FILLINGS OF UNIT COTANGENT BUNDLES

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Abstract. We study the topology of exact and Stein fillings of the canonical contact structure on the unit cotangent bundle of a closed surface Σ_g, where g is at least 2. In particular, we prove a uniqueness theorem asserting that any Stein filling must be s-cobordant rel boundary to the disk cotangent bundle of Σ_g. For exact fillings, we show that the rational homology agrees with that of the disk cotangent bundle, and that the integral homology takes on finitely many possible values, including that of $DT^*\Sigma_g$: for example, if $g-1$ is square-free, then any exact filling has the same integral homology and intersection form as $DT^*\Sigma_g$.

1. Introduction

The unit cotangent bundle ST^*M of a Riemannian manifold M is equipped with a canonical contact structure ξ_{can}, given in local coordinates as the kernel of $\alpha = \sum_i p_i dq_i$. The contact manifold $(ST^*M, \xi_{\text{can}})$ is Stein fillable, with one filling given by the disk cotangent bundle DT^*M, and it is natural to ask whether other such fillings exist. Our goal in this paper is to study the Stein fillings, and more generally the exact symplectic fillings, of $(ST^*M, \xi_{\text{can}})$ in the case where $M = \Sigma_g$ is a surface of genus $g \geq 2$. We will denote the contact manifold $(ST^*\Sigma_g, \xi_{\text{can}})$ by (Y_g, ξ_g).

In the cases $g = 0, 1$ the fillings of (Y_g, ξ_g) are already understood, and we know that in fact any minimal symplectic filling must be diffeomorphic to $DT^*\Sigma_g$. McDuff [McD90] proved this for $(ST^*S^2 = \mathbb{RP}^3, \xi_{\text{can}})$, and then Hind [Hin00] showed that DT^*S^2 is the unique Stein filling up to Stein homotopy. Similarly, Stipsicz [Sti02] proved that a Stein filling of the unit cotangent bundle $ST^*T^2 = T^3$ must be homeomorphic to $DT^*T^2 \cong T^2 \times D^2$, and Wendl [Wen10] showed that all of its minimal strong symplectic fillings are symplectically deformation equivalent to DT^*T^2.

For $g \geq 2$, however, no such uniqueness results for symplectic fillings are possible. This was observed by Li, Mak, and Yasui [LMY14, Proposition 3.3], who noted that in this case McDuff [McD91] constructed a symplectic 4-manifold which strongly fills its disconnected boundary, one of whose components is (Y_g, ξ_g). One can glue a symplectic cap with b_2^+ arbitrarily large to the remaining component to get an arbitrarily large filling of (Y_g, ξ_g); or, as pointed out by Wendl [Wen14], one can even use this to construct a strong symplectic cobordism from any contact 3-manifold to (Y_g, ξ_g).

Despite this, if we require the fillings in question to be exact or Stein then the situation is drastically simpler. Our main results are the following, which appear as Theorem 4.10 and Theorem 3.4 respectively.

Theorem 1.1. *If (W, J) is a Stein filling of $(Y_g, \xi_g) = (ST^*\Sigma_g, \xi_{\text{can}})$, then W is s-cobordant rel boundary to the disk cotangent bundle $DT^*\Sigma_g$.*

In particular, W is homotopy equivalent rel boundary to $DT^*\Sigma_g$.

Theorem 1.2. *If (W,ω) is an exact symplectic filling of $(Y_g,\xi_g) = (ST^*\Sigma_g, \xi_{\mathrm{can}})$, then the homology of W is given by*

$$H_1(W;\mathbb{Z}) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}/d\mathbb{Z}, \qquad H_2(W;\mathbb{Z}) \cong \mathbb{Z}, \qquad H_3(W;\mathbb{Z}) = 0$$

for some integer d such that d^2 divides $g-1$, and the intersection form on $H_2(W)$ is $\langle \frac{2g-2}{d^2} \rangle$.

Remark 1.3. The requirement that $d^2 \mid g-1$ implies that the integral homology and intersection form of an exact filling of (Y_g,ξ_g) are uniquely determined (hence isomorphic to those of $DT^*\Sigma_g$) whenever $g-1$ is square-free. This condition is well-known to hold for a subset of the natural numbers with density $\frac{6}{\pi^2}$.

Remark 1.4. Li, Mak, and Yasui have independently proved a stronger version of Theorem 1.2, namely that every exact filling of (Y_g,ξ_g) has the integral homology and intersection form of $DT^*\Sigma_g$, by similar arguments; the key technique in both cases is originally due to them, as explained below. In other words, every exact filling has $d = 1$.

One notable feature of Theorems 1.1 and 1.2 is that all of the fillings involved have $b_2^+(W)$ positive. As far as we are aware, any classification theorems which have been proved to date for symplectic or Stein fillings of fillable contact 3-manifolds (Y,ξ) have the feature that all of the symplectic fillings have $b_2^+ = 0$. This is true because the classifications usually follow from one of two starting points: either (Y,ξ) has a symplectic cap containing a symplectic sphere of nonnegative self-intersection, or (Y,ξ) is supported by a planar open book.

In the first of these cases, it follows from McDuff [McD90] that any filling embeds into a blow-up of either $\mathbb{CP}^2$ or a ruled surface, and in either case the closed manifold has $b_2^+ = 1$, with H_2^+ generated by the symplectic sphere inside the cap. In the second case, the classifications use work of Wendl [Wen10], who showed that all Stein fillings admit Lefschetz fibrations corresponding to factorizations of the monodromy into positive Dehn twists; but the planarity implies by a result of Etnyre [Etn04] (whose proof relies on [McD90]) that the filling is negative definite. These techniques have been applied successfully to many contact structures on lens spaces [McD90, Lis08, PVHM10, Kal13], links of simple singularities [OO05], and Seifert fibered spaces [Sta15], among others.

The reason we are able to succeed in the absence of either technique is the use of a Calabi-Yau cap, as defined and studied by Li, Mak, and Yasui [LMY14]. We find a Lagrangian Σ_g inside a K3 surface with simply connected complement, and a Weinstein tubular neighborhood of this Lagrangian is symplectomorphic to the disk cotangent bundle of Σ_g, so its complement is a symplectic cap for (Y_g,ξ_g). Gluing this cap to any filling produces a closed 4-manifold X of symplectic Kodaira dimension zero, and the classification of the latter [MS97, Bau08, Li06a] tells us that X must be an integral homology K3. In Section 3 we then deduce Theorem 1.2 from careful application of the Mayer-Vietoris sequence, and following this we use properties of Stein fillings in Section 4 to pin down the fundamental group and prove Theorem 1.1. We note that Li, Mak, and Yasui originally proved that (Y_g,ξ_g) have "finite type" in the language of [LMY14] by using Lagrangian surfaces in T^4 to construct Calabi-Yau caps, proving that they have "finite type" in the terminology of [LMY14]; following this, our construction of Lagrangians in a K3 led to Theorems 1.1 and 1.2, and they simultaneously used a different construction of Lagrangians in a K3 to produce their stronger version of Theorem 1.2 as described in Remark 1.4.

Finally, we remark that we would like to strengthen Theorem 1.1 by showing that any Stein filling W of (Y_g,ξ_g) is homeomorphic to $DT^*\Sigma_g$, but for now this may be out of reach using our techniques. This would require a proof of the topological s-cobordism theorem for s-cobordisms between 4-manifolds with fundamental group $\pi_1(\Sigma_g)$, which is currently only known when the fundamental group is "good" (see Freedman-Quinn [FQ90]), and it is an open question whether surface groups are good.

