SYMMETRY BREAKING FOR EQUIVARIANT MAPS

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ABSTRACT. In this work we state and prove a number of foundational results in the local bifurcation theory of smooth equivariant maps. In particular, we show that stable one-parameter families of maps are generic and that stability is characterised by semi-algebraic conditions on the finite jet of the family at the bifurcation point. We also prove strong determinacy theorems that allow for high order forced symmetry breaking. We give a number of examples, related to earlier work of Field & Richardson, that show that even for finite groups we can expect branches of fixed or prime period two points with submaximal isotropy type. Finally, we provide a simplified proof of a result that justifies the use of normal forms in the analysis of the equivariant Hopf bifurcation.

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1. Introduction

Let Γ be a compact Lie group and V be an irreducible finite dimensional non-trivial representation space for Γ over $\mathbb{R}$ or $\mathbb{C}$. In Field & Richardson [22, 23, 24, 25], a theoretical framework was developed for the local analysis of symmetry breaking bifurcations[1] of one parameter families of smooth Γ-equivariant vector fields on V. This approach was developed further in [16, 21], where genericity and determinacy theorems were proved for bifurcation problems defined on general real or complex irreducible representations. Taken together, these results imply that there is a finite classification of branching patterns of stable families, that the stability of a family is determined by a finite jet at the bifurcation point (finite determinacy), and that branches persist generically under high order symmetry breaking perturbations (strong determinacy). We refer the reader to [19] for a discussion of some of these results and their proofs. Suffice it to say that techniques are typically geometric and depend on ideas from real algebraic geometry, equivariant transversality and resolution of singularities ("blowing-up"). Our aim in this work is to develop an analogous theory for smooth families of Γ-equivariant *maps*. Specifically, we study bifurcations of generic one-parameter families of Γ-equivariant maps defined on an irreducible representation (V, Γ). Rather than looking for branches of relative equilibria (or limit cycles), we search for branches of invariant group orbits. If Γ is *finite*, branches consist of fixed points or points of prime period two and some of our results extend work of Chossat & Golubitsky [8], Peckham & Kevrekidis [33], and Vanderbauwhede [38] to situations where the equivariant branching lemma does not apply. The reader should also note the important work by Ruelle [35] on bifurcations of equivariant maps (and vector fields). Using our results on families of maps we provide a simplified analysis of the effect of breaking normal form symmetries in the equivariant Hopf bifurcation. This approach avoids the use of the somewhat technical normal hyperbolicity results proved in [21, Appendix].

[1] We refer the reader to [28], [24, Introduction] for an overview and background on symmetry breaking and equivariant bifurcation theory.

In more detail, we start in §2 with a review of basic notations and facts about group actions, representations and dynamics of equivariant maps. In §3, following [24], we cover the basic definitions of stable family, branching pattern and determinacy for equivariant maps. We conclude with the definition of strong determinacy which allows for forced symmetry breaking. In §4, we prove genericity and determinacy theorems for one-parameter families of equivariant maps (Theorems 4.5.3, 4.5.7, 4.6.5, 4.7.1). These results are proved using techniques based on equivariant transversality and stratified sets. Essentially, we show that our concept of genericity (or determinacy) can be formulated in terms of transversality conditions to stratified sets. Granted this, genericity and determinacy theorems follow easily using standard transversality theory. We conclude §4 with a version of the invariant sphere theorem [16] for maps (Theorem 4.8.1) and show how this can be used to prove a partial extension of Fiedler's Hopf bifurcation theorem [10] to maps. Using results from [25], we present in §5 a large class of examples based on the series of finite reflection groups $W(B_n)$, $n \geq 2$. In particular, we give many examples where there are stable submaximal branches of fixed points or points of prime period two. In §6, we prove a strong determinacy theorem for families of equivariant maps (Theorem 6.1.1). The methods used here are very similar to of [21, §§7–10] and depend on resolution of singularities arguments and, in the case of non-Abelian non-finite compact Lie groups, recent results of Schwarz [36] on the coherence of orbit strata. We conclude the section by showing how the strong determinacy theorem can be used to justify the use of normal forms in the analysis of period doubling bifurcations for equivariant maps (Theorems 6.3.3, 6.3.5). Finally, in §7, we show how our results on families of equivariant maps can be used to justify the use of normal forms in the equivariant Hopf bifurcation theorem for vector fields (Theorem 7.2.1). Using blowing-up arguments, we reduce the study of the $\Gamma \times S^1$-equivariant Hopf bifurcation to an analysis of a Γ-equivariant family of (blown-up) Poincaré maps.

1.1. **Acknowledgements.** We thank Marty Golubitsky for helpful conversations on period doubling and for telling us of the works by Vanderbauwhede and Peckam & Kevrekidis. Thanks also to Muriel Kœnig and Pascal Chossat for a number of stimulating conversations on the equivariant Hopf bifurcation which encouraged us to write down the simplified proof of the equivariant Hopf bifurcation theorem presented in §7.

2. Technical Preliminaries and Basic Notations

As far as possible we shall follow the notational conventions of [21, 22].

2.1. **Generalities on groups.** Throughout, we shall be considering compact Lie groups Γ. If H is a (closed) subgroup of Γ, we let $N(H)$ denote the *normalizer* of H in Γ and $C(H)$ denote the *centralizer* of H in Γ. Obviously, $C(H) \subset N(H)$. We let H^0 denote the identity component of H.

2.2. **Γ-sets and isotropy types.** Let Γ be a group and X be a Γ-set. If $x \in X$, then $\Gamma \cdot x$ denotes the Γ-orbit of x and Γ_x denotes the isotropy subgroup of Γ at x. We refer to the conjugacy class (Γ_x) of Γ_x in Γ as the *isotropy type* or *orbit type* of x. We let $\mathcal{O}(X, \Gamma)$ denote the set of isotropy types for the Γ-set X. We abbreviate $\mathcal{O}(X, \Gamma)$ to $\mathcal{O}$ if X and Γ are implicit from the context. For $x \in \Gamma$, we let $\iota(x)$ denote the isotropy type of x. If $\tau \in \mathcal{O}(X, \Gamma)$, we let $X_\tau = \{x \in X \mid \iota(x) = \tau\}$ be the set of points of isotropy type τ. We let X^Γ denote the fixed point set for the action of Γ on X. We define the usual partial order on $\mathcal{O}(X,\Gamma)$ by "$\tau > \mu$, if there exists $H \in \tau$, $K \in \mu$ such that $H \supset K$, $H \neq K$".

2.3. **Representations.** Let V be a nontrivial (finite-dimensional) real representation space for Γ. We assume that V has a positive definite Γ-invariant inner product $(\,,)$ with associated norm $|\cdot|$ and regard Γ as acting on V by orthogonal transformations.

If V is a nontrivial complex representation space for Γ, we assume that V has a positive definite Γ-invariant hermitian inner product $<\,,>$ and regard (V,Γ) as a unitary representation. If we let $(\,,)$ denote the real part of $<\,,>$, then $(\,,)$ is a Γ-invariant inner product on V. We let J_V denote the complex structure on V defined by scalar multiplication by $\imath$. If we regard $S^1 \subset \mathbb{C}$ as the group of complex numbers of unit modulus, we may extend the action of Γ on V to an action of $\Gamma \times S^1$ on V. The resulting representation $(V, \Gamma\times S^1)$ is complex and both $<\,,>$ and $(\,,)$ will be $\Gamma\times S^1$-invariant. We reserve the notation S^1 for the group of complex numbers of unit modulus and take the S^1-action on V defined by scalar multiplication.

Suppose that (V,Γ) is a real irreducible representation and let $L_\Gamma(V,V)$ denote the space of all Γ-equivariant $\mathbb{R}$-linear endomorphisms of V. We recall Frobenius' Theorem [30, 7.7] that $L_\Gamma(V,V)$ is isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the quaternions).

Definition 2.3.1. Let (V,Γ) be a nontrivial irreducible real representation.

(1) (V,Γ) is *absolutely irreducible* if $L_\Gamma(V,V)\cong\mathbb{R}$.
(2) (V,Γ) is *irreducible of complex type* if $L_\Gamma(V,V)\cong\mathbb{C}$.
(3) (V,Γ) is *irreducible of quaternionic type* if $L_\Gamma(V,V)\cong\mathbb{H}$.

Remark 2.3.2. If (V,Γ) is irreducible of complex type, we may give V the structure of a complex vector space so that (V,Γ) is irreducible as a complex representation. We take as complex structure on V any element of $L_\Gamma(V,V)$ whose square is $-I_V$. This choice is unique up to multiplication by $\pm I_V$. Elements of $L_\Gamma(V,V)$ will then be complex scalar multiples of the identity map of V. Similar remarks hold for the quaternionic case. ♢

Suppose that (W,Γ) is absolutely irreducible. The action of Γ on W extends to a $\mathbb{C}$-linear action on the complexification $V = W\otimes_R \mathbb{C}$ of W. The representation (V,Γ) is then irreducible as a complex representation. More generally, we recall the following basic result [5] on complex representations.

Lemma 2.3.3. *Let (V,Γ) be an irreducible complex representation. Then one of the following three exclusive possibilities must occur.*

(R) *(V,Γ) is isomorphic to the complexification of an absolutely irreducible representation.*
(C) *If we regard (V,Γ) as a real representation, then (V,Γ) is of complex type.*
(Q) *If we regard (V,Γ) as a real representation, then (V,Γ) is of quaternionic type.*

Definition 2.3.4 ([21, §2])**.** Let (V,Γ) be a complex representation. We say that

(1) (V,Γ) is *complex irreducible* if (V,Γ) is nontrivial, irreducible and not of quaternionic type.
(2) (V,Γ) is *tangential* if (V,Γ) is complex irreducible and $\Gamma\supset S^1$.

