Scaling in the gradient contact process

Gunnar Pruessner[1], Michael Gastner[2], Dániel Zimmerman[3], and Beáta Oborny[3]

[1] *Mathematics Institute, University of Warwick, Coventry, United Kingdom*
g.pruessner@warwick.ac.uk

[2] *Santa Fe Institute, Santa Fe, NM, USA*

[3] *Department of Plant Taxonomy and Ecology, Loránd Eötvös University, Budapest, Hungary*

Abstract

The gradient contact process displays a rich variety of scaling features which are relevant for biological processes and very interesting from a more theoretical point of view. We have investigated the finite size scaling of fluctuations as well as the geometrical features. On the one hand, the aim is to identify biologically relevant observables that would allow us to identify the contact process as the universality class of the underlying microscopic mechanism in the generation of species patterns. Is it possible to infer the universality class of the reproduction mechanism in a biological system by looking at, say, areal photographs? On the other hand, the gradient contact process can be studied as a form of correlated gradient percolation.

Introduction

Figure 1: Example of a borderline, a timberline in Colorado.

Biologists would like to understand the *structure of species boundaries*, such as the timberline shown above. A widely used model in ecology is the *contact process*. Physicists would like to find the universal behaviour of the contact process in nature. The contact process belongs to the directed percolation universality class, which is thought to be very big, but has never been observed in nature.

The uniform Contact Process

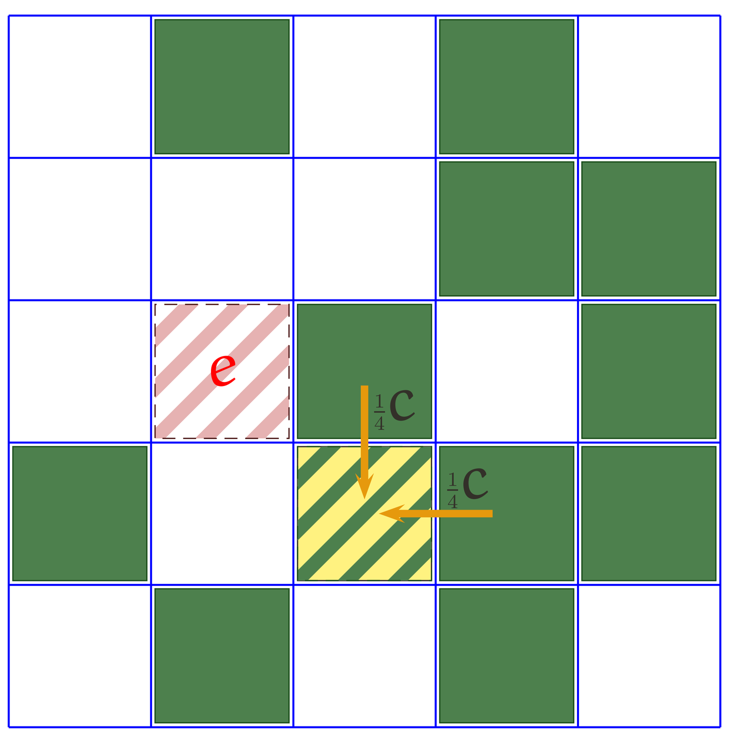

Figure 2: Microscopic updates in the contact process.

Contact Process:
The contact process is a very simple (meta) population model. Sites are either vacant or occupied. Occupied sites become vacant with rate e and vacant sites become occupied with rate zc, where z is the fraction of occupied nearest neighbours. The idea is that each occupied nearest neighbour tries to produce an offspring with rate c at a randomly chosen neighbouring site [1].

The *activity* ρ is the density of occupied sites in the system. Once the lattice is completely empty, there is no way it can become occupied anywhere again, because there is no spontaneous creation. This is the *absorbing state.*