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2. Calabi-Yau caps

In this section, we will construct and study a certain type of concave filling which was originally used by Li, Mak, and Yasui [LMY14] to bound the topology of Stein fillings of a given manifold.

Definition 2.1. *Let (Y, ξ) be a contact manifold. A* Calabi-Yau cap *for (Y, ξ) is a symplectic manifold (W, ω) with concave boundary (Y, ξ) and torsion first Chern class, such that there is a contact form α for ξ and a Liouville vector field X near $Y = \partial W$ satisfying $\alpha = \iota_X\omega|_Y$.*

In this section we will show that (Y_g, ξ_g) admits a simply connected Calabi-Yau cap by finding an embedded Lagrangian Σ_g of genus g inside the elliptic surface $E(2)$, which is a K3 surface. The cap X_g is then the complement of a Weinstein tubular neighborhood of Σ_g, and it is Calabi-Yau because a K3 surface has trivial canonical class. We remark that Li, Mak, and Yasui originally constructed a Calabi-Yau cap for (Y_g, ξ_g) by finding a Lagrangian Σ_g inside the standard symplectic T^4, but this larger cap (and the one they also find in a K3 surface) allows for much stronger restrictions on the possible fillings.

Theorem 2.2. *The elliptic surface $E(2)$ contains a Lagrangian surface Σ_g of genus g such that the complement X_g of a Weinstein tubular neighborhood of Σ_g is simply connected.*

Proof. We express the elliptic fibration $\pi : E(2) \to S^2$ as a fiber sum $E(1)\#_{T^2}E(1)$, where if a and b are a pair of curves in the torus T^2 which intersect exactly once then each fibration $E(1) \to S^2$ has six singular fibers with vanishing cycle a and six with vanishing cycle b, corresponding to the relation $(ab)^6 = 1$ in the mapping class group of the torus. We can think of the base of the fibration π as a connected sum $S^2 = S^2\#S^2$, with one copy of $E(1)$ over each summand.

Let $\gamma \subset S^2$ be a simple closed curve separating the two S^2 summands; then we can arrange for γ to have a small collar neighborhood $A = (-\epsilon, \epsilon) \times \gamma \subset S^2$, with no critical values of π, so that the symplectic form on $\pi^{-1}(A) \cong A \times T^2$ is the product symplectic form induced by area forms on each factor. In particular, if we pick distinct values $t_1, \dots, t_g \in (-\epsilon, \epsilon)$ then the g disjoint tori $T_i = \{t_i\} \times \gamma \times a$ are all Lagrangian.

Now let $c \subset S^2$ be a matching path [Sei08] between two critical points, one in either S^2 summand of $S^2 = S^2\#S^2$, which each have vanishing cycle b. Then c lifts to a Lagrangian sphere $S \subset E(2)$. We can arrange for c to intersect γ transversely in a single point, and if each t_i is sufficiently close to zero it follows that S intersects each T_i transversely in a single point as well, namely the point $a \cap b$ in the fiber above $c \cap (\{t_i\} \times \gamma)$. We surger S and T_i together at each of these points [LS91, Pol91] to produce a Lagrangian Σ_g of genus g;

we note that each surgery depends on an ordering of the Lagrangians, but any choice will suffice. We now take $X_g = E(2) \smallsetminus N(\Sigma_g)$, where $N(\Sigma_g)$ is a small Weinstein neighborhood of Σ_g.

It remains to be seen that X_g is simply connected. Since $E(2)$ is simply connected, $\pi_1(X_g)$ is normally generated by the class of a meridian μ of the Lagrangian Σ_g. We let $c' \subset S^2$ be a path in one of the two S^2 summands with endpoints at a pair of critical values which both have vanishing cycle a, such that c' intersects c once transversely and is disjoint from each of the $\{t_i\} \times \gamma$. Then there is a sphere $S' \subset E(2)$ lying above c' (which need not be Lagrangian) such that $S' \cap S$ is the single point $a \cap b$ in the fiber over $c' \cap c$, hence S' intersects Σ_g transversely in precisely this point. We arrange for S' to intersect $\overline{N(\Sigma_g)}$ in a single meridional disk D about this point, and then $S' \cap X_g$ is a disk $\overline{S' \smallsetminus D}$ with boundary a meridian of Σ_g, so $[\mu] = 0$ and we are done. □

Proposition 2.3. *The cap X_g has Betti numbers $b_2^+(X_g) = 2$ and $b_2^-(X_g) = 19$.*

Proof. Recall that $b_2^+(K3) = 3$ and $b_2^-(K3) = 19$. If D_g is the Weinstein neighborhood of the surface Σ_g, so that D_g is symplectomorphic to the disk cotangent bundle of Σ_g, then $H_2(D_g) = H_2(\Sigma_g) = \mathbb{Z}$, and since Σ_g has self-intersection $2g-2 > 0$ the signature of D_g is 1. By Novikov additivity we have $\sigma(D_g) + \sigma(X_g) = \sigma(K3) = -16$, so X_g has signature -17. It thus suffices to show that $b_2^+(X_g) = 2$.

Let $V_+ \subset H_2(K3;\mathbb{Q})$ be a positive definite 3-dimensional subspace containing the class $[\Sigma_g]$. We can extend $[\Sigma_g]$ to a rational basis of V_+ whose other two classes are orthogonal to $[\Sigma_g]$, and thus integral multiples of those two classes can be represented by surfaces which are disjoint from Σ_g and even avoid the neighborhood D_g. These surfaces span a positive definite subspace of $H_2(X_g;\mathbb{Q})$, so that $b_2^+(X_g) \geq 2$. However, if $b_2^+(X_g) \geq 3$ then adjoining $[\Sigma_g]$ to a basis of a 3-dimensional positive-definite subspace of $H_2(X_g;\mathbb{Q})$ would yield $b_2^+(K3) \geq 4$, which is absurd. □

We can understand the homology of X_g more precisely by considering the Mayer-Vietoris sequence associated to the decomposition $K3 = D_g \cup_{Y_g} X_g$.

Proposition 2.4. *We have $H_1(X_g;\mathbb{Z}) = H_3(X_g;\mathbb{Z}) = 0$ and $H_2(X_g;\mathbb{Z}) \cong \mathbb{Z}^{21} \oplus \mathbb{Z}^{2g}$, where the intersection form on X_g has block form $\left(\begin{smallmatrix} Q_g & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with respect to this decomposition for some nondegenerate form Q_g on $\mathbb{Z}^{21}$.*

Proof. The claim that $H_1(X_g) = 0$ follows immediately from X_g being simply connected, so then $H^1(X_g) = 0$ as well by the universal coefficient theorem. For $H_3(X_g)$, we use the general fact that if X is an orientable n-manifold with nonempty boundary, then $H_{n-1}(X)$ injects into $H_{n-1}(X,\partial X) \cong H^1(X)$: the long exact sequence of the pair $(X,\partial X)$ says that

$$0 \to H_n(X,\partial X) \to H_{n-1}(\partial X) \to H_{n-1}(X) \to H_{n-1}(X,\partial X)$$

is exact, and the map $H_n(X,\partial X) \to H_{n-1}(\partial X)$ is an isomorphism $\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}$ since it carries the relative fundamental class $[X,\partial X]$ to $[\partial X]$. In general, this implies that $H_{n-1}(X)$ is torsion-free since $H^1(X)$ is; in this case, since $H^1(X_g) = 0$ we have $H_3(X_g) = 0$ as well.