Example 2.3.5. Let (V, Γ) be a complex representation. The natural action of S^1 on V commutes with Γ. Set $G = \Gamma \times S^1$. Then (V, G) is a complex representation of G. If (V, Γ) is complex irreducible, then (V, G) is tangential. ♡

2.4. **Isotropy types for representations.** Let (V, Γ) be a finite dimensional real representation. It is well-known and elementary that the set $\mathcal{O}(V, \Gamma) = \mathcal{O}$ of isotropy types is finite. Obviously, we always have $(\Gamma) \in \mathcal{O}$. We define $\mathcal{O}^\star = \mathcal{O} \backslash (\Gamma)$. If τ, $\mu \in \mathcal{O}$, it follows from linearity and slice theory that

$$\tau > \mu \text{ if and only if } V_\tau \subset \partial V_\mu$$

We say that an orbit type τ is *maximal* (respectively, *submaximal*) if (i) $\tau \neq (\Gamma)$ and (ii) $\mu > \tau$ implies $\mu = (\Gamma)$ (respectively, $\tau \neq (\Gamma)$ and τ is not maximal). Given $\tau \in \mathcal{O}$, choose $x \in V_\tau$ and let $N(\Gamma_x)$ denote the normalizer of Γ_x in Γ. Define

$$g_\tau = \dim(\Gamma \cdot x), \quad n_\tau = \dim(N(\Gamma_x)/\Gamma_x)$$

Of course, g_τ and n_τ depend only on τ and not on the choice of x in V_τ.

2.5. **Polynomial Invariants and Equivariants.** Let $P(V)$ denote the $\mathbb{R}$-algebra of $\mathbb{R}$-valued polynomial functions on V and $P(V, V)$ be the $P(V)$-module of all polynomial maps of V into V. For $k \in \mathbb{N}$, we let $P^k(V)$ (respectively, $P^{(k)}(V)$) denote the vector space of all homogeneous polynomials (respectively, polynomials) of degree k. We similarly define the spaces $P^k(V, V)$ and $P^{(k)}(V, V)$. If (V, Γ) is a finite dimensional real representation, we let $P(V)^\Gamma$ denote the $\mathbb{R}$-subalgebra of $P(V)$ consisting of invariant polynomials, and $P_\Gamma(V, V)$ denote the $P(V)^\Gamma$-module of Γ-equivariant polynomial endomorphisms of V. If (V, Γ) has the structure of a complex representation, then $P_\Gamma(V, V)$ may be given the structure of a complex vector space, with scalar multiplication defined by $P \mapsto \lambda P$, $\lambda \in \mathbb{C}$.

In the sequel, we assume that (V, Γ) is nontrivial and either absolutely irreducible or complex irreducible. We let $\mathcal{F} = \{F_1, \dots, F_k\}$ be a minimal set of homogeneous generators for the $P(V)^\Gamma$-module $P_\Gamma(V, V)$ and let $\mathcal{F}_V$ denote the real vector subspace of $P_\Gamma(V, V)$ spanned by $\mathcal{F}$. Let $d_i = \text{degree}(F_i)$, $1 \leq i \leq k$. We order the F_i so that $d_1 \leq d_2 \leq \ldots \leq d_k$. If (V, Γ) is absolutely irreducible, we may suppose that $F_1 = I_V$ and $d_i \geq 2$, $i > 1$. If (V, Γ) is complex irreducible, we may suppose that $F_1 = I_V$, $F_2 = J_V$ and $d_i \geq 2$, $i > 2$.

Lemma 2.5.1. *Suppose that (V, Γ) is a complex representation. We may choose $\mathcal{F}$ so that $\mathcal{F}_V$ is a complex vector subspace of $P_\Gamma(V, V)$. If we set $J = S^1 \cap \Gamma$, then $\mathcal{F}_V$ is invariant under the J-action defined by $\nu_g(F) = g \circ F = F \circ g$, $g \in J$. In particular, $\mathbb{R}^k$ inherits from $\mathcal{F}_V$ the natural structure of a complex J-representation.*

Proof: Let $\mathcal{H}$ be a minimal set of homogeneous generators for $P_\Gamma(V, V)$, regarded as a module over the complex valued invariants. Take $\mathcal{F} = \mathcal{H} \cup \imath\mathcal{H}$. □

Remarks 2.5.2. (1) In the sequel we always assume that if (V, Γ) is complex irreducible then the set $\mathcal{F}$ of generators for $P_\Gamma(V, V)$ satisfies the hypotheses of Lemma 2.5.1. If $\exp(\imath\theta) \in S^1 \cap \Gamma$, we shall let $\nu_\theta : \mathbb{R}^k \to \mathbb{R}^k$ denote the linear isomorphism of $\mathbb{R}^k$ induced by multiplication by $\exp(\imath\theta)$. (2) Lemma 2.5.1 applies when $\Gamma \cap S^1 = \{e\}$. Suppose that $H \neq \Gamma$ is an isotropy group. Regard $N(H)/H$ as a subgroup of $O(V^H)$ and set $H^\star = N(H)/H \cap S^1$. Since the S^1 action on V^H is a restriction of the S^1-action on V, and $F|V^H$ is $N(H)/H$-equivariant, $F \in \mathcal{F}_V$, it follows that $\mathcal{F}_V^H = \{F|V^H \mid F \in \mathcal{F}_V\}$ is $H^\star$-invariant. ♢

2.6. **Smooth families of equivariant maps.** Let $C^\infty_\Gamma(V\times\mathbb{R},V)$ denote the space of smooth (that is, C^∞) Γ-equivariant maps from $V\times\mathbb{R}$ to V, where Γ acts on $V\times\mathbb{R}$ as $(v,\lambda)\mapsto(gv,\lambda)$. Let $C^\infty(V\times\mathbb{R})^\Gamma$ denote the space of smooth $\mathbb{R}$-valued invariant functions on $V\times\mathbb{R}$. We give $C^\infty_\Gamma(V\times\mathbb{R},V)$ and $C^\infty(V\times\mathbb{R})^\Gamma$ the C^∞ topology. Let $f\in C^\infty_\Gamma(V\times\mathbb{R},V)$. It follows either from the theory of closed ideals of differentiable functions (see [12]) or from Schwarz' theorem on smooth invariants (see [34]) that we may write

$$f_\lambda(x)=f(x,\lambda)=\sum_{i=1}^k f_i(x,\lambda)F_i(x),$$

where $f_i\in C^\infty(V\times\mathbb{R})^\Gamma$, $1\le i\le k$.

2.7. **Normalized families.** Suppose $f\in C^\infty_\Gamma(V\times\mathbb{R},V)$ and (V,Γ) is irreducible. Clearly, $x=0$ is a fixed point of $f(x,\lambda)=x$. We refer to $x=0$ as the *trivial branch* of fixed points for f.

Lemma 2.7.1. *Suppose that $f\in C^\infty_\Gamma(V\times\mathbb{R},V)$, $\lambda_0\in\mathbb{R}$ and Df_{λ_0} has no eigenvalues of unit modulus. We may choose a neighborhood U of $(0,\lambda_0)$ in $V\times\mathbb{R}$, such that if $(x,\lambda)\in U$ and $f_\lambda(x)\in\Gamma\cdot x$ then $x=0$.*

Proof: Suppose that (V,Γ) is a complex representation (the proof when (V,Γ) is absolutely irreducible is similar). It follows from the irreducibility of (V,Γ) that $Df_\lambda=\sigma_f(\lambda)I_V$, where $\sigma_f:\mathbb{R}\to\mathbb{C}$ is smooth. Without loss of generality, suppose $|\sigma_f(\lambda)|>1$. It follows by continuity that we can choose a compact neighborhood J of λ_0 such that $|\sigma_f(\lambda)|>1$ on J. Hence, for each $\lambda\in J$, $x=0$ is a hyperbolic repelling point of f_λ. Let D_r denote the closed disc, center zero, radius r in V. Since J is compact, it follows from the mean value theorem that we can choose $r>0$, $c>1$, such that for all $x\in D_r$, $\lambda\in J$, we have $\|f(x)\|\ge c\|x\|$. Hence there are no fixed orbits for f in $U=D_r\times J$. □

It follows from Lemma 2.7.1 that bifurcations of the trivial branch of fixed points only occur when $Df_\lambda(0)$ has an eigenvalue on the unit circle. In particular, note that our interest is in finding branches of invariant *group orbits* not just fixed points.

Just as in [22, 23, 24, 21], we restrict attention to families f that have a non-degenerate change of stability of the trivial branch of fixed points at $\lambda=0$:

$$|\sigma_f(0)|=1,\quad |\sigma_f(0)|'\ne 0 \tag{1}$$

Reparametrizing the bifurcation variable λ and noting that we shall only be interested in (generic) behavior of f near the origin of $V\times\mathbb{R}$, it is no loss of generality to restrict attention to the space

$$\mathcal{M}(V,\Gamma)=\{f\in C^\infty_\Gamma(V\times\mathbb{R},V)\mid \sigma_f(\lambda)=\exp(\imath\omega(\lambda))(1+\lambda)\},$$

where $\omega:\mathbb{R}\to\mathbb{R}$ is a smooth map. In case (V,Γ) is absolutely irreducible, we replace the term $\exp(\imath\omega(\lambda))(1+\lambda)$ by $\pm1+\lambda$.

