A field theory for the Wiener Sausage

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Bilbao, First Joint International Meeting
RSME-SCM-SEMA-SIMAI-UMI

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Outline

The other Wiener!

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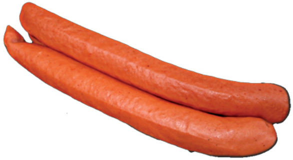

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Wiener process

(named after Norbert Wiener)

Consider a random walker in 2D, leaving a trace:

Think of the random walker (red dot) as the tip of a pen, spilling ink.

What is the area covered in blue (volume of a “Wiener sausage”, traced out in one, two, three dimensions)?

Wiener Sausage

Motivation

- Original problem (average area, 2D) solved by Kolmogoroff and Leontowitsch (1933).
- Famously studied by Spitzer, Kac and Luttinger.
- "Wiener Sausage Volume Moments" by Berezhkovskii, Makhnovskii and Suris (1989).
- Applications …
- Lots of variants and extensions…

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- “Wiener Sausage Volume Moments” by Berezhkovskii, Makhnovskii and Suris (1989).
- Applications in
  - Medicine, *e.g.* tissue “priming”, Dagdug, Berezhkovskii and Weiss (2002).
  - Chemical engineering, *e.g.* agglomerates forming by “sweeping particles”, Eggersdorfer and Pratsinis (2014).
  - Ecology, *e.g.* feeding plankton, Visser (2007).
  - ...
- Lots of variants and extensions...

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Wiener Sausage

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- Applications . . .
- Lots of variants and extensions
  - Presence of traps, *e.g.* Oshanin, Bénichou, Coppey, and Moreau (2002).
  - Surface of the sausage, *e.g.* Rataj, Schmidt and Sporadev (2009).
  - Different boundary conditions, *e.g.* Dagdug, Berezhkovskii and Weiss (2002).
  - . . .

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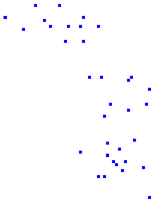
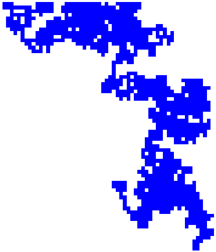
Determine the volume of the Wiener using Statistical Field Theory

Keeping track of a walker's trace is hard.
Easy (-ier, -ish): Walker spatters ink as it walks.



Asymptotic statistics of spatter is that of a continuous trace.

The trajectory of a random walker is self-similar



Wiener Sausage

Poissonian modification

Wiener Sausage observable difficult in a field theory. Therefore:

Poissonian modification

On the lattice: With Poisson rate H walker jumps to a nearest neighbouring site, with rate γ attempts to place immobile offspring at current site.
Deposition suppressed if immobile particle is present already.

Anticipate regularisation: Add extinction rate ϵ' and r for immobile species and walkers respectively.
Mean field approach: $\partial_t \rho_s = \rho_a (1 - \rho_s)\gamma$, where ρ_s number of immobile offspring and ρ_a number density of walkers. (ρ_s is a functional of [the entire history of] ρ_a)
Perturbation theory: $\rho_a(1-\rho_s)\gamma = \gamma\rho_a - \gamma\rho_a\rho_s$.

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Wiener Sausage
Perturbation theory

Perturbation theory: $\rho_a(1-\rho_s)\gamma = \gamma\rho_a - \gamma\rho_a\rho_s$.

Implementation of the suppressed deposition by

- (to first order) allowing unrestricted deposition
- (to second order) removing excess (deposited) particles

The suppression is difficult to deal with.

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Mean field theory in the bulk

If returns (and thus previous deposition) can be ignored, total deposition a is linear in time,

$$\langle a \rangle = \gamma t$$

and Poissonnian moments, $\mathcal{P}^{(a)}(a) = \frac{(\gamma t)^a}{a!} \exp(-\gamma t)$.
Two intertwined Poisson processes for deposition in the presence of extinction, generating function

$$\mathcal{M}^{(a)}(x) = \frac{r/\gamma}{r/\gamma + 1 - \exp(x)}$$

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Outline

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Motivation for a field theory

Motivation for a *field theoretic* study:

- Benefit: Very flexible regarding boundary conditions, additional interactions *etc.*; Very elegant.
- Two species field theory ...
- ... with immobile particles ...
- ... and observables that are spatial integrals.
- "Doable" version of a "heavy duty" field theory.
- Guinea pig example of a **fermionic problem (excluded volume constraint)**.

