Coalescing random walkers

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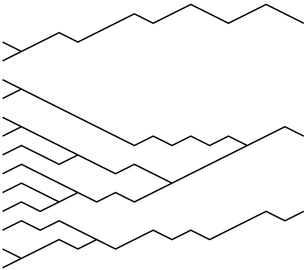
Outline

Introduction

Avalanche size in the totally asymmetric Oslo model maps to the area inscribed by annihilating random walkers.



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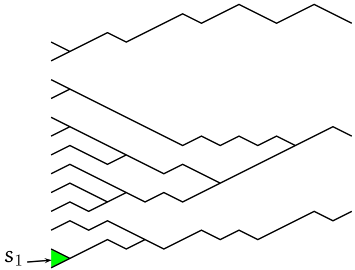

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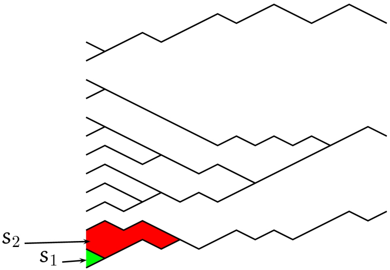

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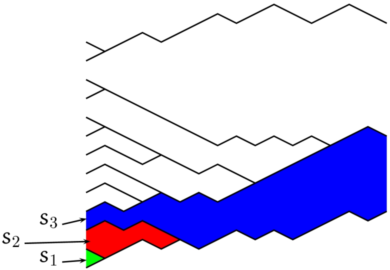

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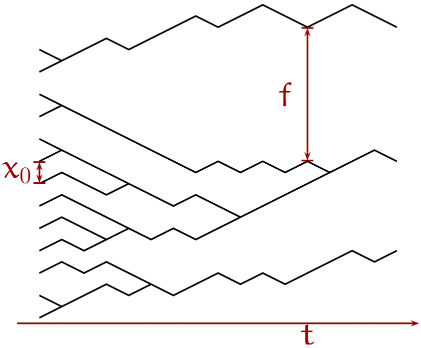
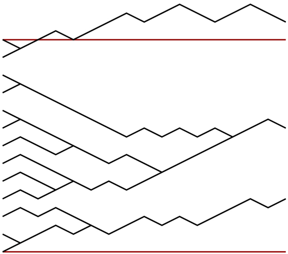
Scaling arguments



Fluctuations f, time t, diffusion constant D (Brownian motion!), initial gap x_0.
Characteristic area $s_c \propto ft \propto (Dt)^{1/2}t \propto t^{3/2}$

Scaling arguments



First moment: N walkers cover area $Nx_0 t$ plus error independent of N, when large enough.
Surprise: Error vanishes like $1/N$, rather than $1/\sqrt{N}$ for independent areas.
This latter term is cancelled by short-lived anticorrelations, as seen in the variance of the estimated average area:

$$\frac{1}{N^2}\sum_{ij}^{N}\left(\langle s_i s_j\rangle - \langle s_i\rangle\langle s_j\rangle\right) = \frac{\sigma^2(s)}{N} + \sum_{\substack{i,j=1\\ i\neq j}}^{N}\left(\langle s_i s_j\rangle - \langle s_i\rangle\langle s_j\rangle\right) = \frac{2}{3N^2}Dt^3 + \dots$$

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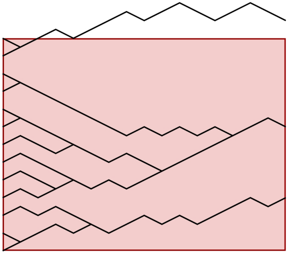

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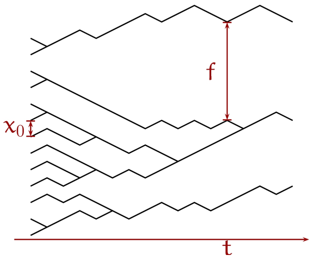
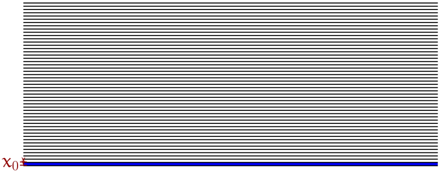
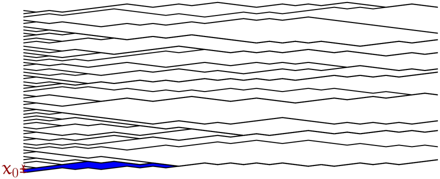


- Assume simple scaling of cluster size distribution $P(s) = s^{-\tau}\mathcal{G}(s/t^{d_f})$.
- Moments thus go like $\langle s^n \rangle \propto t^{d_f(1-\tau+n)}$.
- Characteristic size scales like $s_c \propto \sqrt{D}\, t^{3/2}$, *i.e.* $d_f = 3/2$.
- First moment goes like $\langle s \rangle = x_0 t$, so that $\tau = 4/3$.
- Higher moments thus go like $\langle s^n \rangle = A_n x_0 D^{(1-\tau+n)/2} t^{(3n-1)/2}$.
- All moments (precisely: their leading orders in t) **linear** in x_0?!?