Since $H_1(K3) = H_3(K3) = 0$ and $H_1(X_g) = 0$, a portion of the Mayer-Vietoris sequence is given by

$$0 \to H_2(Y_g) \xrightarrow{i_2} H_2(D_g) \oplus H_2(X_g) \xrightarrow{j} H_2(K3) \xrightarrow{\delta} H_1(Y_g) \xrightarrow{i_1} H_1(D_g) \to 0.$$

Now from $H_1(Y_g) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}/(2g-2)$ we compute $H_2(Y_g) \cong H^1(Y_g) \cong \mathbb{Z}^{2g}$, so that

$$0 \to \mathbb{Z}^{2g} \xrightarrow{i_2} \mathbb{Z} \oplus H_2(X_g) \xrightarrow{j} \mathbb{Z}^{22} \xrightarrow{\delta} \mathbb{Z}^{2g} \oplus \mathbb{Z}/(2g-2) \xrightarrow{i_1} \mathbb{Z}^{2g} \to 0$$

is exact. Any torsion in $H_2(X_g)$ must lie in $\ker(j) = \mathrm{Im}(i_2)$, and since $\mathrm{Im}(i_2) \cong \mathbb{Z}^{2g}$ is torsion-free it follows that $H_2(X_g)$ is as well, hence $H_2(X_g) = \mathbb{Z}^{b_2(X_g)}$. The map $i_1 : \mathbb{Z}^{2g} \oplus \mathbb{Z}/(2g-2) \to \mathbb{Z}^{2g}$ is surjective and its target $\mathbb{Z}^{2g}$ is free, so $\ker(i_1)$ must be precisely the torsion subgroup $\mathbb{Z}/(2g-2)$ of the domain. Thus $\mathrm{Im}(\delta) = \mathbb{Z}/(2g-2)$, and we deduce from the above sequence that

$$0 \to \mathbb{Z}^{2g} \xrightarrow{i_2} \mathbb{Z} \oplus \mathbb{Z}^{b_2(X_g)} \xrightarrow{j} \mathbb{Z}^{22} \to \mathbb{Z}/(2g-2) \to 0$$

is exact. It follows that $\mathrm{Im}(j)$ has index $2g-2$ and that $b_2(X_g) = 21 + 2g$; the latter fact implies that $H_2(X_g) = \mathbb{Z}^{21+2g}$.

Next, we note that the image of the natural map $H_2(Y_g) \to H_2(D_g)$ contributes to $b_2^0(D_g)$, since any surface inside Y_g can be displaced inside a collar neighborhood of ∂D_g; but $H_2(D_g)$ is torsion-free and positive definite, so the map $H_2(Y_g) \to H_2(D_g)$ must be zero. Since i_2 is injective, it follows that $H_2(Y_g) = \mathbb{Z}^{2g}$ injects into $H_2(X_g) = \mathbb{Z}^{21+2g}$. Then $\ker(j) \subset H_2(X_g)$ is isomorphic to $\mathbb{Z}^{2g}$, so that the map $H_2(X_g) \to H_2(K3)$ has rank 21. Since its image is a subgroup of $\mathbb{Z}^{22}$ it is free abelian, hence isomorphic to $\mathbb{Z}^{21}$, and so we have an exact sequence

$$0 \to H_2(Y_g) \to H_2(X_g) \to \mathbb{Z}^{21} \to 0$$

which splits because $\mathbb{Z}^{21}$ is free. Thus we have a direct sum decomposition

$$H_2(X_g) \cong \mathbb{Z}^{21} \oplus H_2(Y_g)$$

in which the $H_2(Y_g)$ summand lies in the kernel of the intersection form. But we have seen that $b_2^+(X_g) + b_2^-(X_g) = 21$, so the intersection form must be nondegenerate on the $\mathbb{Z}^{21}$ summand and the proof is complete. □

Remark 2.5. The kernel of $j : H_2(D_g) \oplus H_2(X_g) \to H_2(K3)$ is the $H_2(Y_g) \cong \mathbb{Z}^{2g}$ summand of $H_2(X_g)$, so it restricts to an injective map

$$j' : H_2(D_g) \oplus \mathbb{Z}^{21} \to H_2(K3)$$

whose domain is the lattice $\mathbb{Z} \oplus \mathbb{Z}^{21}$ with intersection form $\left(\begin{smallmatrix} 2g-2 & 0 \\ 0 & Q_g \end{smallmatrix}\right)$ in block form, and j' embeds this lattice as an index-$(2g-2)$ sublattice of $H_2(K3) \cong \mathbb{Z}^{22} \cong 3H \oplus -2E_8$.

3. The topology of exact fillings

We can now use the Calabi-Yau cap (X_g, ω_g) provided by Theorem 2.2 to understand the topology of fillings of (Y_g, ξ_g).

Proposition 3.1. *Let (W, ω) be an exact symplectic filling of (Y_g, ξ_g). Then the closed symplectic manifold*

$$(Z, \omega_Z) = (W, \omega) \cup_{(Y_g, \xi_g)} (X_g, \omega_g)$$

is an integer homology K3, with $H_1(Z; \mathbb{Z}) = H_3(Z; \mathbb{Z}) = 0$ and $H_2(Z; \mathbb{Z}) = \mathbb{Z}^{22}$.

Proof. We can easily verify that $K_Z \cdot [\omega_Z] = 0$, where K_Z is the canonical class of (Z, ω_Z). Indeed, we can express it as a sum $K_Z|_W \cdot [\omega] + K_Z|_{X_g} \cdot [\omega_g]$, and in the first term we have $[\omega] = 0$ since the form ω is exact, while in the second term we have $K_Z|_{X_g} = 0$ because X_g

has trivial canonical class. Moreover, we have $b_2^+(Z) \geq b_2^+(X_g) = 2$, with the latter equality provided by Proposition 2.3.

Since $b_2^+(Z) \geq 2$ and $K_Z \cdot [\omega_Z] = 0$, it follows from Taubes [Tau95] that the only Seiberg-Witten basic classes on Z are $\pm K_Z$. Further work of Taubes [Tau96] then shows that $K_Z = 0$, hence 0 is the only basic class: indeed, K_Z is Poincaré dual to an embedded symplectic surface Σ, and we have $K_Z \cdot [\omega_Z] = \int_\Sigma \omega_Z \geq 0$ with equality only if $[\Sigma] = 0$. It follows that Z must be symplectically minimal, since otherwise the blow-up formula [FS95] implies that there would be at least two basic classes.

We have now shown that (Z, ω_Z) is minimal with trivial canonical class, and this proves that its symplectic Kodaira dimension [Li06b] is zero. By work of Morgan–Szabó [MS97], Bauer [Bau08], and Li [Li06a], it follows that Z has the rational homology of a K3 surface, an Enriques surface, or a T^2-bundle over T^2. The latter two cases imply $b_2(Z) = 10$ and $b_2(Z) \leq 6$ respectively, and neither of these can happen – we already know that $b_2(Z) \geq b_2^-(Z) \geq b_2^-(X_g) = 19$ – so Z is a rational homology K3.