Excluded volumes are difficult in field theories. May require fermionic treatment (painful).
Idea: Introduce carrying capacity C, whereby deposition rate drops linearly in the occupation, $1 - \rho_s/C$. **Cheating?**

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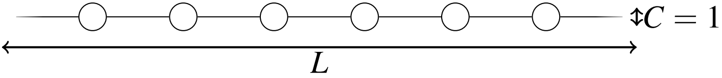
Implementation of the carrying capacity



- One dimensional lattice, length L, carrying capacity C.
- Sites within each column equivalent (**particles per column**).
- When jumping, probability to hit a neighbouring, occupied site is its occupation over carrying capacity C.
- Field-theory now easy (fermionicity is “spurious”).
- carrying capacity C in system of size L corresponds to carrying capacity 1 on $L \times C$ lattice.

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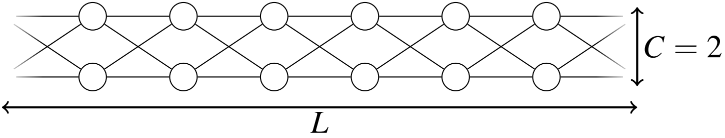

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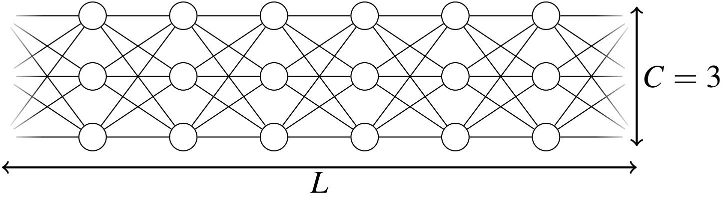

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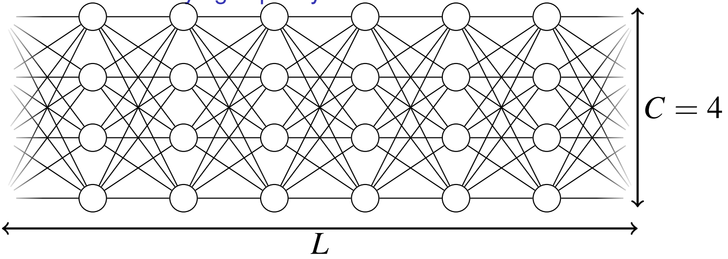

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Using a field theory

Step by step:

1. Write down master equation (with carrying capacity).
2. Rewrite in terms of operators (Doi-Pelitti).
3. Extract propagators and vertices to create diagrams.
4. Dimensional analysis, extract relevant couplings, demonstrate renormalisability.
5. Calculate relevant diagrams, renormalise, extract exponents and other universal quantities.

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Master equation: Bilinear parts — Easy

$$\partial_t \mathcal{P}(\ldots, n, m, \ldots) = \sum_{\mathbf{x}}$$

$$\underbrace{-rn\mathcal{P}(\ldots, n, m, \ldots) + r(n+1)\mathcal{P}(\ldots, n+1, m, \ldots)}_{\text{extinction}}$$

$$\underbrace{-\epsilon' m\mathcal{P}(\ldots, n, m, \ldots) + \epsilon'(m+1)\mathcal{P}(\ldots, n, m+1, \ldots)}_{\text{extinction}}$$

$$\underbrace{-\frac{H}{q}\sum_{\mathbf{e}} n(\mathbf{x})\mathcal{P}(\ldots, n(\mathbf{x}), \ldots, n(\mathbf{x}+\mathbf{e}), \ldots)}_{\text{hoping away}}$$

$$\underbrace{+\frac{H}{q}\sum_{\mathbf{e}} n(\mathbf{x}+\mathbf{e})\mathcal{P}(\ldots, n(\mathbf{x})-1, \ldots, n(\mathbf{x}+\mathbf{e})+1, \ldots)}_{\text{hoping here}} + \text{non-linear terms} \ldots$$