In the sequel, we refer to elements of the spaces $\mathcal{M}(V,\Gamma)$ as *normalized* families. If $\theta\in[0,2\pi)$, we define $\mathcal{M}^\theta(V,\Gamma)=\{f\in\mathcal{M}(V,\Gamma)\mid\omega(0)=\theta\}$. If (V,Γ) is absolutely irreducible, we let $\mathcal{M}^+(V,\Gamma)$, $\mathcal{M}^-(V,\Gamma)$ denote the subspaces of $\mathcal{M}(V,\Gamma)$ corresponding to $\sigma_f(0)=1$, $\sigma_f(0)=-1$ respectively.

3. Branching and invariant group orbits

After briefly reviewing some basic definitions about invariant group orbits for equivariant maps, we discuss branching and stability for 1-parameter families of equivariant maps. We assume the reader has some familiarity with the basic definitions and results about normal hyperbolicity described in Hirsch et al. [29].

3.1. **Invariant group orbits and normal hyperbolicity.** Let M be a smooth Γ-manifold. Suppose that $f : M \to M$ is a smooth and Γ-equivariant. We say that a Γ-orbit $\alpha \subset M$ is an *f-invariant orbit* if $f(\alpha) = \alpha$. It follows from equivariance that α is an f-invariant orbit if and only if there exists $x \in \alpha$ such that $f(x) \in \alpha$. If Γ (or α) is finite, each point of α is a periodic point of f and the stability condition we require is that α consists of hyperbolic periodic points. If Γ is infinite, the natural stability condition is to require that f is normally hyperbolic at α. We refer to [13] for basic properties of normally hyperbolic group orbits. As we shall be considering nondegenerate bifurcations off the trivial solution, we may and shall assume that maps are diffeomorphisms – at least on some neighborhood of invariant group orbits.

A spectral characterization of normal hyperbolicity for invariant group orbits is given in [13, 17]. We recall without proof the main definitions and results from [17] that we need.

Lemma 3.1.1 ([17, Proposition 5.2], [14, Lemma D])**.** *Let α be an f-invariant Γ-orbit and U be a Γ-invariant neighborhood of α. Suppose that $f(x) \in N(\Gamma_x)^0 \cdot x$, for some (hence any) $x \in \alpha$. Then there exists a smooth map $\chi : M \to \Gamma$ such that*

1. *$\chi(y) = e$, $y \in M \setminus U$.*
2. *$\chi(y) \in C(\Gamma_y)^0$, all $y \in M$.*
3. *If we define $\tilde{f} : M \to M$ by $\tilde{f}(y) = \chi(y)f(y)$, then $\tilde{f}$ is equivariant and $\tilde{f}|\alpha$ is the identity map.*

Suppose that α is an f-invariant Γ-orbit of isotropy τ and $f(x) \in N(\Gamma_x)^0 \cdot x$, $x \in \alpha$. It follows from Lemma 3.1.1 that we may find a smooth Γ-equivariant map $\tilde{f}$ such that $\tilde{f}|\alpha$ is the identity and $\tilde{f}, f$ induce the same map on M/Γ. Since $\tilde{f}(x) = x$, $x \in \alpha$, $T_x\tilde{f} : T_xM \to T_xM$. Let $\mathrm{spec}(\tilde{f}, x)$ denote the set of eigenvalues, with multiplicities, of $T_x\tilde{f}$. Then $\mathrm{spec}(\tilde{f}, x)$ depends only on $\tilde{f}$ and is independent of $x \in \alpha$ [17]. We define $\mathrm{spec}(\tilde{f}, \alpha)$ to be equal to $\mathrm{spec}(\tilde{f}, x)$, any $x \in \alpha$.

Before giving the next definition, we recall that S^1 acts on $\mathbb{C}$ by scalar multiplication and that $\mathbb{C}/S^1 \cong \mathbb{R}^+$.

Definition 3.1.2. Let $f, \tilde{f}, \alpha$ be as above.

1. The *(reduced) spectrum* $\mathrm{SPEC}(f, \alpha)$ of f along α is the subset of $\mathbb{R}^+$ defined by
$$\mathrm{SPEC}(f, \alpha) = \mathrm{spec}(\tilde{f}, \alpha)/S^1$$
2. The *index* of f along α, $\mathrm{index}(f, \alpha)$, is defined to be the number of elements of $\mathrm{SPEC}(f, \alpha)$ less than 1 (counting multiplicities).

Remarks 3.1.3. (1) It is shown in [13, 17] that the definition $\mathrm{SPEC}(f, \alpha)$ depends only on f and α and not on the choice of $\tilde{f}$. Note also that the reduced spectrum may be defined even if $f(x) \in N(\Gamma_x) \cdot x$, as opposed to $N(\Gamma_x)^0 \cdot x$. We refer to [13] for details. (2) $\mathrm{SPEC}(f, \alpha)$ contains at least g_τ elements equal to one. ♢

Proposition 3.1.4 ([13]). *Let $f : M \to M$ be a smooth Γ-equivariant map and α be an f-invariant Γ-orbit of isotropy type τ. Then α is normally hyperbolic if and only if $1 \in \mathrm{SPEC}(f, \alpha)$ has multiplicity g_τ.*

We conclude this subsection by recalling the following lemma giving a decomposition of f into 'tangent and normal' components near an invariant Γ-orbit.

Lemma 3.1.5 ([17, Lemma 6.2]). *Let α be an f-invariant Γ-orbit and $\alpha(f) \geq 1$ be the smallest integer such that $f^{\alpha(f)}|\alpha$ is Γ-equivariantly isotopic to the identity. Suppose that U, U' are open Γ-invariant tubular neighborhoods of α such that $\overline{U}, \overline{f(U)} \subset U'$. Denote the corresponding families of slices determined by U, U' by $\mathcal{S} = \{\mathcal{S}_x | x \in \alpha\}$, $\mathcal{S}' = \{\mathcal{S}'_x | x \in \alpha\}$, respectively. There exist smooth maps $\rho : U \to \Gamma$, $h : U \to U'$ satisfying:*

1. *$f^{\alpha(f)}(y) = \rho(y)h(y)$, $y \in U$.*
2. *$h : U \to U'$ is an equivariant embedding.*
3. *$h : S_x \to S'_x$, all $x \in \alpha$.*
4. *$\rho(y) \in C(\Gamma_y)$, all $y \in U$.*
5. *α is normally hyperbolic if and only if each (any) $x \in \alpha$ is a hyperbolic fixed point of $h|S_x$.*

3.2. **Branches of invariant Γ-orbits.** Next we discuss families of maps and branches of invariant group orbits. Most of what we say is a natural extension of the corresponding definitions for vector fields given in [24, 21]. We assume throughout that (V, Γ) is either absolutely irreducible or complex irreducible.

Definition 3.2.1. Given $f \in \mathcal{M}(V, \Gamma)$, let

$$\begin{aligned} \mathbf{I}(f) &= \{(x, \lambda) \mid f_\lambda(x) \in \Gamma \cdot x\} \\ \mathbf{B}(f) &= \{(x, \lambda) \in \mathbf{I}(f) \mid \Gamma \cdot x \text{ is not normally hyperbolic}\} \end{aligned}$$

Both $\mathbf{I}(f)$ and $\mathbf{B}(f)$ are closed Γ-invariant subsets of $V \times \mathbb{R}$. We refer to $\mathbf{B}(f)$ as the *bifurcation* set of f and $\mathbf{D}(f) = (\mathbf{I}(f), \mathbf{B}(f))$ as the *bifurcation diagram* of f.

For each $\tau \in \mathcal{O}(V, \Gamma)$, choose $H \in \tau$ and set $\Delta_\tau = \Gamma/H$. Every Γ-orbit of isotropy type τ is smoothly Γ-equivariantly diffeomorphic to Δ_τ.

Definition 3.2.2. Let $f \in \mathcal{M}(V, \Gamma)$ and $\tau \in \mathcal{O}(V, \Gamma)$. A branch of invariant Γ-orbits (of isotropy type τ) for f at zero consists of a C^1 Γ-equivariant map

$$\phi = (\rho, \lambda) : [0, \delta] \times \Delta_\tau \to V \times \mathbb{R}$$

such that λ is independent of $u \in \Delta_\tau$ and

1. $\phi(0, u) = (0, 0)$, all $u \in \Delta_\tau$.
2. For all $t \in (0, \delta]$, $\alpha_t = \rho(t, \Delta_\tau)$ is an $f_{\lambda(t)}$-invariant Γ-orbit of isotropy τ.
3. For every $u \in \Delta_\tau$, the map $\phi_u : [0, \delta] \to V \times \mathbb{R}$, $t \mapsto \phi(t, u)$ is a C^1-embedding.

If, in addition, we can choose $\delta > 0$ so that

4. For all $t \in (0, \delta]$, $f_{\lambda(t)}$ is a normally hyperbolic at α_t,

we refer to ϕ as a branch of normally hyperbolic invariant Γ-orbits for f at zero.

Remark 3.2.3. Typically, parametrizations ϕ satisfying Definition 3.2.2 are smooth. In fact, if ϕ is smooth, satisfies (1,2) and ϕ_u has initial exponent $1 < p < \infty$, then we may define a new parametrization by $\tilde{\phi}(t, u) = \phi(t^{\frac{1}{p}}, u)$. Although $\tilde{\phi}$ is no longer smooth, it satisfies all the conditions of the definition. Subsequently, when we address the problem of constructing explicit parametrizations, we always construct smooth parametrizations with (minimal) finite initial exponent, possibly greater than one. Whether we work with smooth or C^1-parametrizations, the main point is that the direction of branching should be well-defined at $t = 0$. ♢

Let $f \in \mathcal{M}(V, \Gamma)$. We regard two branches of invariant Γ-orbits for f as *equivalent* if they differ only by a (local) reparametrization. If ϕ is a branch, we let $[\phi]$ denote its equivalence class and we identify $[\phi]$ with the germ of the image of ϕ at the origin of $V \times \mathbb{R}$.