Finally, if $H_1(Z)$ is nontrivial, then it is torsion and so the kernel of the abelianization map $\pi_1(Z) \to H_1(Z)$ has finite index in $\pi_1(Z)$. If the corresponding finite cover $(\tilde{Z}, \tilde{\omega}_Z) \to (Z, \omega_Z)$ has degree $n = |H_1(Z)|$, then $(\tilde{Z}, \tilde{\omega}_Z)$ has symplectic Kodaira dimension zero and hence signature at least -16 [Bau08, Li06a], whereas $\sigma(\tilde{Z}) = -16n$, so we must have $n = 1$ and thus $H_1(Z) = 0$. It follows from Poincaré duality and the universal coefficient theorem that $H_3(Z) \cong H^1(Z) = 0$, and that $H_2(Z) \cong H^2(Z)$ is torsion-free since $H_1(Z) = 0$ is. □

Remark 3.2. In the above argument, we see that Z has even intersection form since $K_Z = 0$ is a characteristic class. If we can show that $\pi_1(Z) = 1$, then Z will be a simply connected, even, smooth 4-manifold with $b_2^+(Z) = 3$ and $b_2^-(Z) = 19$, implying that it is homotopy equivalent and hence homeomorphic to a K3 surface [Fre82].

For example, if (W, J) is a Stein filling of (Y_g, ξ_g) then the inclusion $Y_g \hookrightarrow W$ induces a surjection $\pi_1(Y_g) \to \pi_1(W)$, and the cap X_g is simply connected, so van Kampen's theorem says that $\pi_1(Z) = \pi_1(W) *_{\pi_1(Y_g)} 1 = 1$. Thus if (W, J) is a Stein filling then Z is homeomorphic to a K3 surface.

Corollary 3.3. *If (W, ω) is an exact symplectic filling of (Y_g, ξ_g), then W has the same Betti numbers as the disk cotangent bundle $DT^*\Sigma_g$, namely $b_3(W) = 0$, $b_2^+(W) = 1$ and $b_2^-(W) = b_2^0(W) = 0$, and $b_1(W) = 2g$.*

Proof. We glue the cap (X_g, ω_g) to (W, ω) to form Z, which is a homology K3 and thus has signature -16. Novikov additivity says that $-16 = \sigma(W) + \sigma(X_g)$, and from Proposition 2.3 we conclude that $\sigma(W) = 1$. In particular, we have $b_2^+(W) \geq 1$.

Now we consider the Mayer-Vietoris sequence for $Z = W \cup_{Y_g} X_g$ with coefficients in $\mathbb{Q}$: since $b_2(Z) = 22$ and $b_2(Y_g) = b_1(Y_g) = 2g$, the part of the sequence between $H_3(Z; \mathbb{Q}) = 0$ and $H_1(Z; \mathbb{Q}) = 0$ has the form

$$0 \to \mathbb{Q}^{2g} \to \mathbb{Q}^{b_2(W)} \oplus \mathbb{Q}^{21+2g} \to \mathbb{Q}^{22} \to \mathbb{Q}^{2g} \to \mathbb{Q}^{b_1(W)} \oplus \mathbb{Q}^0 \to 0.$$

Since we already know that $b_2(W) \geq 1$, an easy exercise shows that in fact $b_2(W) = 1$ and $b_1(W) = 2g$; then $\sigma(W) = 1$ implies that $b_2^+(W) = 1$ and $b_2^-(W) = b_2^0(W) = 0$ as claimed. Similarly, between $H_4(W; \mathbb{Q}) \oplus H_4(X_g; \mathbb{Q}) = 0$ and $H_3(Z; \mathbb{Q}) = 0$, we have

$$0 \to \mathbb{Q} \to \mathbb{Q} \to \mathbb{Q}^{b_3(W)} \oplus \mathbb{Q}^0 \to 0$$

and this can only be exact if $b_3(W) = 0$. □

Theorem 3.4. *If (W,ω) is an exact filling of (Y_g,ξ_g), then for some integer d such that d^2 divides $g-1$ we have $H_3(W;\mathbb{Z})=0$; $H_2(W;\mathbb{Z})\cong\mathbb{Z}$, with intersection form $\langle\frac{2g-2}{d^2}\rangle$; and $H_1(W;\mathbb{Z})\cong\mathbb{Z}^{2g}\oplus\mathbb{Z}/d\mathbb{Z}$.*

Proof. We recall from the proof of Proposition 2.4 that $H_3(W)$ is torsion-free, and so $b_3(W)=0$ implies that $H_3(W)=0$.

Now we write $Z=W\cup_{Y_g}X_g$, with Z an integer homology K3, and consider the Mayer-Vietoris sequence over $\mathbb{Z}$:

$$0\to H_2(Y_g)\xrightarrow{i}H_2(W)\oplus H_2(X_g)\xrightarrow{j}H_2(Z)\xrightarrow{\delta}H_1(Y_g).$$

We know that $H_1(Y_g)=\mathbb{Z}^{2g}\oplus\mathbb{Z}/(2g-2)$, hence $H_2(Y_g)=H^1(Y_g)=\mathbb{Z}^{2g}$, and similarly we know the homology of X_g from Proposition 2.4. Any torsion in $H_2(W)$ must lie in $\ker(j)=\mathrm{Im}(i)$ since $H_2(Z)=\mathbb{Z}^{22}$ is free, but $\mathrm{Im}(i)\cong\mathbb{Z}^{2g}$ is also free, so $H_2(W)$ is torsion-free and thus $H_2(W)=\mathbb{Z}$ by Corollary 3.3.

Since $H_2(W)$ is positive definite and both $H_2(Y_g)$ and $H_2(W)$ are torsion-free, the map $H_2(Y_g)\to H_2(W)$ is zero, and we know that $H_2(X_g)$ decomposes as $\mathbb{Z}^{21}\oplus H_2(Y_g)$. Thus we can split off the $H_2(Y_g)\xrightarrow{\sim}H_2(Y_g)$ component of i in the above sequence, leaving us with

$$0\to\mathbb{Z}\oplus\mathbb{Z}^{21}\to H_2(Z)\xrightarrow{\delta}H_1(Y_g).$$

Let $\Lambda\subset H_2(Z)$ be the image of $\mathbb{Z}\oplus\mathbb{Z}^{21}$; then Λ is a sublattice of rank 22, so it has finite index, which must equal $|\mathrm{Im}(\delta)|$. But from $H_1(Y_g)=\mathbb{Z}^{2g}\oplus\mathbb{Z}/(2g-2)$ it follows that $\mathrm{Im}(\delta)$ is a subgroup of $\mathbb{Z}/(2g-2)$, which then has order $\frac{2g-2}{d}$ for some integer $d\geq 1$. Since $H_1(X_g)=H_1(Z)=0$, the portion $H_2(Z)\xrightarrow{\delta}H_1(Y_g)\to H_1(W)\to 0$ of the sequence shows that $H_1(W)$ is isomorphic to $H_1(Y_g)/\mathrm{Im}(\delta)\cong\mathbb{Z}^{2g}\oplus\mathbb{Z}/d$.

Let $e_1,\dots,e_{22}$ be an integral basis of Λ, where e_1 generates the direct summand $H_2(W)\cong\mathbb{Z}$ and $e_2,\dots,e_{22}$ is an integral basis of $\mathbb{Z}^{21}\subset H_2(X_g)$, and form a matrix A whose columns are the elements $e_1,\dots,e_{22}$ expressed in an integral basis of $H_2(Z)\cong\mathbb{Z}^{22}$. Letting Q_Z be the intersection form on $H_2(Z)$ in this latter basis, then, the intersection form on Λ in the basis $\{e_i\}$ is given by $Q_\Lambda=A^TQ_ZA$, and we have $\det(Q_\Lambda)=\pm\left(\frac{2g-2}{d}\right)^2$ since Q_Z is unimodular and $|\det(A)|=[H_2(Z):\Lambda]=\frac{2g-2}{d}$.