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Master equation: Non-linear parts — Difficult

$$\partial_t \mathcal{P}(\ldots, n, m, \ldots) = \sum_{\mathbf{x}} \boxed{\text{bilinear terms} \ldots} +$$

$$\underbrace{-\gamma n \left(1 - \frac{m}{C}\right) \mathcal{P}(\ldots, n, m, \ldots)}_{\text{deposition}} \; \underbrace{+\gamma n \left(1 - \frac{m-1}{C}\right) \mathcal{P}(\ldots, n, m-1, \ldots)}_{\text{deposition}}$$

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Field Theory

Doi-Pelitti technique

1) Introduce raising and lowering operators

$$a^\dagger |n\rangle = |n+1\rangle \quad \text{and} \quad a|n\rangle = n|n-1\rangle$$
$$b^\dagger |n\rangle = |n+1\rangle \quad \text{and} \quad b|n\rangle = n|n-1\rangle$$

2) Introduce state-vector / generating function

$$|\Psi\rangle(t) = \sum_{\{n,m\}} \mathcal{P}(\dots,n,m,\dots) \prod_{\mathbf{x}} a^{\dagger\, n}(\mathbf{x}) \prod_{\mathbf{x}} b^{\dagger\, m}(\mathbf{x}) |0\rangle$$

Expectation $\langle \bullet \rangle = \langle \Psi_0 | \bullet | \Psi \rangle$ with suitable left vector $\langle \Psi_0 |$.

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Field Theory

Doi-Pelitti technique

3) Doi-shift operators to simplify diagrammatic expansion:

$$a^\dagger = 1 + \tilde{a} \qquad \text{and} \qquad b^\dagger = 1 + \tilde{b}$$

4) Rewrite master equation

$$\partial_t \mathcal{P}(\ldots, n, m, \ldots) = \sum_{\mathbf{x}} \boxed{\text{bilinear terms} \ldots} +$$

$$\boxed{-\gamma n \left(1 - \frac{m}{C}\right) \mathcal{P}(\ldots, n, m, \ldots)} \boxed{+\gamma n \left(1 - \frac{m-1}{C}\right) \mathcal{P}(\ldots, n, m-1, \ldots)}$$

as (term-by-term messy):

$$\partial_t \left|\Psi\right\rangle(t) = \text{bilinear terms} +$$

$$\sum_{\mathbf{x}} \gamma\, \tilde{b}(\mathbf{x})\, a^\dagger(\mathbf{x}) a(\mathbf{x}) \quad -\frac{\gamma}{C}\, \tilde{b}(\mathbf{r}) b^\dagger(\mathbf{r}) b(\mathbf{r})\, a^\dagger(\mathbf{r}) a(\mathbf{r})$$

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Field Theory

Doi-Pelitti technique

5) Introduce Liouvillian:

$$\partial_t \left|\Psi\right\rangle (t) = \sum_{\mathbf{x}} \boxed{\text{bilinear terms} \ldots}$$

$$+\gamma\, \tilde{b}(\mathbf{x})\, a^\dagger(\mathbf{x})a(\mathbf{x}) \qquad -\frac{\gamma}{C}\, \tilde{b}(\mathbf{r})b^\dagger(\mathbf{r})b(\mathbf{r})\, a^\dagger(\mathbf{r})a(\mathbf{r})$$

$$\mathcal{L}_1 = -\gamma\tilde{\psi}\phi^*\phi \qquad +\frac{\gamma}{C}\, \tilde{\psi}\psi^*\psi\phi^*\phi$$

6) Path integral re-formulation

$$\int \mathcal{D}\tilde{\phi}\mathcal{D}\phi\mathcal{D}\tilde{\psi}\mathcal{D}\psi \exp\left(-\int đ^d k đ\omega\,(\mathcal{L}_0 + \mathcal{L}_1)\right)$$

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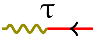
Field Theory

- Extract bare propagators:

$$\left\langle \phi(\mathbf{k},\omega)\tilde{\phi}(\mathbf{k}',\omega') \right\rangle_0 =$$ 

$$\left\langle \psi(\mathbf{k},\omega)\tilde{\psi}(\mathbf{k}',\omega') \right\rangle_0 =$$ 

$$\left\langle \psi(\mathbf{k},\omega)\tilde{\phi}(\mathbf{k}',\omega') \right\rangle_0 =$$ 

- Allow for different renormalisation of initially identical couplings.
- Dimensional analysis: upper critical dimension $d_c = 2$.

