Self Organised Criticality

in the third decade after BTW

Gunnar Pruessner

Department of Mathematics
Imperial College London

London, 14 Feb 2012

Outline

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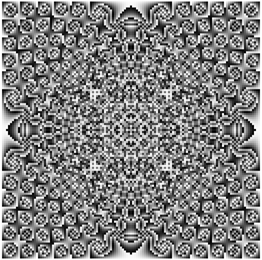
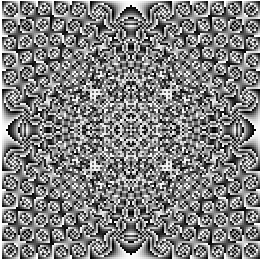
Prelude: The physics of fractals



Question: Where does scale invariant behaviour in nature come from?

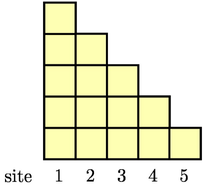
Answer: Due to a phase transition, self-organised to the critical point.

Prelude: The physics of fractals



- Anderson, 1972: *More is different*
  Correlation, cooperation, emergence
- $1/f$ noise "everywhere" (van der Ziel, 1950; Dutta and Horn, 1981)
- Kadanoff, 1986: *Fractals: Where's the Physics?*
- Bak, Tang and Wiesenfeld, 1987: *Self-Organized Criticality: An Explanation of $1/f$ Noise*

The BTW Model



The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Intended as an explanation of $1/f$ noise.
- Generates(?) scale invariant event statistics.
- **The physics of fractals.**

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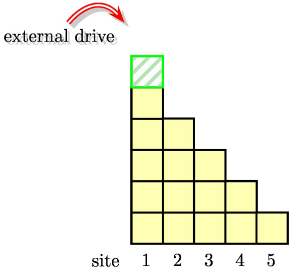

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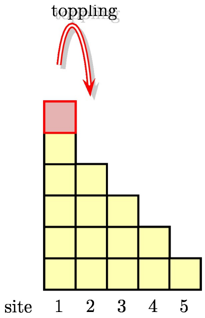

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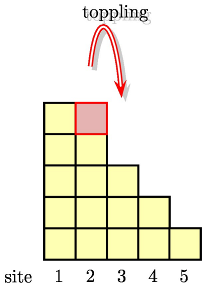

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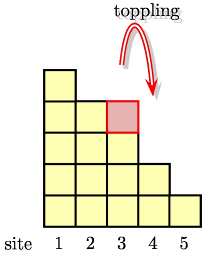

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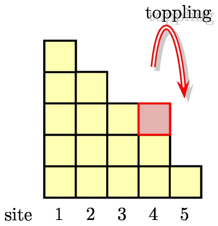

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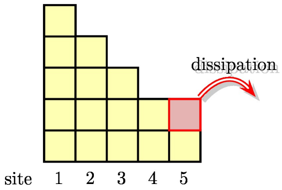

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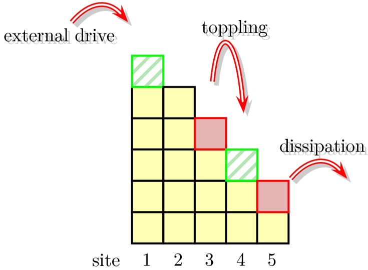

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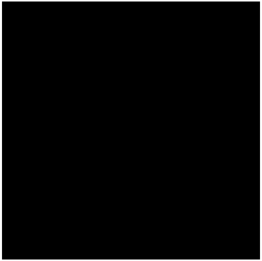

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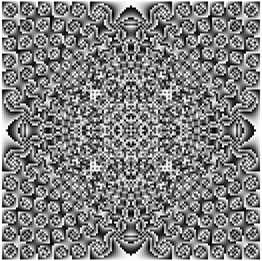

Key ingredients for SOC models:

- Separation of time scales.
- Interaction.
- Thresholds (non-linearity).
- Observables: Avalanche sizes and durations.

