1)/4 - \zeta$$

where now $\zeta = 2$ if $p = 3$ mod 4, $\zeta = 3/2$ if χ_2 is non-trivial, and $\zeta = 5/2$ if χ_2 is trivial. Note also that the dimension of $M_{1/2}(\Gamma_1(4p), \chi)$ might be 0 or (if χ is trivial or equal to χ_p) 1.

So, for example, if $p = 3$ mod 4 then the space of modular forms of level $4p$, weight 3/2 and character χ (assumed even) has dimension $(p+9)/4$, and the space of cusp forms has dimension $(p-7)/4$, unless χ is either trivial or χ_p, in which case it has dimension $(p-3)/4$. Note that in the case $p = 3$ every even Dirichlet character of level 12 is either trivial or χ_3, so none of our spaces have negative dimension!

I wonder if there is an "explicit formula" for, say, the cuspidal eigenforms in weight 3/2, trivial character and level $4p$ with $p = 3$ mod 4; this space has dimension $(p-3)/4$. On the other hand I guess there aren't explicit formulae for the level p weight 2 eigenforms, and their dimension is also pretty regular....

9 A computer experiment to try and find an interesting eigenform.

Let's start with the cusp form $\eta(24z)$ of weight 1/2 and level 576, and let's cube it. We get a form $f = q^3 - 3q^{27} + 5q^{75} - 7q^{147} + \ldots$. Now let's hit this with T_{25}. To $O(q^{962})$ we get $6f$. That's good enough for me! Now let's try T_{49}. We get $-8f$. Now T_{121}. We get $-12f$. And T_{13^2} seems to have eigenvalue 14. So in fact it seems to me that f^3 is an eigenvector! Note also that it "looks old" doesn't it: we could have taken a factor of 3 out. Then presumably we would have got something of level 192 and trivial character, and the signs of the eigenvalues of T_{p^2} look to me like they are to do with p mod 4.

So still no interesting eigenform! Let's go to $\eta(24z)^5$. Aah geez this still looks like it's an eigenvector for T_{p^2} for all $p \geq 5$! Let ω_p denote the eigenvalue. It seems to me that these are the ω_p:

```
5:-6
7:16
11:12
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```
13:38
17:126
19:-20
23:168
29:-30
31:88
37:254
41:-42
```