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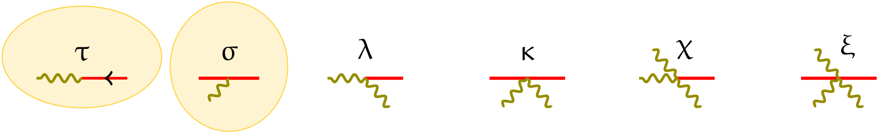
Field Theory

Interaction vertices

Different couplings to allow different renormalisation

$$\mathcal{L}_1 = -\tau\tilde{\psi}\phi - \sigma\tilde{\psi}\tilde{\phi}\phi + \lambda\tilde{\psi}\psi\phi + \kappa\tilde{\phi}\tilde{\psi}\psi\phi + \chi\tilde{\psi}^2\psi\phi + \xi\tilde{\phi}\tilde{\psi}^2\psi\phi$$

Diagrams:



Field Theory
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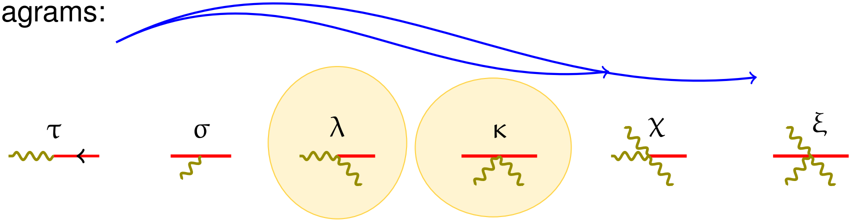

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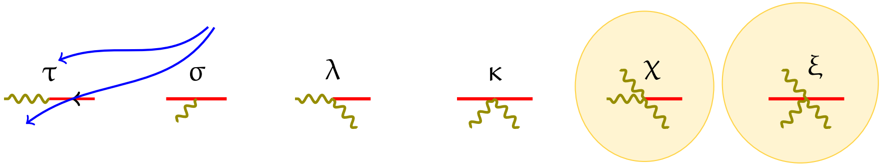

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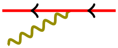
Field Theory

Tree level in the bulk ($d > 2$)

Deposition is suppressed in the presence of deposits.
Without that, deposits could be found all along the walker's trajectory (multiple deposits at revisited sites):



This diagram is present at tree level. Although it cannot be integrated out, its contribution to correlation functions can be determined easily.

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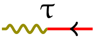
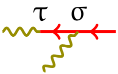
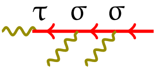
Field Theory

Tree level in the bulk ($d > 2$)

- Tree level = no loops (return asymptotically irrelevant)
- Non-linearities present at tree level.
- n'th moment of the sausage volume a dominated[1] by trees with n branches:

$\langle a \rangle =$ 

$\langle a^2 \rangle =$ 

$\langle a^3 \rangle =$ 

Reproduces Poissonian results above...

[1] Lower order terms from other trees.

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Field Theory

Tree level in finite systems

In finite systems,

- Fourier integrals turn into sums.
- Loss of translational invariance results in vertices becoming sums.
- Example

$$\langle a \rangle = \frac{8\tau}{\pi^4 D} L^2 \sum_{n\text{odd}} \frac{1}{n^4} = \frac{\tau L^2}{12D}$$

- Higher orders increasingly messy, *e.g.*
$2\pi \sum_{\substack{nml\\ \text{odd}}} \frac{1}{n^3}\frac{1}{m}\frac{1}{l}\frac{1}{2\pi}\left(\frac{1}{n+m-l}+\frac{1}{n-m+l}+\frac{1}{-n+m+l}-\frac{1}{n+m+l}\right)=\frac{1}{6}\left(\frac{\pi}{2}\right)^6$
- Ignoring return, sausage volume is linear in residence time, whose moments can be extracted from recurrence relations of moment generating functions.