On the other hand, we can write Q_Λ in block form with respect to this basis as $\begin{pmatrix} e_1\cdot e_1 & 0\\ 0 & Q_g\end{pmatrix}$, where Q_g is the nondegenerate intersection form on $\mathbb{Z}^{21}\subset H_2(X_g)$; note that $|\det(Q_g)|$ does not depend on W but only on the cap X_g. From this it is clear that $\det(Q_\Lambda)=(e_1\cdot e_1)\det(Q_g)$, so it follows that $(e_1\cdot e_1)|\det(Q_g)|=\left(\frac{2g-2}{d}\right)^2$. In the case where W is the disk cotangent bundle $DT^*\Sigma_g$ we have $e_1\cdot e_1=\Sigma_g^2=2g-2$ and $d=1$ (see Remark 2.5), so it follows that $|\det(Q_g)|$ is equal to $2g-2$. We conclude that

$$e_1\cdot e_1=\frac{\left(\frac{2g-2}{d}\right)^2}{2g-2}=\frac{2g-2}{d^2}.$$

Since Z is a homology K3 it has an even intersection form, so $e_1\cdot e_1$ must be an even integer and we have $d^2\mid g-1$, completing the proof. □

Corollary 3.5. *If $g-1$ is square-free then any exact filling of (Y_g,ξ_g) has the same homology and intersection form as the disk cotangent bundle $DT^*\Sigma_g$.*

4. The topology of Stein fillings

4.1. **The homology of a Stein filling.** In this section we further investigate the topology of a filling (W, ω) of (Y_g, ξ_g) which is not only exact but Stein; in this case we denote it by (W, J) to avoid confusion. In this case W has a handle decomposition consisting of only 0-, 1-, and 2-handles, from which it follows classically that the inclusion $i : Y_g \hookrightarrow W$ induces a surjection $\pi_1(Y_g) \xrightarrow{i_*} \pi_1(W)$. We note that since Y_g is a circle bundle over Σ_g with Euler number $2g-2$, its fundamental group has presentation

$$\pi_1(Y_g) = \left\langle a_1, \dots, a_g, b_1, \dots, b_g, t \;\middle|\; \prod_{i=1}^{g} [a_i, b_i] = t^{2g-2}, [a_i, t] = [b_i, t] = 1 \right\rangle,$$

where t represents a circle fiber and is central. We will define $2g+1$ distinguished elements of $\pi_1(W)$ by

$$\alpha_j = i_*(a_j), \qquad \beta_j = i_*(b_j), \qquad \tau = i_*(t)$$

for $j = 1, \dots, g$. Since i_* is surjective, we know that τ is central and that these $2g+1$ elements generate $\pi_1(W)$; in fact, it turns out that $\alpha_1, \dots, \alpha_g$ and $\beta_1, \dots, \beta_g$ suffice.

Proposition 4.1. *Suppose that (W, J) is a Stein filling of (Y_g, ξ_g) as above, and let $H \subset \pi_1(Y_g)$ denote the subgroup generated by $a_1, \dots, a_g, b_1, \dots, b_g$. If $i_* : \pi_1(Y_g) \to \pi_1(W)$ is the inclusion-induced map, then $i_*|_H$ is surjective; in other words, $i_*(H) = \pi_1(W)$, and so $\pi_1(W)$ is generated by the elements $\alpha_1, \dots, \alpha_g$ and $\beta_1, \dots, \beta_g$.*

Proof. It is not hard to check that H is normal of index $2g-2$, since the only other generator in the above presentation is central (namely t) and the quotient $\pi_1(Y_g)/H$ is $\langle t \mid t^{2g-2} = 1 \rangle$. Since i_* is surjective, it is also easy to see that $i_*(H)$ is a normal subgroup of $\pi_1(W)$. Moreover, i_* induces a map

$$\pi_1(Y_g)/H \to \pi_1(W)/i_*(H)$$

between the respective quotients, and this map is surjective, so since $\pi_1(Y_g)/H$ is a finite cyclic group generated by $[t]$ it follows that $\pi_1(W)/i_*(H)$ is also a finite cyclic group which is generated by $[\tau]$. Thus $i_*(H)$ is a normal subgroup of $\pi_1(W)$ of some finite index $k \geq 1$ which divides $|\pi_1(Y_g)/H| = 2g-2$.

Let $p : \tilde{W} \to W$ be a finite k-fold covering such that $p_*(\pi_1(\tilde{W})) = i_*(H)$. Then $\tilde{W}$ is also a Stein domain, and its boundary $\tilde{Y} = \partial\tilde{W}$ is a k-fold cover of $Y_g = \partial W$, which must be connected since Stein domains have connected boundary. Thus $G = (p|_{\tilde{Y}})_*(\pi_1(\tilde{Y}))$ is an index-k subgroup of $\pi_1(Y_g)$. The cover $\tilde{Y} \to Y_g$ is normal, implying that G is moreover a normal subgroup of $\pi_1(Y_g)$: indeed, the cover $\tilde{W} \to W$ is normal since $i_*(H)$ is a normal subgroup of $\pi_1(W)$, so its deck transformations act transitively on each fiber, and these restrict to deck transformations of $\tilde{Y}$, so the latter act transitively on fibers of $\tilde{Y}$.

We now consider the commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{i}} & \tilde{W} \\ {\scriptstyle p|_{\tilde{Y}}}\downarrow & & \downarrow{\scriptstyle p} \\ Y_g & \xrightarrow{i} & W \end{array}$$

where i and $\tilde{i}$ are the respective inclusion maps of each manifold into the Stein domain which it bounds, and thus induce surjections on the respective fundamental groups. We have $p_*(\pi_1(\tilde{W})) = i_*(H)$ by construction, and since $\tilde{i}_*(\pi_1(\tilde{Y})) = \pi_1(\tilde{W})$ we can write

$$i_*(H) = p_*(\tilde{i}_*(\pi_1(\tilde{Y}))) = i_*((p|_{\tilde{Y}})_*(\pi_1(\tilde{Y}))) = i_*(G).$$

Thus if $t^j \in G$ for some j, then we have $\tau^j \in i_*(G) = i_*(H)$, and so k divides j.

Now we consider the composition $\varphi : \tilde{Y} \xrightarrow{p} Y_g \to \Sigma_g$, where we are now using p to denote the restriction $p|_{\tilde{Y}}$. The preimage of a point $x \in \Sigma_g$ is a k-fold cover of the circle fiber above x in Y_g, which we identify with t (at least up to conjugation, since we should pick a base point). If this preimage is disconnected, then one of its components is a circle $\gamma \subset \tilde{Y}$ which is an l-fold cover of the circle fiber in Y_g for some $1 \leq l < k$. Thus $p_*(\gamma)$ is conjugate to t^l; but G is normal and does not contain t^l, so it cannot actually contain $p_*(\gamma)$ either. We conclude that $\varphi^{-1}(x)$ is a circle, and hence that $\tilde{Y}$ is also a circle bundle over Σ_g. Its Euler number is then $\frac{2g-2}{k}$, though we only need that it is nonzero: if it were zero, then the image under p of a section would give a section of Y_g, which has nonzero Euler number.