Now those numbers look interesting! And we spot a level 144 weight 4 newform with T_p-eigenvalue ω_p: indeed the following is the q-expansion of a level 144 weight 4 trivial character cusp form:

```
q - 6*q^5 + 16*q^7 + 12*q^11 + 38*q^13 + 126*q^17 - 20*q^19 + 168*q^23 - 89*q^25
    - 30*q^29 + 88*q^31 - 96*q^35 + 254*q^37 - 42*q^41 + 52*q^43 - 96*q^47 -
    87*q^49 - 198*q^53 - 72*q^55 - 660*q^59 - 538*q^61 - 228*q^65 - 884*q^67 +
    792*q^71 + 218*q^73 + 192*q^77 + 520*q^79 - 492*q^83 - 756*q^85 - 810*q^89 +
    608*q^91 + 120*q^95 + 1154*q^97 + O(q^100)
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So some kind of miracle has occurred: the T_{p^2}-eigenvalue of some weight 5/2 form is also the T_p-eigenvalue of a weight 4 form! This is a special case of Shimura's theorem.

10 What Shimura did.

So Shimura came up with the following extraordinary thing. Say $\kappa \geq 3$ is odd, χ has level N (a multiple of 4) and f is non-zero of level N, weight $\kappa/2$ and character χ. Say f is an eigenform for T_{p^2} for all primes p with eigenvalue ω_p. Then there's also a modular form of weight $\kappa - 1$ and character χ^2 with T_p-eigenvalue ω_p for all p (at least away from the level); the level of this "Shimura lift" is $N/2$ (a theorem of Niwa and Cipra).

Conceptually it looks to me, very vaguely, that the following is what is happening: attached to a weight $\kappa/2$ eigenform there is some kind of "Galois representation" with H-T weights 0 and $\kappa/2 - 1$. Its symmetric square has H-T weights 0, $\kappa/2 - 1$ and $\kappa - 2$; the non-integral part is somehow removed by some fudge factor, and the eigenvalue of T_{p^2} is the trace on what is left; so by Fontaine-Mazur you expect a weight $\kappa - 2$ form.

What is so weird though is—why does one expect the symmetric square to be reducible?? I guess that in some sense the Serre-Stark theorem is an instance of this: the only eigenforms one sees have T_{p^2}-eigenvalue $\psi(p)(1 + p^{-1})$.

11 Very vague musings.

Maybe to a weight $\kappa/2$ eigenform one might hope to attach a Galois representation to $\mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ of some group G which is a central extension of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ by some group $\{1, z\}$ of order 2? At the very least G would have to have a character of the form "square root of cyclotomic". And the H-T weights of the Galois rep would be 0 and $\kappa/2 - 1$. So the symmetric square would be 3-dimensional and perhaps would for some reason contain a 1-dimensional factor and the resulting 2-dimensional quotient would have HT weights 0 and $\kappa - 2$, giving us the classical form.

But I can't figure out how to make this rigorous. If $\{1, z\}$ were a normal subgroup of G then it would be central automatically, and now if G had a character ξ which was the square root of cyclo then for an irreducible 2-d rep of G which was non-trivial on z, z would have to act via -1 so twisting by ξ would now give us a classical rep again. Hmmph.

Given a 2-dimensional representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ we somehow want to add a 1-dimensional representation and then hope that the resulting thing is Symm^2 of some 2-d rep of some bigger group. I can't see how this could work.

12 Eisenstein series.

I don't know a general reference, but I know the story for level 4. From the formulae in Cohen-Oesterle we see that the dimension of the Eisenstein space in weight $k/2$, for any $k \geq 5$, is 2. Moreover Shimura (in Antwerp 1) writes down two Eisenstein series. He computes the q-expansion of one, and leaves the other one as an exercise for the reader. I'll do the exercise. The first Eisenstein series, up to a non-zero constant depending on k, is $\sum_{n\geq 0} a_n q^n$, with a_n some ghastly mess, but a_{r^2t} for any $r \geq 1$ and t squarefree basically "only depends on a_t", and for $t > 0$ squarefree we have

$$a_t = t^{k/2-1}L((k-1)/2, \phi_t)$$

with ϕ_t the Dirichlet character $m \mapsto (t/m)$ if k is 1 mod 4 and $m \mapsto (t/m)(-1/m)$ if k is 3 mod 4 (note that this character is even if $k = 1$ mod 4 and odd if $k = 3$ mod 4, so we're evaluating the L-functions of even Dirichlet characters at positive even integers, and the L-functions of odd Dirichlet characters at positive odd integers, which is as it should be. Note that I've taken out more fudge factors than Shimura because I'm not interested in powers of π (that depend only on k) floating around.

The other Eisenstein series, we'll have to do ourselves. Here's the definition: the following sum is over $n, m \in \mathbf{Z}$ with $n > 0$ odd and $(m, n) = 1$, and it's (up to a constant depending only on k)

$$F_k(z) = \sum_{m,n} (-m/n)\epsilon_n^k(-n/z + 4m)^{-k/2}z^{-k/2}$$