Field Theory

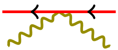
Full theory for the bulk ($d < 2$)

- Walker walks:  $= \dfrac{1}{-\imath\omega + D\mathbf{k}^2}$

- …and leaves behind a trace in the form of branched-off particles

 $= b^\dagger(\mathbf{x}) a^\dagger(\mathbf{x}) a(\mathbf{x})$

- No deposition if a particle is there already

 $= b^\dagger(\mathbf{x}) b(\mathbf{x}) a^\dagger(\mathbf{x}) a(\mathbf{x})$

- Substrate particles stuck on the lattice:  $= \dfrac{1}{-\imath\omega + \epsilon'}$

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Field Theory

Meaning of vertices

This diagram probes the lattice for deposits (and suppresses further deposition):

Without it, no loops can be formed $\longrightarrow$ tree level theory.

Interaction of the walker with its past trace.

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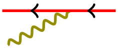
Outline

Field Theory

Interaction diagrams

Calculate features of the Wiener sausage using **renormalisation**.
Deposit along the trajectory



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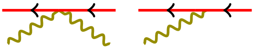

Loop = interaction = signature of collective phenomenon

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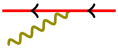

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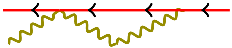

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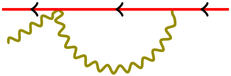

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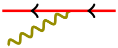

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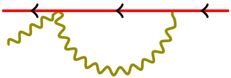


$$= \int \bar{d}\omega' \bar{d}^d k' \, \frac{1}{-\imath\omega' + D\mathbf{k}'^2} \frac{1}{\imath\omega' + \epsilon'}$$

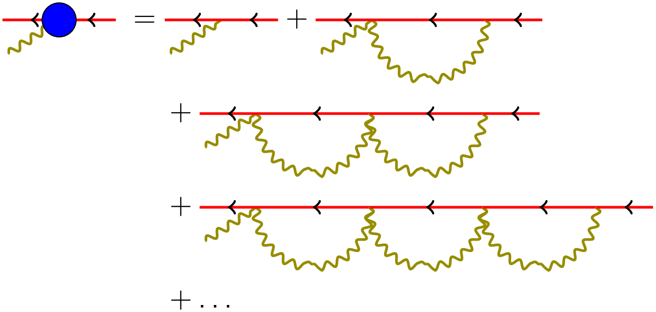
Physical origin of UV divergence: Time spent[2] per volume element diverges at $d \geqslant 2 = d_u$, upper critical dimension. Above: interaction irrelevant, size of sphere enters.

[2] Lingering, not returning, $\int \mathrm{d}t \, (4Dt\pi)^{-d/2} \exp\left(-(x-x')^2/(4Dt)\right)$.

Field Theory

Renormalisation

At the heart of the theory is the **renormalisation** of the following process:



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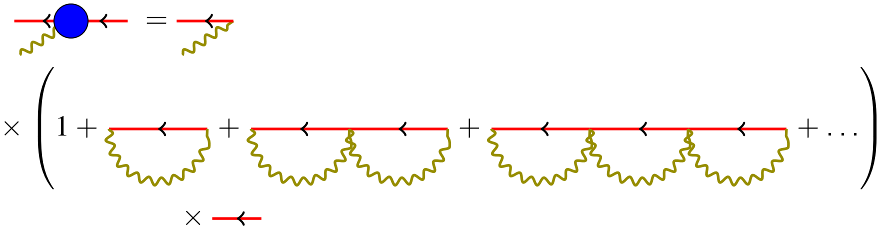