Moment scaling with initial gap





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As long as characteristic size does not involve x_0, all moments are linear in x_0.

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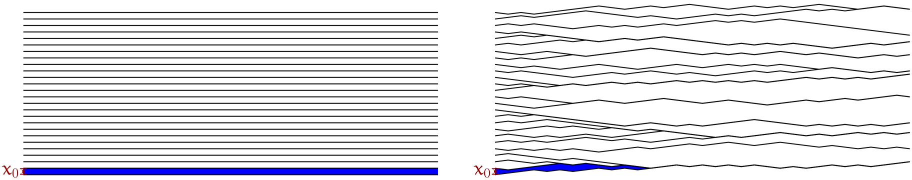

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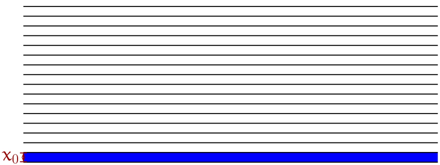

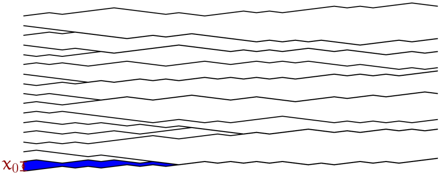

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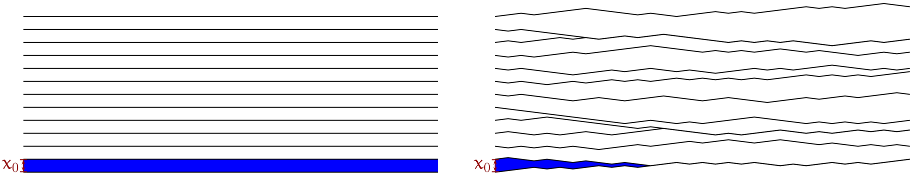

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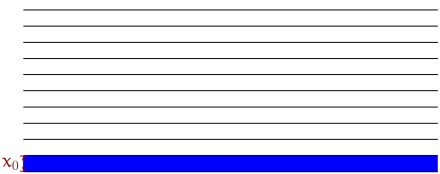

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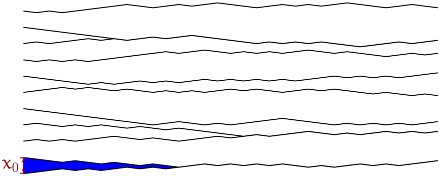

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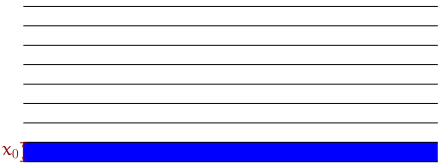

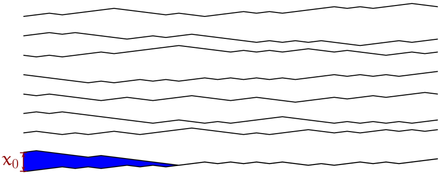

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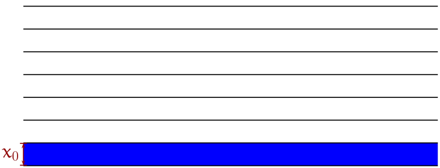

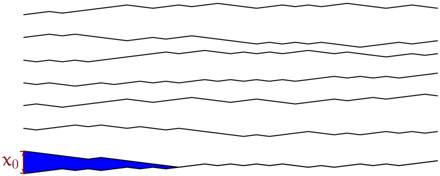

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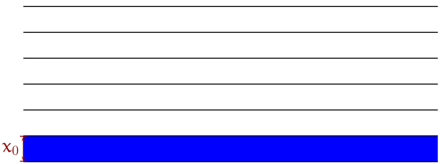

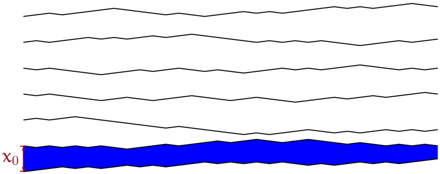

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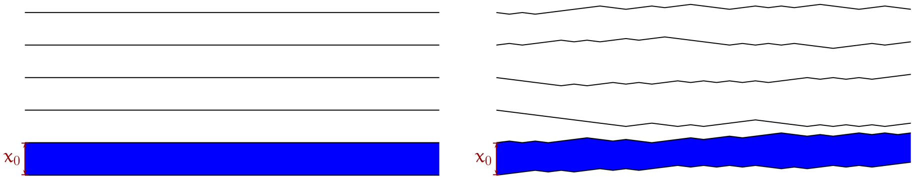