$1/f$ noise — a red herring? I

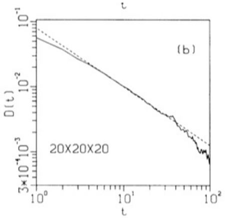

FIG. 3. Distribution of lifetimes corresponding to Fig. 2. (a) For the 50×50 array, the slope $\alpha \approx 0.42$, yielding a "$1/f$" noise spectrum $f^{-1.58}$; (b) 20×20×20 array, $\alpha \approx 0.90$, yielding an $f^{-1.1}$ spectrum

From: Bak, Tang, Wiesenfeld, 1987

- Power spectrum $P(f) \propto 1/f$, thus correlation function (via Wiener Khinchin)

$1/f$ noise — a red herring? II

- Dimensional analysis:

$$\int \mathrm{d}f\, 1/f^{\alpha} e^{-2\pi\imath ft} = \ldots \propto t^{\alpha-1} = \text{const}$$

- $1/f$ noise suggests long time correlations
- Initially, SOC was intended an explanation of $1/f$ noise.
- Initially the BTW model was thought to display $1/f$ noise.
- Jensen, Christensen and Fogedby: "Not quite."
- Today: Little interest in $1/f$.
- Today: Power laws in other observables.
- Today: Scaling questioned.

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Why is SOC important?

SOC today: Non-trivial scale invariance in avalanching (intermittent) systems as known from ordinary critical phenomena, but without the need of external tuning of a control parameter to a non-trivial value.

Emergence!

- Explanation of emergent,
- …cooperative,
- …long time and length scale
- …phenomena,
- …as signalled by power laws.

Why is SOC important?

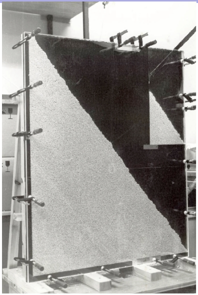
SOC today: Non-trivial scale invariance in avalanching (intermittent) systems as known from ordinary critical phenomena, but without the need of external tuning of a control parameter to a non-trivial value.

Universality!

- Understanding and classifying natural phenomena
- ...using *Micky Mouse Models*
- ...on a small scale (in the lab or on the computer).
- (Triggering critical points?)
- But: Where is the evidence for scale invariance in nature (dirty power laws)?

Experiments:

Granular media, superconductors, rain. . .



Photograph courtesy of V. Frette, K. Christensen, A. Malthe-Sørenssen, J. Feder, T. Jøssang and P. Meakin.

- Large number of experiments and observations:
- Earthquakes suggested by Bak, Tang and Wiesenfeld.
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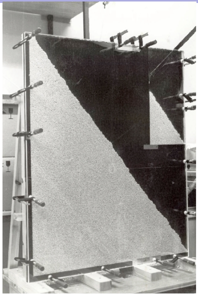

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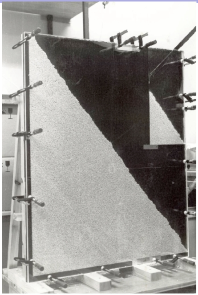

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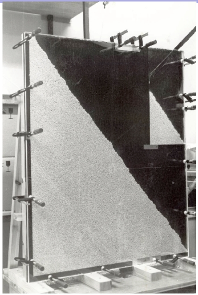

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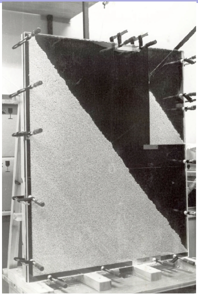

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Outline

1. SOC: The early programme
2. More models
   - Better Models: The Manna model
   - Collapse with Oslo
   - Exponents in 1,2,3D
3. Tools in SOC
4. Field theory for SOC
5. Any Answers?

More models

- Initial intention for more models: Expand BTW universality class.
- Later: Provide more evidence for SOC as a whole.
- More models…

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More models

The failure of SOC?