with $\epsilon_n = 1$ for $n = 1$ mod 4 and i for $n = 3$ mod 4, and the $-k/2$th power being computed as the $-k$th power of the square root, with the square root taken to have either positive real part, or zero real part and positive imaginary part. Now one has to perform a gory calculation. The product of the square roots might not be the square root of $(-n + 4mz)$ because the product of the square roots is certainly in the upper half plane, whereas the square root of $-n + 4mz$ is in the lower half plane for $m < 0$. What one does is rewrites the sum as three sums: $m = 0$, $m > 0$ and $m < 0$. The $m = 0$ sum only involves $n = 1$ and hence gives us $(-1/z)^{-k/2}z^{-k/2}$, which is $((-1/z)^{1/2}z^{-1/2})^{-k}$ and the product of the square roots is in the upper half plane and it's a square root of -1, so it's i, and we get i^{-k} (which will be $\pm i$ depending on k mod 4).

The sum for $m > 0$ we evaluate as (the sum over $m, n > 0$ with n odd and $(m, n) = 1$

$$\sum_{m,n>0} (-m/n)\epsilon_n^k(-n + 4mz)^{-k/2}.$$

The sum for $m < 0$ we evaluate similarly (note now that the product of the square roots is minus the square root of the product) and, *after changing variables* $n \mapsto -n$ *and* $m \mapsto -m$ we get

$$\sum_{m>0,n<0} (m/-n)(-1)\epsilon_{-n}^k(n - 4mz)^{-k/2}.$$

Now in this second sum we have $(n - 4mz)^{1/2} = (-n + 4mz)^{1/2}i^{-1}$ and subbing this in we get

$$\sum_{m>0,n<0} (m/-n)(-1)i^k\epsilon_{-n}^k(-n + 4mz)^{-k/2}.$$

Now we can put it all back together and get i^{-k} plus

$$\sum_{m>0,n\in\mathbf{Z},(m,n)=1} c_{m,n}(-n + 4mz)^{-k/2}$$

where $c_{m,n}$ is zero if n is even, $(-m/n)\epsilon_n^k$ if $n > 0$ is odd, and $(m/-n)(-1)i^k\epsilon_{-n}^k$ if $n < 0$ is odd. Now a miraculuous calculation shows that $c_{m,n}$ only depends on n modulo $4m$ (even allowing sign changes for n) and hence we can rewrite as

$$i^k + \sum_{m>0} \left(\sum_{n\in\mathbf{Z},(n,m)=1} c_{m,n}(-n + 4mz)^{-k/2} \right)$$

and setting $n = r + 4mh$ with $0 \le r < 4m$ we get

$$i^k + \sum_{m>0} \sum_{0 \le r < 4m} c_{m,r} \sum_{h \in \mathbf{Z}} (-r - 4mh + 4mz)^{-k/2}$$

and we can do the sum over h (it's in Shimura) and we get

$$c + \sum_{m>0} \sum_{0<r<4m} c_{m,r} m^{-k/2} \sum_{n \ge 1} n^{k/2-1} e^{2\pi i n (z - r/4m)}$$

(up to a constant depending only on k) so it's $\sum_{n \ge 0} a_n q^n$ with, for $n > 0$ (and the sum over odd r only)

$$a_n = n^{k/2-1} \sum_{m>0} m^{-k/2} \sum_{0<r<4m} (-m/r) \epsilon_r^k e^{-2\pi i n r/4m}.$$

This will no doubt have something to do with L-functions :-/

One thing is for sure though, and that's that the Eisenstein series that Shimura does bash out has special values of L-functions showing up in its coefficients.

Cohen noted in 1975 (although maybe it was well-known before then) that using the functional equation we can switch to the other side of the L-function; for example he checked that for every integer $r \ge 2$ there was a weight $r + \frac{1}{2}$ modular form of level 4 such that the coefficient of q^n, for $n = (-1)^r D$ with D the discriminant of a quadratic field, was $L(1 - r, \chi_D)$. So we get one form which encodes $L(1 - r, \chi_D)$ for all D.

13 Waldspurger.

I've not seen Waldspurger's paper "Sur les coefficients de Fourier des formes modulaires de poids demi-intier" yet, but judging from what Tunnell says about it, Waldspurger takes these L-function observations up a level by doing an analogous thing with cusp forms! His result, vaguely speaking, is that if f has weight $k/2$ and is an eigenform, and F is its Shimura lift to weight $k - 1$, and if F is now cuspidal, then the L-functions of quadratic twists of F, evaluated at the central point, are all (up to harmless factors) *squares* of Fourier coefficients of f!

For example the special values at 1 of quadratic twists of an elliptic curve seems to be related in quite a concrete way to the coefficients of a weight 3/2 modular form whose Shimura lift is the weight 2 form attached to the curve in question. In an Inventiones paper from 1983, Tunnell observed that Waldspurger's generalisation could be concretely applied to $E : y^2 = x^3 + x$, giving an explicit weight 3/2 modular form such that the coefficient of q^d was some explicit constant times $L(E^d, 1)$ where E is the twist of E by d (probably d has to be odd and squarefree, but he has another form which works when d is even). As a result he got a very computationally effective way of proving that a number wasn't a congruent number (because Coates-Wiles shows that if the L-function is non-vanishing then the curve has rank zero) and furthermore, if the curve has rank zero, he gets a conjectural formula for the Tate-Schaferevich group of the curve that again is very computationally efficient.

14 More on Tunnell's forms.

Let's compute all the weight 1/2 and weight 3/2 forms for all levels dividing 128, and character either trivial or even of conductor 8 (these are the only two even characters that square to the identity).

First note that my weight 1 computations indicate that there are no weight 1 forms (of any character at all) of level 2^n for $n < 7$, and at level $2^7 = 128$ there is one, with character the odd character of conductor 8.