Example 3.2.4. Define $c_+, c_- : [0, \infty) \to V \times \mathbb{R}$ by $c_\pm(s) = (0, \pm s)$, $s \in [0, \infty)$. Then $c_\pm$ define two trivial branches of fixed points for any $f \in \mathcal{V}(V, \Gamma)$. ♡

3.3. **The branching pattern.**

Definition 3.3.1 ([24, §1, §3])**.** Let $f \in \mathcal{M}(V, \Gamma)$. The *branching pattern* $\Xi(f)$ of f is the set of all equivalence classes of *non-trivial* branches of invariant orbits for f. Each point in $\Xi(f)$ is labelled with the isotropy type of the associated branch.

3.4. **Stabilities.** We refine our definition of the branching pattern to take account of stabilities. Suppose that $\phi = (\rho, \lambda)$ is a branch of normally hyperbolic invariant group orbits of isotropy type τ for $f \in \mathcal{M}(V, \Gamma)$. By continuity, $\mathrm{index}(f_{\lambda(t)}, \rho(t, \Delta_\tau))$ is constant, $t > 0$, and we define the index of ϕ, $\mathrm{index}(\phi)$, to be the common value of the indices of the non-trivial invariant orbits along the branch. We define $\mathrm{index}([\phi]) = \mathrm{index}(\phi)$ and note that $\mathrm{index}([\phi])$ depends only on the the equivalence class of ϕ. If all the branches of f are normally hyperbolic, then *index* is a well defined $\mathbb{N}$-valued map on $\Xi(f)$.

Lemma 3.4.1. *Let $f \in \mathcal{M}(V, \Gamma)$ and suppose that ϕ is a branch of normally hyperbolic invariant Γ-orbits. Then ϕ is either a supercritical or subcritical branch.*

Proof: Similar to that of Lemma 3.4.2 [21]. □

If $[\phi] \in \Sigma(f)$ is a normally hyperbolic branch we define $\mathrm{sign}([\phi])$ to be $+1$ if the branch is supercritical and -1 if it is subcritical. It follows from Lemma 3.4.1 that sign is well-defined on the set of normally hyperbolic branches.

3.5. **Branching conditions.** Following [24], we consider the following *branching conditions* on $f \in \mathcal{M}(V, \Gamma)$:

B1 There is a finite set $\phi_1, \dots, \phi_{r+2}$ of branches of invariant Γ-orbits for f, with images $C_1, \dots, C_{r+2}$, such that

(1) $\Xi(f) = \{[\phi_1], \dots, [\phi_r]\}$, $[\phi_{r+1}] = [c^+]$, $[\phi_{r+2}] = [c^-]$.
(2) There is a neighborhood N of $(0, 0)$ in $V \times \mathbb{R}$ such that if $(x, \lambda) \in N$ and $\Gamma \cdot x$ is f_λ-invariant then
$$\Gamma \cdot x \times \{\lambda\} \subset \cup_{j=1}^{r+2} C_j$$
(3) If $i \neq j$, then $C_i \cap C_j = \{(0, 0)\}$.

B2 Every $[\phi] \in \Xi(f)$ is a branch of normally hyperbolic f-invariant Γ-orbits.

Definition 3.5.1 ([24])**.** A family $f \in \mathcal{M}(V, \Gamma)$ is *weakly regular* if f satisfies the branching condition **B1**. If, in addition, f satisfies the branching condition **B2**, we say that f is *regular*.

Remark 3.5.2. If f is regular then $(0, 0) \in V \times \mathbb{R}$ is an *isolated* point of $\mathbf{B}(f)$. ♢

3.6. **The signed indexed branching pattern.**

Definition 3.6.1. Suppose $f \in \mathcal{M}(V,\Gamma)$ is regular. The *signed indexed branching pattern* $\Xi^\star(f)$ of f consists of the set $\Xi(f)$, labelled by isotropy types, together with the maps $\mathrm{index} : \Xi(f) \to \mathbb{N}$ and $\mathrm{sign} : \Xi(f) \to \{\pm 1\}$.

Every regular family f has a well-defined signed indexed branching pattern $\Xi^\star(f)$ which describes the stabilities of the f-invariant Γ-orbits on some neighborhood of zero.

Definition 3.6.2. Let $f, g \in \mathcal{M}(V,\Gamma)$ be weakly regular. We say that $\Xi(f)$ is isomorphic to $\Xi(g)$ if there is a bijection between $\Xi(f)$ and $\Xi(g)$ preserving isotropy type. If f, g are regular, we say that $\Xi^\star(f)$ is isomorphic to $\Xi^\star(g)$ if $\Xi(f)$ is isomorphic to $\Xi(g)$ by an isomorphism preserving the sign and index functions.

3.7. **Stable families.**

Definition 3.7.1 (cf. [24, §2])**.** A family $f \in \mathcal{M}(V,\Gamma)$ is *stable* if there is an open neighborhood U of f in $\mathcal{M}(V,\Gamma)$ consisting of regular families such that for every continuous path $\{f_t \mid t \in [0,1]\}$ in U with $f_0 = f$ there exist a compact Γ-invariant neighborhood A of zero in $V \times \mathbb{R}$ and a continuous equivariant isotopy $K : A \times [0,1] \to V \times \mathbb{R}$ of embeddings satisfying

(1) $K_0 = Id_A$.
(2) $K_t(\mathbf{I}(f) \cap A) = K_t(A) \cap \mathbf{I}(f_t)$, $t \in [0,1]$.

Let $\mathcal{S}(V,\Gamma)$ denote the subset of $\mathcal{M}(V,\Gamma)$ consisting of stable families.

Proposition 3.7.2.
(1) *$\mathcal{S}(V,\Gamma)$ is an open subset of $\mathcal{M}(V,\Gamma)$.*
(2) *If f, f' lie in the same connected component of $\mathcal{S}(V,\Gamma)$ then f and f' have isomorphic signed indexed branching patterns.*

Proof: We refer the reader to [24, §2]. □

3.8. **Determinacy.** We conclude this section by extending the definitions of determinacy and strong determinacy given in [21, §3] to families of equivariant maps. As usual, if $f \in \mathcal{M}(V,\Gamma)$, $q \in \mathbb{N}$, we let $j^q f_0(0)$ denote the q-jet of f_0 at $(0,0)$.

Definition 3.8.1. Γ-equivariant bifurcation problems on V are (generically) finitely determined if there exists $q \in \mathbb{N}$ and an open dense semi-algebraic subset $\mathcal{S}(q)$ of $P^{(q)}_\Gamma(V,V)$ such that if $f \in \mathcal{M}(V,\Gamma)$ and $j^q f_0(0) \in \mathcal{S}(q)$ then f is stable.

Remarks 3.8.2. (1) We say Γ-equivariant bifurcation problems on V are (generically) q-determined if q is the smallest positive integer for which we can find $\mathcal{S}(q)$ satisfying the conditions of Definition 3.8.1. For this value of q, we let $\mathcal{R}(q)$ denote the maximal semi-algebraic open subset of $P^{(q)}_\Gamma(V,V)$ satisfying the conditions of Definition 3.8.1. Granted this Definition of $\mathcal{R}(q)$, we shall say that f is *q-determined* if $j^q f_0(0) \in \mathcal{R}(q)$. (2) Let $f \in \mathcal{M}(V,\Gamma)$ be q-determined and set $Q = j^q f_0(0) - j^1 f_0(0)$. Define $J^Q \in \mathcal{M}(V,\Gamma)$ by $J^Q(x,\lambda) = Df_\lambda(0)(x) + Q(x)$. Then f and J^Q have isomorphic signed indexed branching patterns. ♢

We may give refined definitions of stability and determinacy that allow for perturbations by maps which are only equivariant to some finite order.

Given $f \in C^\infty(V \times \mathbb{R}, V)$, $d \geq 1$, let $f^{[d]} = j^d f(0,0) \in P^{(d)}(V \times \mathbb{R}, V)$.

Let H be a closed subgroup of Γ and let H act on $V \times \mathbb{R}$ and V by restriction of the action of Γ. For $d \geq 1$, define

$$\mathcal{M}_0^{[d]}[\Gamma : H] = \{f \in C_H^\infty(V \times \mathbb{R}, V) \mid f^{[d]} \in P_\Gamma^{(d)}(V \times \mathbb{R}, V)\}$$

If $H = \{e\}$, set $\mathcal{M}_0^{[d]}[\Gamma : H] = \mathcal{M}_0^{[d]}[\Gamma]$.

Definition 3.8.3. Let $H \subset \Gamma$ be a closed subgroup, N be a smooth compact H-manifold and $f \in \mathcal{M}_0^{[1]}[\Gamma : H]$. Let $1 \leq r \leq \infty$. A C^r-branch of normally hyperbolic invariant submanifolds of type N for f consists of a C^1 H-equivariant map $\phi = (\rho, \lambda) : [0, \delta] \times N \to V \times \mathbb{R}$ satisfying the following conditions:

1. $\phi(0, x) = (0, 0)$, all $x \in N$.
2. The map $\lambda : [0, \delta] \times N \to \mathbb{R}$ depends only on $t \in [0, \delta]$.
3. For each $t \in (0, \delta]$, $\rho_t(N) = N_t$ is a normally hyperbolic submanifold of V for $f_{\lambda(t)}$.
4. $\phi|(0, \delta] \times N$ is a C^r H-equivariant embedding and for all $(t, x) \in [0, \delta] \times N$, $\frac{\partial \phi}{\partial t}(t, x) \neq 0$.