- Zhang Model (1989) [scaling questioned]
- Dhar-Ramaswamy Model (1989) [solved, directed]
- Forest Fire Model (1990, 1992) [no proper scaling]
- Manna Model (1991) [solid!]
- Olami-Feder-Christensen Model (1992) [scaling questioned, $\alpha \approx 0.05$ (localisation), $\alpha = 0.22$ (jump)]
- Bak-Sneppen Model (1993) [scaling questioned]
- Zaitsev Model (1992)
- Sneppen Model (1992)
- Oslo Model (1996) [solid!]
- Directed Models: Exactly solvable (lack of correlations)

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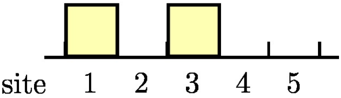
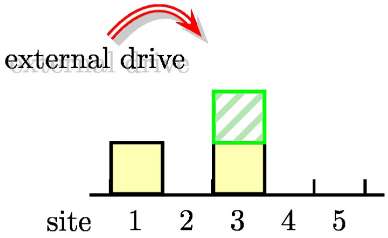
Manna Model



Manna Model (1991)

- Critical height model.
- Stochastic.
- Bulk drive.
- Envisaged to be in the same universality class as BTW.

Manna Model



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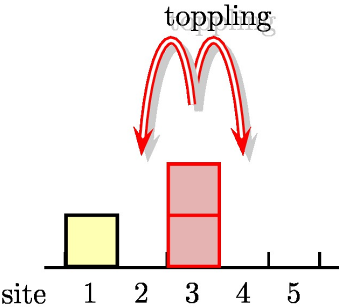

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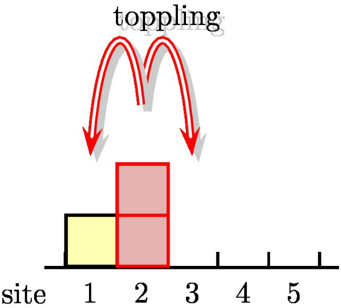

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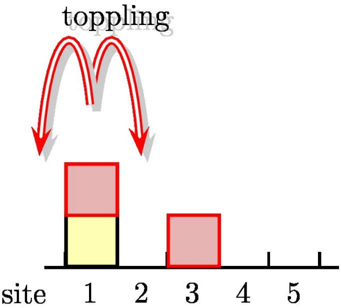

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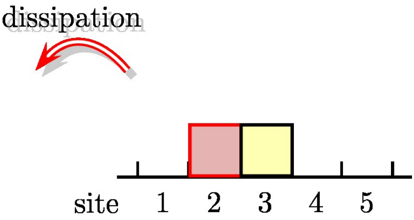

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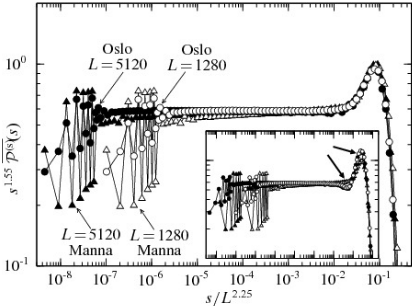
Collapse with Oslo

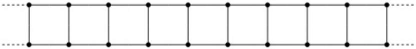


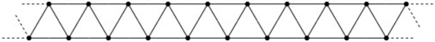
The Manna Model is in the same universality class as the Oslo model.

Manna on different lattices

One and two dimensions


(a)The simple chain. $L = 10, N = 10$.


(b)The rope ladder. $L = 10, N = 20$.


(c)The next nearest neighbour (nnn) chain. $L = 10, N = 20$.

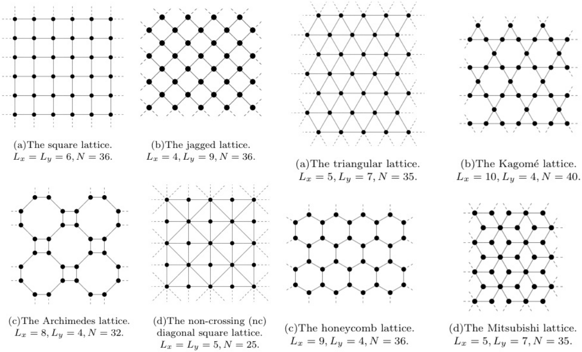

(d)The Futatsubishi lattice. $L = 7, N = 22$.