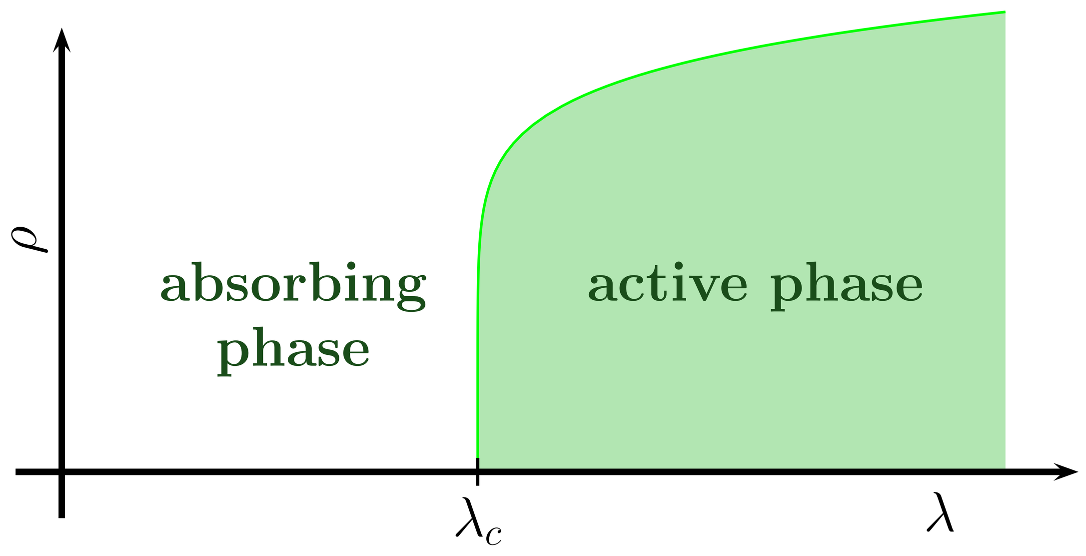

Figure 3: The parameter $\lambda = c/e$ drives a continuous transition from an absorbing phase to an active phase.

The tuning parameter is the ratio $\lambda = c/e$. It drives a (continuous) absorbing state phase transition: For $\lambda < \lambda_c$ the activity vanishes $\rho = 0$ (absorbing phase), for $\lambda > \lambda_c$ activity is sustained, $\rho > 0$ (active phase).

The Gradient Contact Process

Gradient Contact Process:
The ordinary contact process has homogeneous λ throughout the system. To model a spatially changing environment, λ is made space dependent. Simplest scenario: λ changes linearly in position x, so that there is a x_c with $\lambda(x_c) = \lambda_c$.

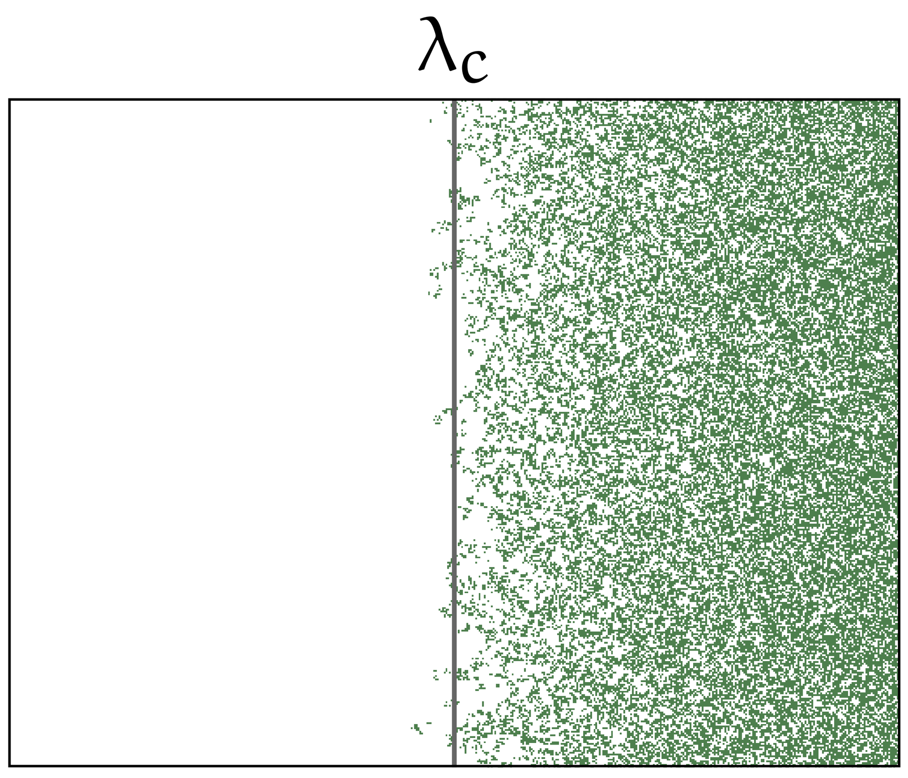

Figure 4: Snapshot of a gradient contact process with λ increasing from left to right.

What are the scaling and the geometrical properties of such a system?

DP scaling features

Determine characteristic scale and apply standard finite size scaling argument.

Characteristic scale:
The characteristic scale at any point in the system is the range to the left and to the right, so that all points within this range have a (bulk) correlation length at least as big as the range.