Remarks 3.8.4. (1) We emphasize that we only require the manifolds $\rho_t(N)$ in Definition 3.8.3 to be C^r. Of course, in Definition 3.2.2, the invariant manifolds are Γ-orbits and therefore smoothly embedded. (2) In the usual way, we regard branches as equivalent if they differ only by a local C^r reparametrization. (3) Let $f \in \mathcal{M}_0^{[1]}[\Gamma : H]$ and $\phi : [0, \delta] \times N \to V \times \mathbb{R}$ be a C^r-branch of normally hyperbolic invariant submanifolds of type N for f. For $t > 0$, let $\{W^{ss}(N_t, x) \mid x \in N_t\}$ denote the strong stable foliation of $W^s(N_t)$. The the dimension of $W^{ss}(N_t, x)$ is independent of x and $t \in (0, \delta]$ and we define $\text{index}(\phi) = \dim(W^{ss}(N_t, x))$, $t \in (0, \delta]$. Finally, for possibly smaller $\delta > 0$, we may show that ϕ is either sub- or supercritical. ♢

Suppose that $f \in \mathcal{M}(V, \Gamma)$ is regular. Choose a Γ-invariant neighborhood A of zero in $V \times \mathbb{R}$ such that

$$A \cap \mathbf{I}(f) = \bigcup_{i \in I} E_i$$

where each E_i is a (the image of) branch of normally hyperbolic Γ-orbits. We call $\mathcal{E} = \{E_i \mid i \in I\}$ a *local representation* of $\mathbf{I}(f)$ at zero. Given $E \in \mathcal{E}$, let $\tau(E)$ denote the isotropy type of the branch E.

Definition 3.8.5. Let $f \in \mathcal{M}(V, \Gamma)$ be stable and $\mathcal{E} = \{E_i \mid i \in I\}$ be a local representation of $\mathbf{I}(f)$ at zero. Let H be a closed subgroup of Γ and $d \in \mathbb{N}$. We say f is (d, H)-stable if there exists an open neighborhood U of f in $\mathcal{M}_0^{[d]}[G : H]$ such that for every continuous path $\{f_t \mid t \in [0, 1]\}$ in U with $f_0 = f$, there exists an H-invariant compact neighborhood A of zero in $V \times \mathbb{R}$ and a continuous H-equivariant isotopy $K : A \times [0, 1] \to V \times \mathbb{R}$ of embeddings such that

1. $K_0 = Id_A$.
2. For every $E \in \mathcal{E}$, $t \in [0, 1]$, $K_t(A \cap E)$ is a branch of normally hyperbolic submanifolds of type $\Delta_{\tau(E)}$ for f_t.

Remarks 3.8.6. (1) If $H = \{e\}$ in Definition 3.8.5, we say f is *strongly d-stable.* (2) In (2) of Definition 3.8.5, we implicitly assume that the branch is C^r for some $r \geq 1$. The differentiability class does not play a major role in our results and the strong determinacy theorem that we prove holds for all r, $1 \leq r < \infty$. ♢

Definition 3.8.7. We say Γ-equivariant bifurcation problems on V are (generically) *strongly determined* if there exist $d \in \mathbb{N}$ and an open dense semi-analytic subset $\mathcal{S}(d) \subset P_\Gamma^{(d)}(V,V)$ such that if $f \in \mathcal{M}(V,\Gamma)$ and $j^d f_0(0) \in \mathcal{S}(d)$ then f is strongly d-stable.

Remarks 3.8.8. (1) We shall say that Γ-equivariant bifurcation problems on V are (generically) strongly d-determined if d is the smallest positive integer for which we can find $\mathcal{S}(d)$ satisfying the conditions of Definition 3.8.7. For this value of d, we let $\mathcal{N}(d)$ denote the maximal semi-analytic open subset of $P_\Gamma^{(d)}(V,V)$ satisfying the conditions of Definition 3.8.7. We say that f is *strongly d-determined* if $j^d f_0(0) \in \mathcal{N}(d)$. (2) Let H a closed subgroup of Γ. We say that Γ-equivariant bifurcation problems on V are (generically) strongly H-determined if there exist $d \in \mathbb{N}$ and an open and dense semi-analytic subset $\mathcal{S}(d) \subset P_\Gamma^{(d)}(V,V)_0$ such that if $f \in \mathcal{M}(V,\Gamma)$ and $j^d f_0(0) \in \mathcal{S}(d)$ then f is (d,H)-stable. Modulo statements about H-equivariance of isotopies (see Definition 3.8.5), it is clear that strong-determinacy implies strong H-determinacy for all closed subgroups H of Γ. ♢

4. Genericity theorems

In this section, we prove our basic genericity and determinacy theorems for Γ-equivariant bifurcation problems on V. The approach we use is broadly similar to that followed in [16, Appendix], [22, §5] and, in particular, [21, §4]. Our results hold for general absolutely or complex irreducible representations of compact Lie groups. We assume known basic facts about semi-algebraic sets and Whitney stratified sets. We refer the reader to [22], [21, §4] for a brief review of these topics and to Costi [9], Mather [32] and Gibson et al. [26] for more detailed presentations.

4.1. **Invariant and equivariant generators.** Let $\mathcal{F} = \{F_1, \dots, F_k\}$ be a minimal set of homogeneous generators for the $P(V)^\Gamma$-module $P_\Gamma(V,V)$ and $\mathcal{P} = \{p_1, \dots, p_\ell\}$ be a minimal set of homogeneous generators for the $\mathbb{R}$-algebra $P(V)^\Gamma$. Set $P = (p_1, \dots, p_\ell) : V \to \mathbb{R}^\ell$ and note that $P : V \to \mathbb{R}^\ell$ may be regarded as the *orbit map* of V onto $V/\Gamma \subset \mathbb{R}^\ell$.

4.2. **The varieties Σ, Ξ.** Define $F : V \times \mathbb{R}^k \to V$ by $F(x,t) = \sum_{i=1}^k t_i F_i(x)$ and let

$$\begin{aligned} \Sigma &= \{(x,t) \in V \times \mathbb{R}^k \mid F(x,t) \in T_x N(\Gamma_x) \cdot x\} \\ \Xi &= \{(x,t) \in V \times \mathbb{R}^k \mid P(x) = P(F(x,t))\} \end{aligned}$$

Obviously Ξ is a Γ-invariant real algebraic subset of $V \times \mathbb{R}^k$ and $F(x,t) \in \Gamma \cdot x$ if and only if $(x,t) \in \Xi$. Moreover, if $(x,t) \in \Xi$, then $F(x,t) \in N(\Gamma_x) \cdot x$.

Remarks 4.2.1. (1) The variety Σ was defined in [22, 16, 21] and plays a basic role in the codimension 1 bifurcation theory of smooth Γ-equivariant vector fields. We refer to these works, especially [21, §4], for properties of Σ. Much of what we do in this section will be a relatively straightforward extension of this theory to Ξ. (2) If (V,Γ) is complex irreducible, then we assume (see Lemma 2.5.1, Remarks 2.5.2) that $\mathcal{F}$ is chosen so that $\mathcal{F}_V$ has the structure of a complex vector space. In particular, k will be even and we may identify $\mathbb{R}^k$ with $\mathbb{C}^m$, where $k = 2m$. ♢

Example 4.2.2. Suppose that Γ is finite. Define $\Xi^+ = \{(x,t) \in V \times \mathbb{R}^k \mid F(x,t) = x\}$. Obviously, $\Xi^+ \subset \Xi$. However, it is not generally true that $\Xi^+ = \Xi$. For example, if $\Gamma = \mathbb{Z}_2$ acts non-trivially on $\mathbb{R}$, then $P(x) = x^2$ and $F(x,t) = tx$. Hence Ξ is the zero variety of $x^2(t^2-1)$ while Ξ^+ is the zero variety of $x(t-1)$. ♡

For each $\tau \in \mathcal{O}(V,\Gamma)$, we let Ξ_τ denote the subset of Ξ consisting of points of isotropy type τ. Clearly, Ξ is the disjoint union over $\mathcal{O}(V,\Gamma)$ of the sets Ξ_τ. Since (V,Γ) is a non-trivial irreducible representation, $\Xi_{(\Gamma)} = \{0\} \times \mathbb{R}^k$.

4.3. **Geometric properties of Ξ.** Suppose that $H \in \tau \in \mathcal{O}^\star$. Then $N(H)/H$ acts on the fixed point space V^H.