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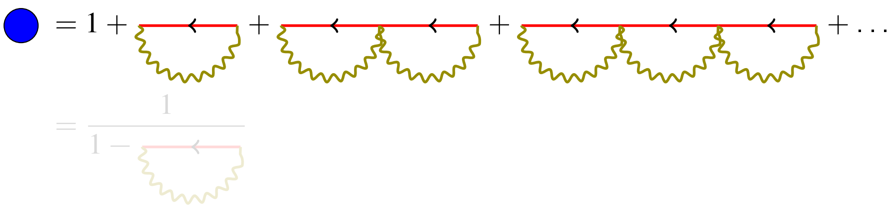
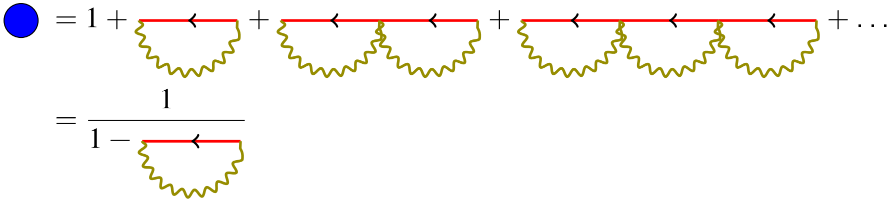


Field Theory

Renormalisation

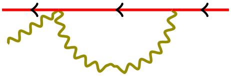
At the heart of the theory is the **renormalisation** of the following process:



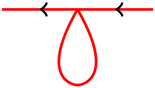
Field Theory

Renormalisation: What are the loops

What physical process do the loops



correspond to? Trajectory intersecting itself (contract along wriggly line):



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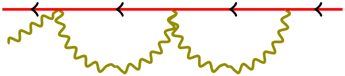
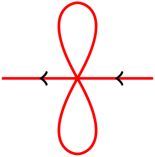
Field Theory

Renormalisation: What are the loops

What physical process do the loops



correspond to? Trajectory intersecting itself twice (contract along wriggly line):



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Outline

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Field Theory

Results

Focus on first moment of sausage volume as a function of time.

- In **one** dimensions: Length covered proportional to square root of time, $\langle a \rangle = \frac{\tau}{\kappa} 4 \sqrt{\frac{tD}{\pi}}$. **Exact amplitude!**
- In **two** dimensions: Area covered linear in time, t (modulo logarithmic corrections, $t/\ln(t)$).
- **In general:** $\langle a^m \rangle \propto t^{md/2}$.
- Next: Finite size scaling
- In three dimensions and higher: Volume linear in time, t.
- ...random walker may never return.
- Well known results (Leontovich and Kolmogorov, Berezhkovskii, Makhnovskii and Suris)...
- ...but, hey, what a nice playground for field theory (fermionicity, renormalisation, calculating moments easily ...sort of).

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Outline

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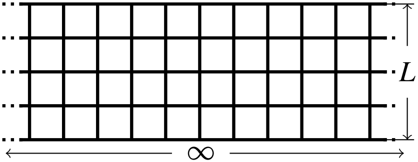
Field Theory

Extension: Regular lattice with open boundary conditions



Nonlinearity changes in finite systems from

$$\kappa \int đ\omega_{1,2,3,4} \int đ^d k_{1,2,3,4}\, \phi^\dagger(\mathbf{k}_1)\psi^\dagger(\mathbf{k}_2)\phi(\mathbf{k}_3)\psi(\mathbf{k}_4)$$

$$\delta(\omega_1+\omega_2+\omega_3+\omega_4)\delta^d(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$$

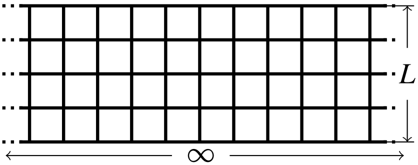
which originates from

$$\delta^d(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4) = \int \mathrm{d}^d r\, e^{-\imath \mathbf{r}\mathbf{k}_1} e^{-\imath \mathbf{r}\mathbf{k}_2} e^{-\imath \mathbf{r}\mathbf{k}_3} e^{-\imath \mathbf{r}\mathbf{k}_4}$$

to …

Field Theory

Extension: Regular lattice with open boundary conditions



...