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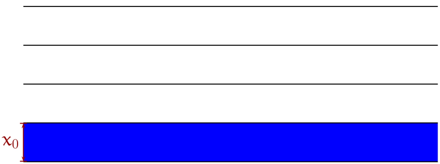

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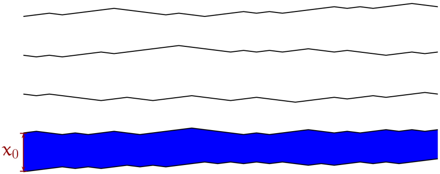

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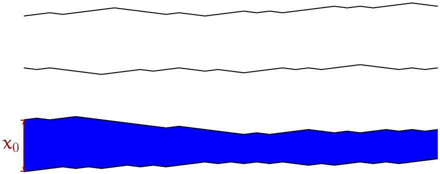

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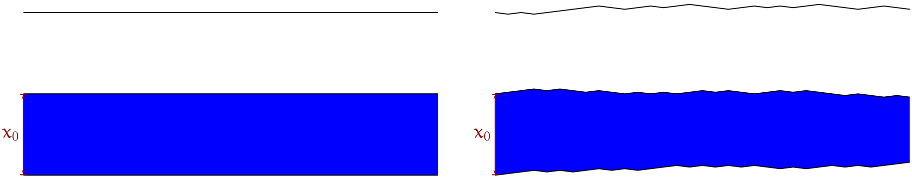

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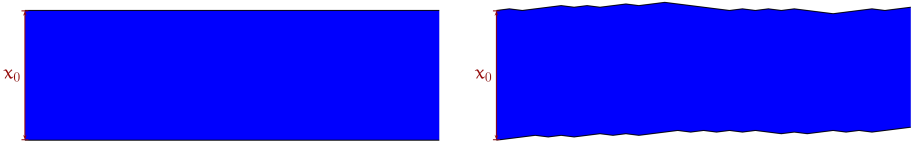

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Perturbation theory

Calculation of moments in closed form is difficult.
Perturbation theory in powers of t actually not possible (integrals violate assumptions about small and large parameters). Overlooking that and ignoring divergent coefficients, gives

$$\left\langle s^2 \right\rangle = c_0 t^{5/2} + c_1 t^{3/2} + c_2 t^{1/2} + c_3 t^{-1/2} + \dots \ .$$

Series expansion of the exact result:

$$\left\langle s^2 \right\rangle = c_0 t^{5/2} + c_1 t^{3/2} \mathbf{- d_0 t} + c_2 t^{1/2} \mathbf{- d_1} + c_3 t^{-1/2} + \dots \ .$$

Exact results

Exact second moment is messy:

$$\left\langle s^2 \right\rangle = \frac{1}{180}\left(\frac{x_0^3}{D}\right)^2 \left[\left(1 + 28\frac{tD}{x_0^2} + 132\left(\frac{tD}{x_0^2}\right)^2\right)\sqrt{\frac{4tD}{\pi x_0^2}}\exp\left(-\frac{x_0^2}{4tD}\right) - \left(1 + 30\frac{tD}{x_0^2}\right)\operatorname{erfc}\left(\sqrt{\frac{x_0^2}{4tD}}\right) + 60\left(\frac{tD}{x_0^2}\right)^2\left(3 + 2\frac{tD}{x_0^2}\right)\operatorname{erf}\left(\sqrt{\frac{x_0^2}{4tD}}\right)\right]$$

Correlation functions

How to calculate the correlation function $\langle s_i s_{i+1} \rangle$?

(Idea by Alan Bray)

- Observation:
  $\left\langle (s_i + s_{i+1})^2 \right\rangle = \left\langle s_i^2 \right\rangle + \left\langle s_{i+1}^2 \right\rangle + 2 \left\langle s_i s_{i+1} \right\rangle$.
- $\left\langle s_i^2 \right\rangle = \left\langle s_{i+1}^2 \right\rangle$ known from above.
- $\left\langle (s_i + s_{i+1})^2 \right\rangle$ is the second moment of the area for the initial gap *doubled*, $x_0 \to 2x_0$.
- Thus:

$$\langle s_i s_{i+1} \rangle = \frac{1}{2} \left\langle s^2 \right\rangle (2x_0) - \left\langle s^2 \right\rangle (x_0) \ .$$

- Similar for higher correlation functions $\left\langle s_i s_{i+j} \right\rangle$.
  Note: Leading order terms, linear in x_0, cancel.

Summary

Area between coalescing random walkers.

- Scaling arguments suggest the right behaviour.
- Anti-correlations visible when estimating first moment.
- All moments linear in initial gap (to leading order in t).
- Perturbation theory fails.
- First and second moment known exactly.
- Correlation function can be calculated by merging areas.

Published in:
Peter Welinder, Gunnar Pruessner and Kim Christensen *Multiscaling in the sequence of areas enclosed by coalescing random walkers*, New J. Phys. **9**, 149-1–18 (2007).

Thank you!