From: Huynh, G P, Chew, 2011

The Manna Model has been investigated numerically in great detail.

Manna on different lattices

One and two dimensions



(a)The square lattice. $L_x = L_y = 6, N = 36$.
(b)The jagged lattice. $L_x = 4, L_y = 9, N = 36$.
(c)The Archimedes lattice. $L_x = 8, L_y = 4, N = 32$.
(d)The non-crossing (nc) diagonal square lattice. $L_x = L_y = 5, N = 25$.

(a)The triangular lattice. $L_x = 5, L_y = 7, N = 35$.
(b)The Kagomé lattice. $L_x = 10, L_y = 4, N = 40$.
(c)The honeycomb lattice. $L_x = 9, L_y = 4, N = 36$.
(d)The Mitsubishi lattice. $L_x = 5, L_y = 7, N = 35$.

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Manna on different lattices

One and two dimensions

lattice	d	D	τ	z	α	D_a	τ_a	$\mu_1^{(s)}$	$-\Sigma_s$	$-\Sigma_t$	$-\Sigma_a$
simple chain	1	2.27(2)	1.117(8)	1.450(12)	1.19(2)	0.998(4)	1.260(13)	2.000(4)	0.27(2)	0.27(3)	0.259(14)
rope ladder	1	2.24(2)	1.108(9)	1.44(2)	1.18(3)	0.998(7)	1.26(2)	1.989(5)	0.24(2)	0.26(5)	0.26(2)
nnn chain	1	2.33(11)	1.14(4)	1.48(11)	1.22(14)	0.997(15)	1.27(5)	1.991(11)	0.33(11)	0.3(2)	0.27(5)
Futatsubishi	1	2.24(3)	1.105(14)	1.43(3)	1.16(6)	0.999(15)	1.24(5)	2.008(11)	0.24(3)	0.23(9)	0.24(5)
square	2	2.748(13)	1.272(3)	1.52(2)	1.48(2)	1.992(8)	1.380(8)	1.9975(11)	0.748(13)	0.73(4)	0.76(2)
jagged	2	2.764(15)	1.276(4)	1.54(2)	1.49(3)	1.995(7)	1.384(8)	2.0007(12)	0.764(15)	0.76(5)	0.77(2)
Archimedes	2	2.76(2)	1.275(6)	1.54(3)	1.50(3)	1.997(10)	1.382(11)	2.001(2)	0.76(2)	0.78(6)	0.76(3)
nc diagonal square	2	2.750(14)	1.273(4)	1.53(2)	1.49(2)	1.992(7)	1.381(8)	2.0005(12)	0.750(14)	0.75(4)	0.76(2)
triangular	2	2.76(2)	1.275(5)	1.51(2)	1.47(3)	2.003(11)	1.388(12)	1.997(2)	0.76(2)	0.71(6)	0.78(3)
Kagomé	2	2.741(13)	1.270(4)	1.53(2)	1.49(2)	1.993(8)	1.381(9)	1.9994(12)	0.741(13)	0.75(5)	0.76(2)
honeycomb	2	2.73(2)	1.268(6)	1.55(4)	1.51(4)	1.990(13)	1.376(14)	2.000(2)	0.73(2)	0.79(8)	0.75(3)
Mitsubishi	2	2.75(2)	1.273(6)	1.54(3)	1.50(4)	1.999(12)	1.387(12)	1.998(2)	0.75(2)	0.77(7)	0.77(3)

From: Huynh, G P, Chew, 2011

The Manna Model has been investigated numerically in great detail.