From the above we see that $b_1(\tilde{Y}) = 2g$, and since $H_1(\tilde{Y})$ surjects onto $H_1(\tilde{W})$ it follows that $b_1(\tilde{W}) \leq 2g$, hence

$$\chi(\tilde{W}) = 1 - b_1(\tilde{W}) + b_2(\tilde{W}) \geq 1 - 2g.$$

But we also know that $\chi(\tilde{W}) = k\chi(W) = k(2-2g)$, so we have $k(2-2g) \geq 1-2g$, or equivalently $(k-1)(2-2g) \geq -1$. Since $2-2g \leq -2$, this can only hold if $k = 1$; but k is the index of $i_*(H)$ in $\pi_1(W)$, so the two must be equal. □

Theorem 4.2. *Let (W, J) be a Stein filling of (Y_g, ξ_g). Then W has the same integral homology and intersection form as the disk cotangent bundle $DT^*\Sigma_g$. In particular, we have $H_1(W) \cong \mathbb{Z}^{2g}$, and the intersection form on $H_2(W) \cong \mathbb{Z}$ is $\langle 2g-2 \rangle$.*

Proof. In light of Theorem 3.4 we know that $H_1(W) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}/d\mathbb{Z}$ for some $d \geq 1$, and that it suffices to show that $d = 1$. Now according to Proposition 4.1, the fundamental group $\pi_1(W)$ is generated by the $2g$ elements $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$, hence its abelianization $H_1(W)$ is also generated by the corresponding homology classes. However, if $d > 1$ then any presentation of $\mathbb{Z}^{2g} \oplus \mathbb{Z}/d\mathbb{Z}$ requires at least $2g+1$ generators, so we must have $d = 1$. □

Theorem 4.2 tells us the first group homology of $\pi_1(W)$, since $H_1(\pi_1(W); \mathbb{Z}) = H_1(W; \mathbb{Z})$. The second homology of $\pi_1(W)$ will also be useful later:

Proposition 4.3. *If (W, J) is a Stein filling of (Y_g, ξ_g), then $H_2(\pi_1(W); \mathbb{Z}) \cong \mathbb{Z}$.*

Proof. Let $\pi = \pi_1(W)$, and recall that $H_2(W) \cong \mathbb{Z}$. The group homology $H_2(\pi; \mathbb{Z}) = H_2(K(\pi, 1); \mathbb{Z})$ is classically known to be isomorphic to the cokernel of the Hurewicz map

$$h : \pi_2(W) \to H_2(W),$$

which is $\mathbb{Z}/\operatorname{Im}(h)$, so $H_2(\pi; \mathbb{Z})$ is $\mathbb{Z}$ if the Hurewicz map is zero and finite otherwise. The $2g$ elements $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ generate π by Proposition 4.1, so their images generate $H_1(W; \mathbb{Z}) = \mathbb{Z}^{2g}$ and are thus linearly independent over $\mathbb{Q}$.

Supposing that h is nonzero, we have $H_2(\pi; \mathbb{Q}) = 0$. According to Stallings [Sta65, Theorem 7.4], the linear independence of the α_i and β_i in $H_1(\pi)$ and the vanishing of $H_2(\pi; \mathbb{Q})$ guarantee that $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ form a basis of a free subgroup of π, and we conclude that π is the free group F_{2g}. The element $\prod_{j=1}^g [\alpha_j, \beta_j]$ of π is central, since it equals τ^{2g-2} and τ is central; but free groups have trivial center, so $\prod_{j=1}^g [\alpha_j, \beta_j] = 1$ and thus π is a nontrivial quotient of F_{2g}. Since finitely generated free groups are Hopfian, a nontrivial quotient of F_{2g} cannot be isomorphic to F_{2g}, and we have a contradiction. □

Since the class $[t]$ of the circle fiber generates the torsion summand of $H_1(Y_g)$, and Theorem 4.2 says that $H_1(W)$ is torsion-free, we see that $[\tau] = 0$ in $H_1(W)$. Thus τ lies in the commutator subgroup of $\pi_1(W)$. In Section 4.2 we will see that in fact $\tau = 1$ in $\pi_1(W)$.

4.2. **The fundamental group of a Stein filling.** Let (W, J) denote a Stein filling of (Y_g, ξ_g) as usual. Our goal in this section is to explicitly determine its fundamental group:

Theorem 4.4. *The fundamental group $\pi_1(W)$ is isomorphic to $\pi_1(\Sigma_g)$.*

Our strategy will be to first show that $\pi_1(W)$ must be an extension of $\pi_1(\Sigma_g)$ by a cyclic group and then use what we know about its group homology to show that this cyclic group is in fact trivial.

Summarizing what we know so far about $\pi_1(W)$, we have seen that it is a quotient of

$$\left\langle \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \tau \;\middle|\; \prod_{i=1}^{g} [\alpha_i, \beta_i] = \tau^{2g-2}, [\alpha_i, \tau] = [\beta_i, \tau] = 1 \right\rangle,$$

where the central element τ is the image of a circle fiber $t \in \pi_1(Y_g)$. Moreover, $H_1(W; \mathbb{Z}) = \mathbb{Z}^{2g}$ is generated by the elements α_i and β_i, and the central element τ belongs to the commutator subgroup of $\pi_1(W)$. Thus there is a surjection

$$p : \pi_1(\Sigma_g) = \left\langle \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g \;\middle|\; \prod_{i=1}^{g} [\alpha_i, \beta_i] = 1 \right\rangle \to \pi_1(W)/\langle \tau \rangle$$

through which the abelianization map $\mathrm{ab} : \pi_1(\Sigma_g) \to \mathbb{Z}^{2g}$ factors as

$$\pi_1(\Sigma_g) \xrightarrow{p} \pi_1(W)/\langle \tau \rangle \xrightarrow{\mathrm{ab}} \mathbb{Z}^{2g}.$$

Such a factorization exists for any surjection $p : \pi_1(\Sigma_g) \to \pi_1(W)/\langle \tau \rangle$: since $\pi_1(\Sigma_g) \xrightarrow{\mathrm{ab} \circ p} \mathbb{Z}^{2g}$ is a map to an abelian group, it factors as $\pi_1(\Sigma_g) \xrightarrow{\mathrm{ab}} \mathbb{Z}^{2g} \xrightarrow{\psi} \mathbb{Z}^{2g}$, and ψ is a surjection $\mathbb{Z}^{2g} \to \mathbb{Z}^{2g}$ since $\mathrm{ab} \circ p$ is onto, so it is an isomorphism and then $\mathrm{ab} : \pi_1(\Sigma_g) \to \mathbb{Z}^{2g}$ is equal to $(\psi^{-1} \circ \mathrm{ab}) \circ p$.

If we fix a surjection $\varphi : \mathbb{Z}^{2g} \to \mathbb{Z}/n\mathbb{Z}$ for some $n > 1$, then we have a collection of surjective maps of the form

$$\begin{array}{lllll} \pi_1(W) & & & & \\ & \searrow & & & \\ & & \pi_1(W)/\langle \tau \rangle \xrightarrow{\mathrm{ab}} \mathbb{Z}^{2g} \xrightarrow{\varphi} \mathbb{Z}/n\mathbb{Z} & & \\ & \nearrow_{p} & & & \\ \pi_1(\Sigma_g) & & & & \end{array}$$

and the kernels of the maps $\pi_1(W) \to \mathbb{Z}/n\mathbb{Z}$ and $\pi_1(\Sigma_g) \to \mathbb{Z}/n\mathbb{Z}$ define normal, n-fold cyclic covers W' and $\Sigma_{g'}$ of W and Σ_g respectively, where $2 - 2g' = n(2 - 2g)$.