$$\kappa \int đ\omega_{1,2,3,4} \int đ^{d-1}k_{1,2,3,4} \sum_{nmkl} \phi^\dagger(\mathbf{k}_1)\psi^\dagger(\mathbf{k}_2)\phi(\mathbf{k}_3)\psi(\mathbf{k}_4) \delta(\omega_1+\omega_2+\omega_3+\omega_4)\delta^{d-1}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)U_{nmkl}$$

with

$$U_{nmkl} = \frac{2}{L}\int dz\ \sin zq_n \sin zq_m \sin zq_k \sin zq_l$$

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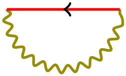
Field Theory

Extension: Regular lattice with open boundary conditions



$$= \kappa^2 \left(\frac{2}{L}\right)^2 \sum_{ab} \int \text{đ}\omega' \, \text{đ}^{d-1} k'$$

$$\times \frac{1}{-\imath\omega' + D\mathbf{k}'^2 + Dq_a^2} \frac{1}{\imath\omega' + \epsilon'} U_{nmab} U_{ablk}$$

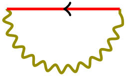
where $q_n = n\pi/L$, $n = 1, 2, \ldots$ are modes in the finite direction.
$U_{nmlk} = (2/L) \int_0^L \mathrm{d}x \, \sin(q_n x) \sin(q_m x) \sin(q_l x) \sin(q_k x)$ accounts for lack of translational variance.

Problem: Renormalisation scheme requires the RHS to be expressed as a multiple of κU_{nmlk}.

Solution: *Deviation* **of RHS from multiple of κU_{nmlk} sub-leading (as found in Casimir systems).**

Field Theory

Extension: Regular lattice with open boundary conditions



$$= \kappa^2 \left(\frac{2}{L}\right)^2 \sum_{ab} \int đ\omega' \, đ^{d-1}k' \times \frac{1}{-\imath\omega' + D\mathbf{k}'^2 + Dq_a^2} \frac{1}{\imath\omega' + \epsilon'} U_{nmab} U_{ablk}$$

where $q_n = n\pi/L$, $n = 1, 2, \ldots$ are modes in the finite direction.
$U_{nmlk} = (2/L)\int_0^L \mathrm{d}x \, \sin(q_n x)\sin(q_m x)\sin(q_l x)\sin(q_k x)$ accounts for lack of translational variance.
General result in $d < 2$: $\langle a^m \rangle \propto m!\tau\sigma^{m-1}\left(\frac{L}{\pi}\right)^{md}\kappa^{-m}$.
Finite size L has the effect of a lowest mode, $q_1 = \pi/L$.
Large L like $d \to d-1$ for periodic BC (crossover).

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Field Theory

More exotic extension: Challenges for dealing with “exotic” lattices

- Lack of conservation (U_{nmkl} instead of $\delta()$)
- New interaction (U_{nmkl} possibly not renormalising to U_{nmkl})
- Different spectrum

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Field Theory

More exotic extension: Fractal lattices

What is the minimal adjustment to go from regular lattices to networks and fractals?

$$\int đ^{\mathbf{d}} k \; \frac{1}{-\imath\omega' + D\mathbf{k}'^2 + Dq_n^2} \cdots$$

Eigenvalues $\mathbf{k}$ of d dimensional lattice are themselves a d dimensional lattice. Spectral dimension $d_s = 2d_f/d_w$ (regular lattice $d_s = d$).
Works only if (bare) propagator itself does not renormalise ($\eta = 0$).
So: Wiener sausage volume $\propto t^{d_s/2}$.
Note: Known return time distribution in networks $\propto t^{-d_s/2}$.

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Wiener Sausage

More exotic extension: Numerics for fractals

Good support for Wiener sausage on fractals

Lattice	fractal d_f	spectral d_s	$d_s/2$	measured
SSTK	1.464	1.16	0.58	0.58
CRAB	1.584	1.23	0.61	0.59
ARROW	1.584	1.36	0.68	0.65
SITE	2	1.55	0.77	0.76

What about Networks?

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Wiener Sausage on Networks

What is needed

- Field theory on networks: Spectrum and structure of eigenvectors for any network.
- At least spectral dimension.
- Exact solution of the Wiener sausage on any network.
- At least numerics for that.

Thank you!