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Manna on different lattices

Three dimensions

Lattice	$\overline{q}$	$\overline{q^{(v)}}$	$\langle z \rangle$	D	τ	z	α	D_a	τ_a	$\mu_1^{(s)}$	$-\Sigma_s$	$-\Sigma_t$	$-\Sigma_a$
SC	6	1	[0.622325(1)]	3.38(2)	1.408(3)	1.779(7)	1.784(9)	3.04(5)	1.45(4)	2.0057(5)	1.38(2)	1.395(16)	1.36(13)
BCC	8	4	[0.600620(2)]	3.36(2)	1.404(4)	1.777(8)	1.78(1)	2.99(2)	1.444(18)	2.0030(5)	1.36(2)	1.390(19)	1.33(6)
BCCN	14	5	[0.581502(1)]	3.38(3)	1.408(4)	1.776(9)	1.783(11)	3.01(3)	1.44(3)	2.0041(6)	1.38(3)	1.39(2)	1.32(7)
FCC	12	4	[0.589187(3)]	3.35(4)	1.402(8)	1.765(16)	1.78(2)	3.1(2)	1.48(14)	2.0035(11)	1.35(4)	1.37(4)	1.5(5)
FCCN	18	5	[0.566307(3)]	3.38(4)	1.408(7)	1.781(14)	1.787(18)	3.00(4)	1.44(3)	2.0051(8)	1.38(4)	1.40(3)	1.32(9)
Overall				3.370(11)	1.407(2)	1.777(4)	1.783(5)	3.003(14)	1.442(12)	2.0042(3)		1.380(13)	

From: Huynh, G P, 2012

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Outline

Tools in SOC

- (Extensive) numerics (BTW, FFM, BS, Manna, Oslo).
- Analytical tools:
  - Exact solutions (so far: directed models only).
  - Mappings to known (understood?) phenomena.
  - Growth processes and field theories.

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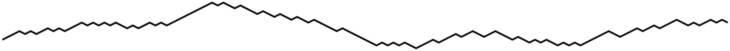
Link to growth phenomena

Generic scale invariance

Stochastic evolution of sandpile surface.



$$\partial_t \phi(\mathbf{r},t) = (\nu_{\|}\partial_{\|}^2 + \nu_{\perp}\partial_{\perp}^2)\phi + \eta(\mathbf{r},t)$$

- *Generic* scale invariance (Hwa and Kardar, 1989, and Grinstein, Lee and Sachdev 1990)
- No mass term $-\epsilon\phi$ on the right $\longrightarrow$ conservative dynamics (finiteness generates ϵ).
- Anisotropy (boundaries?) required in the presence of conserved noise.
- Non-trivial exponents in the presence of non-linearities and non-conserved noise.

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Effect of a mass term

Mass term

$$\partial_t \phi = \nu \nabla^2 \phi - \epsilon \phi + \ldots + \eta$$

represents disspation

$$\partial_t \int_V \mathrm{d}^d x\, \phi = \text{surface terms} - \epsilon \int_V \mathrm{d}^d x\, \phi$$

and correlation length

$$\phi = \ldots e^{-|x|\sqrt{\epsilon/\nu}} .$$

But: How can a renormalised $\epsilon = 0$ be maintained without trivialising the phenomenon?

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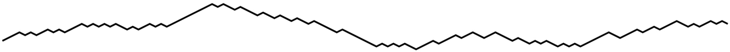
Field theories for Manna and Oslo

Number of charges interpreted as an interface.



- **Manna model** has a Langevin equation
- **Oslo model** implements **quenched Edwards Wilkinson equation** $\longrightarrow$ interfaces!
- Field theories for both still unclear.
- Mechanism of self-organisation still unclear.
- Link to known universality classes.
- Link to **directed percolation**?

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Any Answers?

- Does SOC exist in computer models? Yes. Manna and Oslo models are robust and universal.
- Does SOC exist in nature or experiments? Probably: Superconductors, granular media, earthquakes, precipitation
- Is SOC ubiquitous? Apparently not.
- Is SOC understood? Yes, it looks good!
- Is it worth understanding? Certainly: Understanding of long-range correlations in nature and criticality without tuning.

Thanks!

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