Remark 4.3.1. For future reference, note that if $-I \in \Gamma$ and $H \in \tau \in \mathcal{O}^\star$, then $-I \in N(H)/H$, where we regard $N(H)/H$ as a subgroup of the orthogonal group of V^H. Similarly, if (V,Γ) is complex irreducible and $\Gamma \supset S^1$, then $N(H)/H \supset S^1$, where the S^1 action on V^H is the restriction of the action of S^1 on V. In general, $N(H)/H$ may contain $-I$ or S^1 without the same being true for Γ. ♢

Lemma 4.3.2. *Suppose that* $(x_0,t_0) \in \Xi^H_\tau$. *Then* $D_2F_{(x_0,t_0)} : \mathbb{R}^k \to V^H$ *is surjective.*

Proof: Since $x_0 \in V^H_\tau$, $\{F_1(x_0),\dots,F_k(x_0)\}$ spans V^H. Since the matrix of $D_2F_{(x_0,t_0)}$ equals $[F_1(x_0),\dots,F_k(x_0)]$, $D_2F_{(x_0,t_0)}$ is onto. □

Lemma 4.3.3. *For each* $\tau \in \mathcal{O}(V,\Gamma)$*,* Ξ_τ *is a* Γ*-invariant smooth semi-algebraic submanifold of* $V \times \mathbb{R}^k$ *of dimension* $k + g_\tau$*.*

Proof: Since $\Xi_\tau = \Xi \cap (V \times \mathbb{R}^k)_\tau$, it is obvious that Ξ_τ is a Γ-invariant semi-algebraic subset of $V \times \mathbb{R}^k$. In order to show that Ξ_τ is smooth, it suffices to show that the map $G : V^H \times \mathbb{R}^k \to V^H/N(H)$ defined by $G(x,t) = P(F(x,t)) - P(x)$ is a submersion at (x_0,t_0). This follows from Lemma 4.3.2, since the orbit map restricts to a submersion $P : V^H_\tau \to V^H_\tau/N(H)$ [2]. □

Lemma 4.3.4. *Let* γ*,* $\tau \in \mathcal{O}(V,\Gamma)$*. Then*

(1) $\Xi_\tau \cap \overline{\Xi}_\gamma = \emptyset$ *if* $\gamma > \tau$*.*
(2) *dim*$(\Xi_\tau \cap \overline{\Xi}_\gamma) < g_\tau + k$*, if* $\gamma < \tau$*.*

Proof: Similar to that of Lemma 4.3.4 [21]. □

In future we regard $\mathbb{R}^k$ as embedded in $V \times \mathbb{R}^k$ as the subspace $\{0\} \times \mathbb{R}^k$. We let $\mathbb{R}^{k-1}$, $\mathbb{R}^{k-2}$ denote the subspaces of $\mathbb{R}^k$ defined by $t_1 = 0$ and $t_1, t_2 = 0$ respectively.

If (V,Γ) is absolutely irreducible, we let $\mathbf{C}_+$, $\mathbf{C}_- \subset \mathbb{R}^k$ respectively denote the hyperplanes defined by $t_1 = +1$, $t_1 = -1$. We set $\mathbf{C} = \mathbf{C}_+ \cup \mathbf{C}_-$. If (V,Γ) is complex irreducible, we let $\mathbf{C} \subset \mathbb{R}^k$ denote the cylinder $t_1^2 + t_2^2 = 1$. If (V,Γ) is complex, we let $\mathbf{C}_\theta$ denote the codimension 2 subspace $\{\theta\} \times \mathbb{R}^{k-2}$.

Lemma 4.3.5. *Suppose that* $\tau \in \mathcal{O}^\star$*. Then* $\partial\Xi_\tau \cap \mathbb{R}^k \subset \mathbf{C}$*.*

Proof: We prove in case (V,Γ) is complex irreducible. Set $(h_1,\dots,h_\ell)(x,t) = P(F(x,t)) - P(x)$. Since the invariant p_1 is the square of the Euclidean norm on V and $F(x,t) = (t_1 + \imath t_2)x + O(\|x\|^2)$, it follows that

$$h_1(x,t) = (t_1^2 + t_2^2 - 1)\|x\|^2 + O(\|x\|^3) \tag{2}$$

Suppose that (x^n,t^n) is a sequence of points of Ξ_τ converging to the point $(0,t) \in V \times \mathbb{R}^k$. Substituting in (2), dividing by $\|x^n\|^2$, and letting $n \to \infty$, we deduce that $t_1^2 + t_2^2 = 1$. □

Example 4.3.6. Take the standard action of SO(2) on $\mathbb{C}$. A basis for the SO(2)-equivariant polynomial maps of $\mathbb{C}$ is given by $F_1(z) = z$, $F_2(z) = \imath z$. Let (e) denote the conjugacy class of the identity element. The variety Ξ is the zero set of $((t_1^2-1)+t_2^2)|z|^2$. Consequently, $\Xi_{(\mathrm{SO}(2))} = \mathbb{R}^2$ and $\Xi_{(e)} = \{(z,t) \mid z \neq 0, t_1^2+t_2^2 = 1\}$. Obviously, $\partial\Xi_{(e)}$ meets $\mathbb{R}^2$ along the circle $t_1^2 + t_2^2 = 1$. ♡

Lemma 4.3.7. *Let $H \in \tau \in \mathcal{O}^\star$. Suppose (x^n, t^n) is a sequence of points in Ξ_τ^H converging to $(0,t) \in \{0\} \times \mathbb{R}^k$, (γ^n) is a sequence of points of $N(H)/H \subset O(V^H)$ converging to $\gamma \in N(H)/H$ and that for $n \geq 1$ we have*

$$\sum_{j=1}^{k} t_j^n F_j(x^n) = \gamma^n x^n$$

Then

(a) *If (V,Γ) is absolutely irreducible, then $t_1 = \pm 1$ and $\gamma^2 = I$. If $t_1 = +1$, then $\gamma = I \in N(H)/H$. If $t_1 = -1$ and $-I \in N(H)/H$, then $\gamma = -I$.*

(b) *If (V,Γ) is complex irreducible, then $t_1 + \imath t_2 = \exp(\imath\theta)$, for some $\theta \in [0, 2\pi)$. If $\theta = 2\pi/p$, for some $p \in \mathbb{N}$, then $\gamma^p = I$. If $\exp(\imath\theta) \in S^1 \cap N(H)/H$, then $\gamma = \exp(\imath\theta)$.*

Proof: Given $t \in \mathbb{R}^k$, define $f_t : V \to V$ by $f_t(x) = F(x,t)$. Provided $t_1 \neq 0$, f_t will be a Γ-equivariant diffeomorphism on some Γ-invariant neighborhood D of $0 \in V$. Restrict f_t to the $N(H)/H$-space V^H. Following [21, §9], [20], we (polar) blow-up V^H along the non-principal $N(H)/H$-orbit strata. If $\Pi : \tilde{V}^H \to V^H$ denotes the blowing-down map, we set $E = \Pi^{-1}(0)$. Let $\tilde{f}_t$ denote the lift of $f_t|D$ to $\tilde{V}^H \cap \Pi^{-1}(D)$. The restriction of $\tilde{f}_t$ to E equals the lift of $Df_t(0)$ to E. Suppose now that (V,Γ) is absolutely irreducible and the hypotheses of the lemma are satisfied with $t_1 = 1$. It follows that $\tilde{f}_{t_n}|E$ converges to the identity map. Since $N(H)/H$ acts freely on $E \subset \tilde{V}^H$, we see that $\gamma = I$. If $t_1 = -1$, then the same argument proves that $\tilde{f}_{t_n}^2|E$ must converge to the identity and so $\gamma^2 = I$. If $-I \in N(H)/H$, then $f_{-t_n}(x^n) = (-\gamma^n)(x^n)$, where $-\gamma^n \in N(H)/H$. Hence $-\gamma = I$, since $-t_1^n \to 1$. The other cases are similarly proved. □

Remark 4.3.8. Let (V,Γ) be an absolutely irreducible representation. With the notation of Lemma 4.3.7, suppose that $x^n/|x^n| \to u$, where u lies in the unit sphere of V^H. If $t_1 = -1$ then u must be fixed by $-\gamma$ and so $-\gamma$ must restrict to the identity map on the line through u. In particular $\gamma(\mathbb{R}u) = \mathbb{R}u$. Consequently, if $N(H)/H$ acts freely on the projective space of V^H, then we can never have $t_1 = -1$. Similar remarks hold for the case of complex irreducible representations. ♢

Example 4.3.9. Let $\Gamma = \mathbf{D}_3$ denote the dihedral group of order 6 acting in the standard way on $\mathbb{C} = \mathbb{R}^2$. As bases for $P(\mathbb{C})^\Gamma$, $P_\Gamma(\mathbb{C},\mathbb{C})$, we take $\{|z|^2, \mathrm{Re}(z^3)\}$ and $\{z, \bar{z}^2\}$ respectively. The action of Γ on $\mathbb{C}$ has three isotropy types: $\tau_0 = (\Gamma)$, $\tau_1 = (\mathbb{Z}_2)$ and $\tau_2 = (e)$. It is easy to verify directly that $\overline{\Xi}_{\tau_1}$ meets $\mathbb{R}^2$ along the line $t_1 = 1$. On the other hand $\overline{\Xi}_{\tau_2} \cap \mathbb{R}^2$ consists of the line $t_1 = -1$ together with the isolated point $(1,0)$. Indeed, set $z(\rho) = \imath\rho\exp(\imath\rho)u(\rho)$. Using the implicit function theorem together with the defining equations for Ξ_{τ_2}, it is not hard to show that for all $t_2 \in \mathbb{R}$, we can find smooth maps $u, t_1 : [0,\delta] \to \mathbb{R}(>0)$, such that $t_1(0) = -1$ and $F(t_1(\rho), t_2, z(\rho)) = \overline{z(\rho)}$, $\rho \in [0,\delta]$. Note that this curve of points of period two is tangent to the line in $\mathbb{C}$ on which $z \mapsto \bar{z}$ acts as minus the identity. Later, we prove a general version of this result (see Lemma 4.4.2). ♡

We conclude this subsection with an elementary lemma that will be useful later.