Definition 4.5. *Let (W, J) be a Stein filling of (Y_g, ξ_g), and let $p : \pi_1(\Sigma_g) \to \pi_1(W)/\langle \tau \rangle$ be a surjection. If $\Sigma_{g'} \to \Sigma_g$ and $W' \to W$ are the finite cyclic covers produced by the above construction, then we will say that $(\Sigma_{g'}, W')$ is* induced by *(p, φ).*

Since W' is a finite cover of a Stein manifold it has a natural Stein structure J', so its boundary (Y', ξ') is connected, and as in the proof of Proposition 4.1 it follows that Y' is a normal, n-fold cyclic cover of Y_g.

Lemma 4.6. *If* $(\Sigma_{g'}, W')$ *is induced by* (p, φ)*, then* (W', J') *is a Stein filling of the canonical contact structure* $(Y', \xi') = (Y_{g'}, \xi_{g'})$ *on the unit cotangent bundle of* $\Sigma_{g'}$*.*

Proof. The circle fiber $t \in \pi_1(Y_g)$ is in the kernel of $\pi_1(Y_g) \xrightarrow{i_*} \pi_1(W) \to \mathbb{Z}/n\mathbb{Z}$ since it maps to $\tau \in \pi_1(W)$, so it lifts to a closed curve in Y'. Its preimage in Y' therefore consists of n disjoint circles, so the orbit space Y'/S^1 is an n-fold cover of Σ_g, hence Y' is a circle bundle over $\Sigma_{g'}$. The Euler class of $Y' \to \Sigma_{g'}$ is then n times the Euler class of $Y_g \to \Sigma_g$, namely $-n\chi(\Sigma_g) = -\chi(\Sigma_{g'})$, so in fact Y' is the unit cotangent bundle $Y_{g'}$ of $\Sigma_{g'}$.

Since the contact structure ξ_g is tangent to the fibers of $Y_g \to \Sigma_g$, its cover ξ' is likewise tangent to the fibers of $Y_{g'} \to \Sigma_{g'}$, and the only contact structure on the unit cotangent bundle of $\Sigma_{g'}$ with Legendrian fibers is the canonical one [Gir01, Proposition 3.3] (cf. also [Lut83]). Thus $(Y', \xi') = (Y_{g'}, \xi_{g'})$, and so (W', J') is a Stein filling of $(Y_{g'}, \xi_{g'})$. □

Proposition 4.7. *Suppose that* $(\Sigma_{g'}, W')$ *is induced by* (p, φ)*. Identifying* $\pi_1(\Sigma_{g'})$ *as a subgroup of* $\pi_1(\Sigma_g)$*, the map* p *induces a surjection*

$$p' : \pi_1(\Sigma_{g'}) \to \pi_1(W')/\langle \tau' \rangle$$

such that $\ker(p') = \ker(p)$*.*

Proof. It is clear that $\langle \tau \rangle \subset \pi_1(W')$, viewing the latter as a subgroup of $\pi_1(W)$, since τ is in the kernel of $\pi_1(W) \to \mathbb{Z}/n\mathbb{Z}$. Moreover, if $\tau' \in \pi_1(W')$ denotes the image of the circle fiber $t' \in \pi_1(Y_{g'})$, then since t' projects to the circle fiber $t \in \pi_1(Y_g)$, the covering map $W' \to W$ sends τ' to τ, so we have

$$\langle \tau \rangle \cap \pi_1(W') = \langle \tau' \rangle.$$

Thus the kernel of $\pi_1(W') \hookrightarrow \pi_1(W) \to \pi_1(W)/\langle \tau \rangle$ is $\langle \tau' \rangle$, inducing an injective map

$$\frac{\pi_1(W')}{\langle \tau' \rangle} \hookrightarrow \frac{\pi_1(W)}{\langle \tau \rangle},$$

and it follows that $\pi_1(W')/\langle \tau' \rangle$ has index n in $\pi_1(W)/\langle \tau \rangle$. Since $\pi_1(W')$ is by definition the kernel of the map $\pi_1(W) \to \mathbb{Z}/n\mathbb{Z}$, the group $\pi_1(W')/\langle \tau' \rangle$ sits in the kernel of the surjective $\pi_1(W)/\langle \tau \rangle \xrightarrow{\varphi \circ \mathrm{ab}} \mathbb{Z}/n\mathbb{Z}$, and this kernel has index n so we conclude that

$$\pi_1(W')/\langle \tau' \rangle = \ker(\varphi \circ \mathrm{ab} : \pi_1(W)/\langle \tau \rangle \to \mathbb{Z}/n\mathbb{Z}).$$

Since $\pi_1(\Sigma_{g'})$ is the kernel of $(\varphi \circ \mathrm{ab}) \circ p$, it follows that $p(\pi_1(\Sigma_{g'}))$ lies in $\ker(\varphi \circ \mathrm{ab})$, and so p restricts to a map

$$p' : \pi_1(\Sigma_{g'}) \to \pi_1(W')/\langle \tau' \rangle,$$

which is easily seen to be surjective just as p is. Finally, since p' is the restriction of p to $\pi_1(\Sigma_{g'}) \subset \pi_1(\Sigma_g)$ it follows that $\ker(p') = \ker(p) \cap \pi_1(\Sigma_{g'})$. But $\ker(p) \subset \ker(\varphi \circ \mathrm{ab} \circ p) = \pi_1(\Sigma_{g'})$, and so $\ker(p) = \ker(p')$ as claimed. □

Proposition 4.7 allows us to characterize $\pi_1(W)$ as a cyclic extension of a surface group:

Proposition 4.8. *The fundamental group* $\pi_1(W)$ *is a central extension of* $\pi_1(\Sigma_g)$ *by a cyclic group. More precisely, there is a short exact sequence*

$$1 \to \langle \tau \rangle \to \pi_1(W) \to \pi_1(\Sigma_g) \to 1$$

with the image of $\langle \tau \rangle$ *being central in* $\pi_1(W)$*.*

Proof. It suffices to show that the surjection $p : \pi_1(\Sigma_g) \to \pi_1(W)/\langle\tau\rangle$ is also injective. Supposing otherwise, let x be a nontrivial element of $\ker(p)$. Since surface groups are RFRS [Ago08] (cf. also [Hem72]), there is a descending chain of subgroups

$$\pi_1(\Sigma_g) = G_0 \supset G_1 \supset G_2 \supset \dots$$

such that each G_{i+1} is a normal subgroup of G_i with finite cyclic quotient, defined as the kernel of a map which factors through $G_i \to (G_i)^{\mathrm{ab}}$, and $\bigcap_{i=0}^{\infty} G_i = \{1\}$. This corresponds to a tower of normal, finite cyclic covers

$$\dots \to \Sigma_{g_2} \to \Sigma_{g_1} \to \Sigma_{g_0} = \Sigma_g$$

such that $\pi_1(\Sigma_{g_{i+1}}) = \ker\left(\pi_1(\Sigma_{g_i}) \xrightarrow{\mathrm{ab}} \mathbb{Z}^{2g_i} \xrightarrow{\varphi_i} \mathbb{Z}/n_i\mathbb{Z}\right)$ for some φ_i. Now by induction, since $\mathrm{ab} : \pi_1(\Sigma_g) \to \mathbb{Z}^{2g}$ factors through $p_0 = p : \pi_1(\Sigma_g) \to \pi_1(W_0)/\langle\tau_0\rangle$, with $W_0 = W$, we can construct for each $i \geq 0$ a normal cyclic cover (W_{i+1}, J_{i+1}) of (W_i, J_i) as above, with $(\Sigma_{g_{i+1}}, W_{i+1})$ induced by (p_i, φ_i). By Lemma 4.6, (W_{i+1}, J_{i+1}) is a Stein filling of $(Y_{g_{i+1}}, \xi_{g_{i+1}})$, and Proposition 4.7 provides a surjection

$$p_{i+1} : \pi_1(\Sigma_{g_{i+1}}) \to \pi_1(W_{i+1})/\langle\tau_{i+1}\rangle$$

with $\ker(p_{i+1}) = \ker(p_i)$. Thus $x \in \ker(p_0)$ implies that $x \in \ker(p_i) \subset G_i$ for all i. But since $\bigcap G_i = \{1\}$ it follows that $x \notin G_k$ for some $k \geq 0$, and this is a contradiction. □