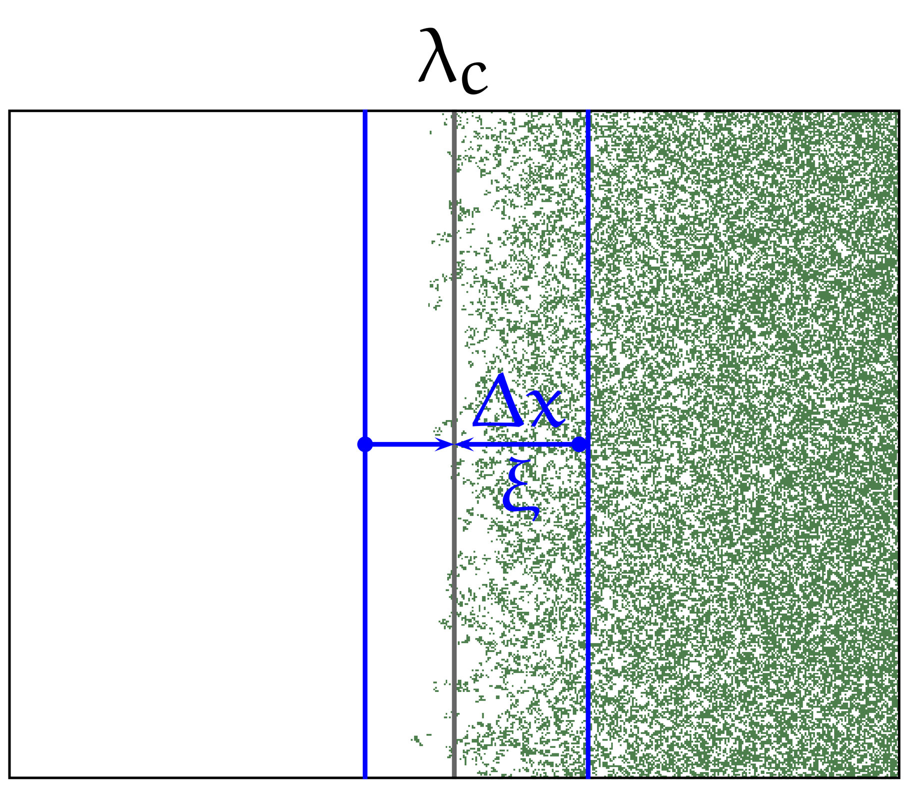

Figure 5: The characteristic scale Δx around λ_c is determined self-consistently by $\xi(\lambda(\Delta x)) = \Delta x$.

The scale Δx is determined by $\xi(\lambda(\Delta x)) = \Delta x$, where the *correlation length* ξ is assumed to correspond to the *bulk correlation length*, $\xi \propto |\lambda - \lambda_c|^{-\nu}$. The parameter λ changes linearly in space with slope λ', so that $\lambda(x) = \lambda_c + \lambda' x$. As a result, $\Delta x \propto \lambda'^{-\nu/(1+\nu)}$. Now standard finite size scaling arguments can be applied:

$$\Delta x \propto \lambda'^{-\nu/(1+\nu)}$$
$$\rho \propto \lambda'^{\beta/(1+\nu)}$$
$$\sigma^2(\rho) L^d \propto \lambda'^{-\gamma/(1+\nu)}$$

This scaling has been confirmed numerically.

How to probe for this scaling behaviour in static field data, usually single snapshots?