Lemma 4.3.10. *Let* $\mathfrak{m} = \{q \in P(V)^\Gamma \mid q(0) = 0\}$ *and suppose*

$$G(x) = \sum_{i=1}^{k} q_i(x)F_i(x),$$

where $q_i \in \mathfrak{m}$, $1 \leq i \leq k$. *If we define* $\tilde{F} : V \times \mathbb{R}^{k+1} \to V$ *by* $\tilde{F}(x,t) = F(x,t) + t_{k+1}G(x)$, *and let* $\tilde{\Xi} = \{(x,t) \mid P(\tilde{F}(x,t)) = P(x)\}$, *then for all* $\tau \in \mathcal{O}$ *we have*

$$\partial\tilde{\Xi}_\tau \cap \mathbb{R}^{k+1} = \partial\Xi_\tau \cap \mathbb{R}^k \times (\{0\} \times \mathbb{R})$$

Proof: The result follows easily by observing that $(x,(t_1,\ldots,t_{k+1})) \in \tilde{\Xi}_\tau$ if and only if $(x,(t_1 + t_{k+1}q_1(x),\ldots,t_k + t_{k+1}q_k(x))) \in \Xi_\tau$. □

Let $\mathcal{S}$ denote the canonical (minimal) semi-algebraic stratification of Ξ (see [32] or [21, §4]).

Theorem 4.3.11 ([22, Theorem 5.10])**.** *The stratification* $\mathcal{S}$ *induces a semi-algebraic Whitney stratification of each* Ξ_τ. *In particular, each* Ξ_τ *is a union of* $\mathcal{S}$*-strata.*

Proof: The proof is similar to that of [22, Theorem 5.10] or [21, Theorem 4.3.7] and we shall not repeat the details. □

4.4. **The sets** C_τ**.** Given $\tau \in \mathcal{O}$, define $C_\tau = \mathbb{R}^k \cap \overline{\Xi}_\tau$. Clearly $C_{(\Gamma)} = \mathbb{R}^k$. If $\tau \neq (\Gamma)$, then it follows from Lemma 4.3.5 that $C_\tau \subset \mathbf{C}$. If (V,Γ) is absolutely irreducible, we define $C_\tau^+ = C_\tau \cap \mathbf{C}_+$ and $C_\tau^- = C_\tau \cap \mathbf{C}_-$. (Note that C_τ^- may be empty – Example 4.3.9 – but C_τ^+ always contains $(1,0,\ldots,0)$). If (V,Γ) is complex irreducible, then for each point θ lying on the circle $t_1^2 + t_2^2 = 1$, we define $C_\tau^\theta = \mathbf{C}_\theta \cap C_\tau$.

Remark 4.4.1. We recall from [22, 16, 21] that for $\tau \in \mathcal{O}^\star$, we define $A_\tau = \mathbb{R}^k \cap \overline{\Sigma}_\tau$. The sets A_τ are closed semi-algebraic conical subsets of the hyperplane $t_1 = 0$. If (V,Γ) is complex irreducible and tangential then A_τ is invariant under translations by vectors $(0,s,0,\ldots,0) \in \mathbb{R}^k$, $s \in \mathbb{R}$. The same is true if $H \in \tau$ and $S^1 \subset N(H)/H$. ♢

Lemma 4.4.2. *Let* (V,Γ) *be absolutely irreducible. Let* $H \in \tau \in \mathcal{O}$.

(1) *If* $dim(V^H) = 1$, *then* C_τ^+ *is the hyperplane* $t_1 = 1$.
(2) *If there exists* $\gamma \in \Gamma \cap N(H)$, $\mathbf{u} \in V_\tau^H$ *such that the fixed point space of the map* $-\gamma : V^H \to V^H$ *is the line* $\mathbb{R}\mathbf{u}$, *then* C_τ^- *is the hyperplane* $t_1 = -1$.

Proof: Statement (1) is well-known and is essentially just a reformulation of the equivariant branching lemma (see [19, Example 6.8]). It remains to prove (2). It suffices to show that there is an open and dense subset $\mathcal{R}$ of $\mathbb{R}^{k-1}$ such that if $(t_2,\ldots,t_k) \in \mathcal{R}$, then there is a smooth solution $(x(\rho),(t_1(\rho),t_2,\ldots,t_k))$, $\rho \in [0,\rho]$, to $F(x,t) = \gamma x$ with $x(0) = 0$, $t_1(0) = -1$. It follows from Lemma 4.3.10 that if we add the cubic $|x|^2x$ to our generating set $\mathcal{F}$ and define

$$\tilde{F}(x,\tilde{t}) = \tilde{F}(x,(t,t_{k+1})) = F(x,t) + t_{k+1}|x|^2x,$$

then it suffices to show that there is an open and dense subset $\tilde{\mathcal{R}}$ of $\mathbb{R}^k$ such that if $(t_2,\ldots,t_{k+1}) \in \tilde{\mathcal{R}}$, we can find a smooth solution $(x(\rho),(t_1(\rho),t_2,\ldots,t_k,t_{k+1}))$ of $\tilde{F}(x,t) = \gamma x$ satisfying the conditions listed previously.

Write $\sum_{j=2}^{k+1} t_j F_j(x) = Q(x) + C(x) + H(x)$, where $Q(x)$ is the sum of the quadratic terms, $C(x)$ is the sum of the cubic terms and $H(x)$ is the sum of the remaining higher order terms. Let W be the orthogonal complement of $\mathbb{R}\mathbf{u}$ in V^H. Denote the orthogonal projections of V^H on $\mathbb{R}\mathbf{u}$ and W by π_u and π_W respectively. Let $j \in \mathbb{N}$ be the smallest integer such that $\pi_W F_j(\mathbf{u}) \neq 0$ and set $b = d_j$. Note that $j > 1$, since $F_1(\mathbf{u}) = \mathbf{u}$, and that $j \in \{2, \ldots, k\}$, since $\{F_1(\mathbf{u}), \ldots, F_k(\mathbf{u})\}$ span V^H. Let $t_1 = -1 + c\rho^2$, where $c = \pm 1$, and $(t_2, \ldots, t_{k+1}) \in \mathbb{R}^k$, and look for a solution to $\tilde{F}(x, \tilde{t}) = \gamma x$ of the form $x(\rho) = (\rho q(\rho)\mathbf{u}, \rho^b \hat{w}(\rho))$, where $q : [0, \delta] \to \mathbb{R}$, $\hat{w} : [0, \delta] \to W$ and $q(0) > 0$, $\hat{w}(0) \neq 0$. Since $b > 1$, $x(\rho)$ will be tangent to $\mathbb{R}\mathbf{u}$ at $\rho = 0$. Substituting in the equation $\tilde{F}(x, \tilde{t}) = \gamma x$, we find the following conditions on $x(\rho)$ for it to be a solution.

$$\begin{aligned} -\rho q(\rho) &= \\ &\quad (-1 + c\rho^2)\rho q(\rho) + \rho^2 q \pi_u [q Q(\mathbf{u}) + \rho^{b-1} A(\mathbf{u}, \hat{w}) + q^2 \rho C(\mathbf{u})] + O(|\rho|^4) \\ \rho^b \gamma \hat{w}(\rho) &= (-1 + c\rho^2)\rho^b \hat{w}(\rho) + q^b \rho^b \pi_W F_b(\mathbf{u}) + O(|\rho|^{1+b}), \end{aligned}$$

where $A(\mathbf{u}, \hat{w}) = 2DQ(\mathbf{u})(\hat{w})$. Since Q is even and $\gamma \mathbf{u} = -\mathbf{u}$, we have $\pi_u(Q(\mathbf{u})) = 0$. It follows from Lemma 4.3.7 that $\gamma^2 = I$ and so $\gamma | W = I_W$. Dividing the second equation by ρ^b and setting $\rho = 0$, we see that

$$\hat{w}(0) = q(0)^b \pi_W F_b(\mathbf{u})/2.$$

Now suppose $b = 2$ (the proof if $b > 2$ is similar but easier). Dividing the first equation by ρ^3 and substituting for $\hat{w}(0)$ we find that

$$cq(0) + q(0)^3 \Lambda(\mathbf{u}) = 0, \tag{3}$$

where $\Lambda(u) = \pi_u(A(\mathbf{u}, \pi_W F_b(\mathbf{u})/2)) + C(\mathbf{u})$ is independent of $\hat{w}(0)$. Since $\tilde{F}$ contains the term $t_{k+1}|x|^2 x$, there is an open and dense subset $\tilde{\mathcal{R}}$ of $\mathbb{R}^k$ such that, if $(t_2, \ldots, t_{k+1}) \in \tilde{\mathcal{R}}$, then $\Lambda(\mathbf{u}) \neq 0$. From (3), $q(0)^2 = -c/\Lambda(\mathbf{u})$ and so we can choose c so that there is a unique positive value of $q(0)$ satisfying the equation.

The construction of a smooth solution $x(\rho) = (\rho q(\rho)\mathbf{u}, \rho^b \hat{w}(\rho))$ with these initial values of $q, \hat{w}$ is now a routine application of the implicit function theorem. □

Remark 4.4.3. An alternative proof of Lemma 4.4.2(2) can be based on Liapunov-Schmidt reduction and the equivariant branching lemma. See Vanderbauwhede [38], Peckham & Kevrekidis [33] and also [27, Lecture 2]. ♢

As a corollary of Theorem 4.3.11 and the definition of the sets C_τ, we have

Proposition 4.4.4. *Each semi-algebraic set C_τ inherits a Whitney regular stratification $\mathcal{C}_\tau$ from $\mathcal{S}$.*

In the sequel we shall denote the stratification $\mathcal{C}_{(\Gamma)}$ of $\Xi_{(\Gamma)} = \mathbb{R}^k$ by $\mathcal{B}$. As usual, we denote the union of the i-dimensional strata of $\mathcal{B}$ by $\mathcal{B}_i$, $i \geq 0$. It follows from our constructions that

$$\mathcal{B}_k = \mathbb{R}^k \setminus \bigcup_{\tau \neq (\Gamma)} C_\tau \tag{4}$$

$$\mathcal{B}_i \subset \mathbf{C},\ i < k \tag{5}$$

In the next few paragraphs, we undertake a more careful analysis of the sets C_τ. The way we do this is to start by showing that if $\exp(\imath\theta) \in N(H)/H$ we can reduce the study of C_τ^θ to that of C_τ^+. We then show how the structure of C_τ^+ can sometimes be given in terms of the corresponding sets A_τ (see Remark 4.4.1).