Now for $N = 2^n$, $2 \le n \le 7$, there are no weight 1/2 cusp forms of level N, and the number of Eisenstein series in weight 1/2 and trivial character in level $4, 8, 16, 32, 64, 128$ is $1, 1, 2, 2, 3, 3$ (these being θ, $\theta(4z)$ and $\theta(16z)$, the point being that there's no room for any $\theta_\psi(q^t)$ for ψ non-trivial and even, because the conductor of ψ would have to be at most 4), and the number of Eisenstein series of weight 1/2 and even quadratic character is always 1 (it's θ_χ for χ the character) apart from the cases $N = 32, 64, 128$ and even character of conductor 8 where the answer is 2,2,3 resp. (the extra forms are $\theta_\chi(4z)$ and $\theta_\chi(16z)$).

Note in particular that we understand all the Hecke eigenvalues of these forms too (because we're in weight 1/2 so everything is a theta series).

So that's weight 1/2 out of the way. Now let's plough up weight 3/2, starting at level 4 (and trivial character). Here there's only one form (θ^3) and its T_4-eigenvalue is 1 and its T_{p^2}-eigenvalue is $(1+p)$ for p odd (look at the constant coefficient!).

At level 8 there are two modular forms of trivial character (θ^3 and $\theta_2^2\theta$). Now $f := \theta^3$ is an eigenform for all T_{p^2}, but if $g := \theta_2^2\theta$ then $T_4 g = f$, $T_9 g = 4g$, $T_{25} g = 6g$ and so on, so we have a 2-dimensional eigenspace for all the good T_{p^2}, and a new eigenvalue of zero for T_{2^2}, so indeed it looks like we're picking up the level 4 infinite slope Eisenstein oldform.

Let's just stick to trivial character at the minute.

At level 16 with trivial character there are still no cusp forms, and there are four modular forms, because now θ_4 comes into play so we get θ^3, $\theta^2\theta_4$, $\theta\theta_4^2$, and θ_4^3 as well as $\theta_4\theta_2^2$ and so on (there are six combinations but they only span a 4-dimensional space). Each seems to me to be an eigenvector for T_{p^2} for p odd. Perhaps surprisingly, T_4 sends each of these six things to θ^3, hence multiplicity 1 is in some sense failing: each T_{p^2} (including $p = 2$) is simultaneously diagonalisable, all the T_{p^2} for $p > 2$ are scalar $p + 1$ and T_2 has rank 1 and a 3-dimensional kernel.

At level 32 with trivial character: still no cusp forms, but now six modular forms. We're allowed to use θ_8 now; there are ten products of theta series and these generate a 6-dimensional space, so that's it. Again T_{p^2} acts as $p + 1$ for $p > 2$.

At level 64 with trivial character something happens! There's a 10-dimensional space of modular forms and a 1-dimensional space of cusp forms! The obvious products of θ_t ($t = 1, 2, 4, 8, 16$) span a 10-dimensional space so that's all the forms; for the first few p odd we see computationally that T_{p^2} has eigenvalue $p + 1$ ten times if $p = 1 \bmod 4$, but only nine $p + 1$ eigenvalues for $p = -1 \bmod 4$; the tenth eigenvalue is $p - 1$. This is the cusp form, surely: it's a theta series, it's half of $-\theta.\theta_4^2 + \theta_4^3 + 2\theta_{16}\theta_4\theta - 2\theta_{16}\theta_4^2$ which comes out magically to be $\sum \chi_{-1}(n)nq^n$. So there's our first cusp form but it's not very interesting.

Note that because of our bounds on N (N divides 128) this is the only weight 3/2 theta series we'll see (they have level $4r^2$ in general, where r is the conductor of an odd character).

Finally level 128; there are 16 modular forms of which three are cusp forms. I can think of 27 theta series of the form $\theta_r\theta_s\theta_t$ with $r, s, t \in \{1, 2, 4, 8, 16, 32\}$ with an even number of them not powers of 4; all such things are level 128 trivial character forms. They span a space that looks like it's 16-dimensional though, which is good.

This space (the 16-dimensional space) seems to break up into four generalised eigenspaces. The least interesting is 12-dimensional, $T_{p^2} = 1 + p$ for all odd primes $p < 40$ and T_4 is nilpotent, with 7-dimensional kernel, $(T_4)^2$ has 11-dimensional kernel, and $(T_4)^3 = 0$. Next we have a 2-dimensional space on which all Hecke operators appear to act via scalars and the scalar is that of the elliptic curve $X_0(32)$ (that is, $T_4 = 0$, $T_9 = 0$, $T_{25} = -2$ and so on) so no doubt these lift to $X_0(32)$. Next we get a 1-dimensional eigenspace spanned by the weight 3/2 theta series $\sum_n \psi(n)nq^{n^2}$ with ψ of conductor 4; this is surely cuspidal and has $T_4 = 0$ and $T_{p^2} = \pm(1 + p)$ depending on p mod 4. Finally we have θ^3 with $T_4 = 1$ and $T_{p^2} = (1 + p)$ for all odd p.

Note that the weight 2 level 64 newform (which is a twist of $X_0(32)$ by either character of conductor 8) doesn't come from weight 3/2 and level 128 and trivial character.

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Now let's move onto even character of conductor 8. [come back to this one day?]

References

[1] Serre and Stark, *Modular forms of weight 1/2*, Antwerp VI (Springer LNM 627) pages 27–68.