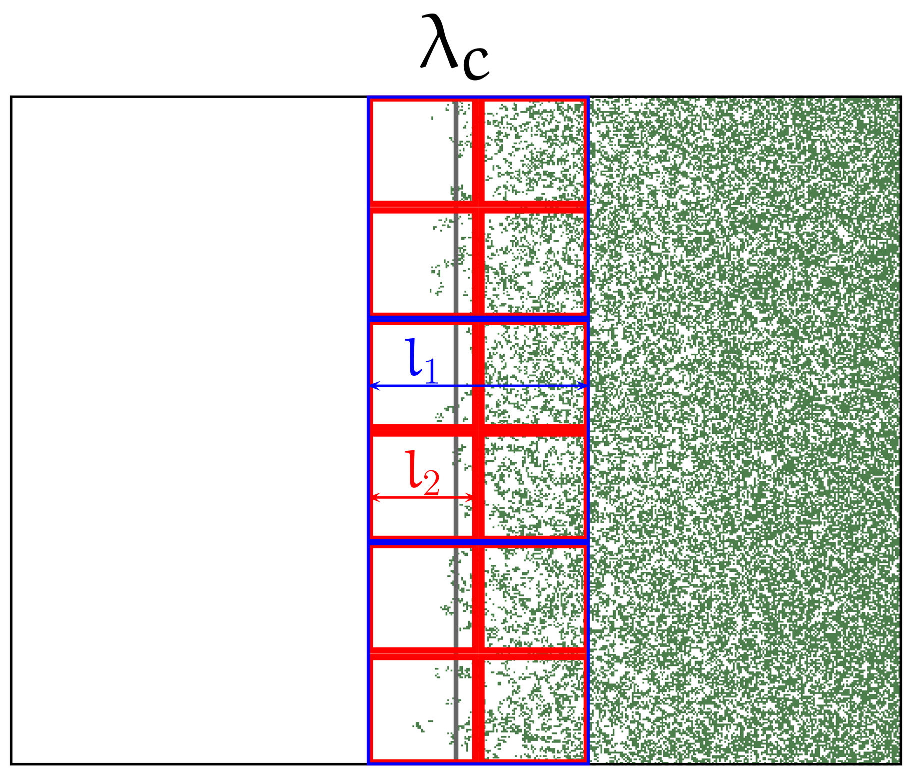

Figure 5: Measuring the fluctuations of activity in blocks of different sizes, say l_1 and l_2.

Block scaling of the fluctuations offers a way to determine the scaling behaviour of a system in a single snapshot. However, instead of ordinary block scaling,

$$\sigma^2(l; L) \propto l^{-2\beta/\nu}$$

one finds

$$\sigma^2(l; L) \propto l^{-\beta/\nu} L^{-\beta/\nu} .$$

The presence of cross-over phenomena makes it very difficult to identify the scaling reliably [2].

Borderline scaling

Two types of borderlines can be identified, by borrowing the definitions from standard (gradient) percolation: The hull is constructed by performing a biased walk along the largest (spanning) cluster. The perimeter is constructed by performing the same type of walk on the complement (the sites not belonging to the largest cluster).

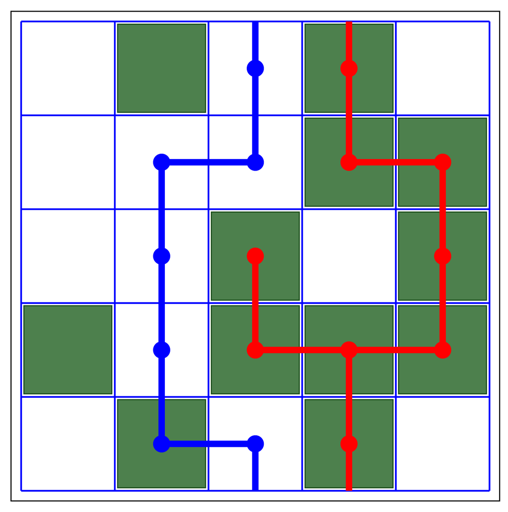

Figure 6a: Tracing out hull and perimeter.

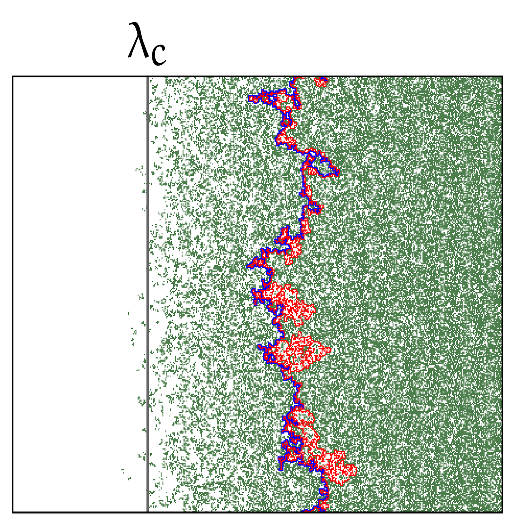

Figure 6b: The long-winded hull and the shorter perimeter in the gradient contact process.

The two lines have two different exponents in gradient percolation: The hull has fractal dimension $7/4$, the perimeter has fractal dimension $4/3$ [3]. Because the borderline is located far away from λ_c, the statistical characteristics in this region are those of independently occupied sites, *unaffected by the particularities of the underlying microscopic process.* One should therefore expect the same exponents as in gradient percolation. This scenario is confirmed numerically.
Preliminary results in a field study by T. Morschhauser, K. Morschhauser, B. Oborny, and D. Zimmermann, seem to support the conjecture.

Summary

We study the scaling behaviour in the gradient contact process in order to identify reliable observables to be investigated in the field. It is difficult to find features signalling the presence of the contact process as the underlying microscopic dynamics: Block-scaling in absorbing state phase transitions suffers strongly from cross-over phenomena and the scaling features of the apparent borderline are governed by ordinary gradient percolation.

References

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