Proof of Theorem 4.4. The circle fiber τ generates a $\mathbb{Z}/n\mathbb{Z}$ subgroup for some $n \geq 0$ (with $n = 0$ if it is nontorsion), so Proposition 4.8 provides a short exact sequence of groups

$$1 \to \mathbb{Z}/n\mathbb{Z} \to \pi_1(W) \to \pi_1(\Sigma_g) \to 1$$

for some $n \geq 0$. We will show that $n = 1$, and thus that $\pi_1(W) \to \pi_1(\Sigma_g)$ is an isomorphism.

The homologies of these groups with $\mathbb{Z}$ coefficients are related by the Lyndon/Hochschild-Serre spectral sequence (see e.g. [Bro82]),

$$E^2_{p,q} = H_p(\pi_1(\Sigma_g); H_q(\mathbb{Z}/n\mathbb{Z};\mathbb{Z})) \quad \Longrightarrow \quad H_{p+q}(\pi_1(W);\mathbb{Z}).$$

Here $H_q(\mathbb{Z}/n\mathbb{Z};\mathbb{Z})$ has a trivial $\pi_1(\Sigma_g)$-action since our extension is central. Since $\pi_1(\Sigma_g)$ has cohomological dimension 2, the E^2 page is supported in the interval $0 \leq p \leq 2$. Moreover, the homology of $\mathbb{Z}/n\mathbb{Z}$ is given by (letting $k \geq 1$):

$$H_q(\mathbb{Z};\mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0,1 \\ 0, & q \geq 2, \end{cases} \qquad H_q(\mathbb{Z}/k\mathbb{Z};\mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0 \\ \mathbb{Z}/k\mathbb{Z}, & q \text{ odd} \\ 0, & q \geq 2 \text{ even.} \end{cases}$$

In either case, the differential $d^2 : E^2_{p,q} \to E^2_{p-2,q+1}$ must be identically zero with the possible exception of the map $\delta : E^2_{2,0} \to E^2_{0,1}$, since otherwise either the source or the target vanishes. Each of the higher differentials $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ must vanish for $r \geq 3$ because either $p > 2$ or $p - r < 0$, so the spectral sequence collapses at the E^3 page and we have

$$E^\infty_{0,2} = 0, \qquad E^\infty_{1,1} = (\mathbb{Z}/n\mathbb{Z})^{2g}, \qquad E^\infty_{2,0} = \ker(\delta : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}).$$

The convergence of this spectral sequence means that these are the associated graded groups of a filtration on $H_2(\pi_1(W);\mathbb{Z})$. But the latter group is $\mathbb{Z}$ by Proposition 4.3, so each associated graded group must be cyclic, and since $E^\infty_{1,1}$ is cyclic we must have $n = 1$. □

4.3. **The homotopy type of a Stein filling.** So far we have shown that if (W, J) is a Stein filling of (Y_g, ξ_g), then W has the homology and intersection form of the disk cotangent bundle $DT^*\Sigma_g$ (Theorem 4.2) and that $\pi_1(W) \cong \pi_1(\Sigma_g)$ (Theorem 4.4), with the circle fiber of $Y_g = \partial W$ being nullhomotopic in W. In this section we will deduce that W is therefore homotopy equivalent, and thus s-cobordant, to $DT^*\Sigma_g$ rel boundary.

Proposition 4.9. *If (W, J) is a Stein filling of (Y_g, ξ_g), then W is aspherical.*

Proof. A decomposition of W into handles of index at most 2, with exactly one 0-handle, necessarily has $2g-1+k$ 1-handles and k 2-handles for some $k \geq 1$ since $\chi(W) = 2-2g$. The corresponding presentation of $\pi_1(W) \cong \pi_1(\Sigma_g)$ has $2g-1+k$ generators and k relations and thus deficiency $2g-1$.

Hillman [Hil97, Proof of Theorem 2] showed that if a presentation P of a group G has deficiency $1+\beta_1(G)$, where β_1 denotes the first L^2-Betti number (see for example [Lüc02]), then the 2-complex corresponding to P is aspherical. In the above case we know that $\beta_1(\pi_1(\Sigma_g)) = 2g-2$, so the 2-complex corresponding to the given presentation of $\pi_1(\Sigma_g)$ is aspherical, and thus W (which retracts onto this complex) is aspherical as well. □

We have now shown that W is a $K(\pi_1(\Sigma_g), 1)$, and so it has the homotopy type of $DT^*\Sigma_g$. Since both are compact 4-manifolds with boundary, we can strengthen this to an assertion about manifolds rel boundary as follows.

Theorem 4.10. *If (W, J) is a Stein filling of (Y_g, ξ_g), then W is s-cobordant rel boundary to the disk cotangent bundle $DT^*\Sigma_g$.*

Proof. It suffices to find a homotopy equivalence $f : DT^*\Sigma_g \to W$ which restricts to a homeomorphism $\partial(DT^*\Sigma_g) \xrightarrow{\sim} \partial W$. Since W is compact and aspherical with $\pi_1(W)$ a surface group, Khan [Kha12, Corollary 1.23] showed that W is *topologically s-rigid*, a condition which implies that if such an f exists then $DT^*\Sigma_g$ is s-cobordant to W.

To construct $f : DT^*\Sigma_g \to W$, following Stipsicz [Sti02], we first take a standard handlebody decomposition of $DT^*\Sigma_g$, with a 0-handle, $2g$ 1-handles, and a single 2-handle, and turn it upside down to build $DT^*\Sigma_g$ from a thickened Y_g by attaching a 2-handle, $2g$ 3-handles, and a 4-handle. We first define f on $Y_g = \partial(DT^*\Sigma_g)$ by picking an identification $\partial(DT^*\Sigma_g) \xrightarrow{\sim} \partial W \hookrightarrow W$ which sends a circle fiber to a circle fiber. We can then extend f over the 2-handle of $DT^*\Sigma_g$: the attaching curve of this handle is identified with the circle fiber in $Y_g = \partial W$ and is thus nullhomotopic in W, so the map of the attaching curve S^1 into W extends to a map $D^2 \to W$ and we identify the 2-handle with this D^2. Similarly,we can extend f over the 3-handles of $DT^*\Sigma_g$ and then over the 4-handle, since in each case f is initially defined along the attaching sphere of a handle and the obstructions to extending it to a map from each handle into W lie in $\pi_2(W) = 0$ and $\pi_3(W) = 0$ respectively.

The map f which we have constructed now induces an isomorphism $f_* : \pi_1(DT^*\Sigma_g) \to \pi_1(W)$, since it induces an isomorphism $\pi_1(\partial(DT^*\Sigma_g)) \xrightarrow{\sim} \pi_1(\partial W)$ which preserves the subgroup generated by the circle fiber, and both groups are quotients of $\pi_1(Y_g)$ by that subgroup. Moreover, f induces an isomorphism on all higher homotopy groups, since these are identically zero, and so f is a homotopy equivalence by Whitehead's theorem. □

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