Our first result shows how the set C_τ transform under change of generating set for $P_\Gamma(V,V)$. Very similar results can be found in [3, §5], [13, 11].

Lemma 4.4.5. *Let $\mathcal{F} = \{F_1,\ldots,F_k\}$, $\mathcal{G} = \{G_1,\ldots,G_k\}$ be two minimal sets of homogeneous generators for $P_\Gamma(V,V)$. Let $\mathcal{B}^{\mathcal{F}}$, $\mathcal{B}^{\mathcal{G}}$ denote the corresponding stratifications of $\mathbb{R}^k$. There is a linear strata preserving isomorphism $L:\mathbb{R}^k\to\mathbb{R}^k$ mapping $\mathcal{B}^{\mathcal{F}}$ onto $\mathcal{B}^{\mathcal{G}}$.*

Proof: Since $\mathcal{F}$ is a minimal set of homogeneous generators for $P_\Gamma(V,V)$, there exist $\alpha_{ij}\in P(V)^\Gamma$ such that

$$G_i(x) = \sum_{j=1}^k \alpha_{ij}(x)F_j(x),\ \ 1\le i\le k.$$

It follows from [12, Lemma 3.4] that the map $L(t) = (\sum_{i=1}^k t_i\alpha_{ij}(0))$ is a linear isomorphism of $\mathbb{R}^k$. Define $H: V\times\mathbb{R}^k\to V\times\mathbb{R}^k$ by $H(x,t) = (x, h_1(x,t),\ldots,h_k(x,t))$, where $h_j(x,t) = \sum_{i=1}^k t_i\alpha_{ij}(x)$, $1\le j\le k$. Then there is a Γ-invariant open neighborhood U of $\mathbb{R}^k$ such that $H|U$ is a Γ-equivariant analytic embedding onto an open neighborhood of $\mathbb{R}^k$. Let $\bar{F}: V\times\mathbb{R}^k\to\mathbb{R}^\ell$ denote the map defined by $\bar{F}(x,t) = P(\sum_{j=1}^k t_jF_j(x)) - P(x)$ and similarly define $\bar{G}$. It follows from our constructions that $\bar{G} = \bar{F}\circ H$. Hence, $H(\bar{G}^{-1}(0)\cap U) = \bar{F}^{-1}(0)\cap H(U)$. Since $H|U$ is an analytic isomorphism and the stratifications $\mathcal{S}^{\mathcal{F}},\mathcal{S}^{\mathcal{G}}$ are minimal, it follows that H is strata preserving. In particular, $H|\mathbb{R}^k = L$ will map $\mathcal{B}^{\mathcal{F}}$ onto $\mathcal{B}^{\mathcal{G}}$. □

Lemma 4.4.6. *Let $H\in\tau\in\mathcal{O}^\star$.*

(a) *If (V,Γ) is absolutely irreducible and $-I\in N(H)/H$, then $C_\tau^- = -C_\tau^+$.*
(b) *Let (V,Γ) be complex irreducible. Suppose that $\mathcal{F}$ is chosen so that $\mathcal{F}_V^H$ is $S^1\cap N(H)/H$-invariant (see Remarks 2.5.2). If $\exp(\imath\theta)\in N(H)/H$, then $\exp(\imath\theta)(C_\tau^0) = C_\tau^\theta$. In particular, if $S^1\subset N(H)/H$, then S^1 acts freely on C_τ and $\exp(\imath\theta)(C_\tau^\phi) = C_\tau^{\theta+\phi}$, all $\theta,\phi\in[0,2\pi)$.*

Proof: The result follows easily from Lemma 4.3.7. □

The next lemma is a straightforward application of standard existence and regularity theory for ordinary differential equations.

Lemma 4.4.7. *We may construct an open Γ-invariant neighborhood U of $\mathbf{C}\subset V\times\mathbb{R}^k$ and a smooth Γ-equivariant map $\Phi: U\times(-2,2)\to V$ such that for each $((x,t),s)\in U\times(-2,2)$, $\Phi((x,t),s)$ is the solution of the ordinary differential equation $\frac{dx}{ds} = F(x,t)$ with initial condition (x,t).*

Following the notation of Lemma 4.4.7, we define $\tilde{F}: U\to V$ by $\tilde{F}(x,t) = \Phi((x,t),1)$. That is, $\tilde{F}$ is the time-one map defined by the solutions of $\frac{dx}{ds} = F(x,t)$. Note that $\tilde{F}$ is a smooth equivariant map.

Lemma 4.4.8. *If (V,Γ) is absolutely irreducible, there exist smooth invariant functions $f_j: U\to\mathbb{R}$, $2\le j\le k$ such that $f_j(0,t) = O(|t_1|)$, $j\ge 2$, and*

$$\tilde{F}(x,t) = \exp(t_1)[x + \sum_{i=2}^k (t_j + f_j(x,t))F_j(x)].$$

Lemma 4.4.9. *Suppose that (V,Γ) is tangential. There exist smooth invariant functions $f_j: U\to\mathbb{R}$, $3\le j\le k$, satisfying*

(a) $f_j(x,t)$ *is independent of* t_2, $j \geq 3$.
(b) $f_j(0,t) = O(|t_1|)$, $j \geq 3$.
(c) $\tilde{F}(x,t) = \exp(t_1 + \imath t_2)[x + \sum_{i=3}^k (t_j + f_j(x,t))F_j(x)]$.

Proofs of Lemmas 4.4.8, 4.4.9: Lemma 4.4.8 is an elementary exercise in ordinary differential equations, the first step of which is to make the transformation $x(t) = \exp(t_1 t)u(t)$. The proof of Lemma 4.4.9 is similar. In this case, we make the transformation $x(t) = \exp((t_1 + \imath t_2)t)u(t)$ and use the S^1-equivariance to obtain an equation $u' = H(u,t)$, where H is independent of t_2. We omit details. □

Lemma 4.4.10. *Suppose that* (V,Γ) *is either absolutely irreducible or tangential. Let* $\tilde{\Xi} = \{(x,t) \in U \mid P(\tilde{F}(x,t)) = P(x)\}$. *If* $\tau \in \mathcal{O}^\star$, *then* $\tilde{\Xi}_\tau$ *is smooth near* $\mathbf{C}$ *and* $C_\tau = \partial\tilde{\Xi}_\tau \cap \mathbf{C}$.

Proof: Suppose first that (V,Γ) is absolutely irreducible. Noting that $\tilde{F}$ agrees with F at terms of lowest order, the result follows easily by a standard 'invariance' argument similar to that used in the proof of Lemma 4.4.5 (see also [3, §5], [11, Proposition]). If (V,Γ) is tangential, we start by observing that the S^1-equivariance implies that $\tilde{\Xi} = \{(x,t) \mid P(\tilde{F}(x,(t_1,0,t_3,\ldots,t_k)) = P(x)\}$. The result then follows, just as before, using the expression for $\tilde{F}$ given by Lemma 4.4.9. □

For $x \in V$, let $d(x) = \dim(V^{\Gamma_x})$. Given $\tau \in \mathcal{O}$, $x \mapsto d(x)$ is constant on V_τ and we set $d_\tau = d(x)$, $x \in V_\tau$. If $t \in \mathbb{R}^k$, we let $t-1$ be the point with coordinates $(t_1 - 1, t_2, \ldots, t_k)$. We similarly define $t+1$ and, more generally, $C \pm 1$ for any subset C of $\mathbb{R}^k$.

Proposition 4.4.11. *Let* $H \in \tau \in \mathcal{O}^\star$.

(a) *If* (V,Γ) *is absolutely irreducible, then* $C_\tau^+ =\in A_\tau + 1$. *In particular,* $C_\tau^+ - 1$ *is a closed semi-algebraic conical subset of* $\mathbb{R}^k$ *and* $k - d_\tau + n_\tau \leq \mathit{dim}(C_\tau^+) \leq k-1$. *If* $-I \in N(H)/H$, *then* $C_\tau^- = A_\tau - 1$.
(b) *Suppose that* (V,Γ) *is complex irreducible and set* $A_\tau^0 = \{t \in A_\tau \mid t_2 = 0\}$. *If* $\exp(\imath\theta) \in S^1 \cap N(H)/H$, *then* $C_\tau^\theta = \exp(\imath\theta)(A_\tau^0 + 1)$.
(c) *If* (V,Γ) *is tangential or* $S^1 \subset N(H)/H$, *then*
$$C_\tau = \{\exp(\imath s)(t+1) \mid t \in A_\tau^0, s \in \mathbb{R}\}$$
and $k - d_\tau + n_\tau \leq \mathit{dim}(C_\tau) \leq k-1$.

Proof: If Γ is finite, (a,b,c) follow easily from Lemma 4.3.7 and [21, Lemma 4.3.9]. Suppose Γ is not finite and (V,Γ) is absolutely irreducible. It follows from Lemmas 4.4.5, 4.4.6, and [21, Lemma 4.3.9] that it suffices to show that $C_\tau^+ - 1 = A_\tau$. Now $C_\tau^+ = \partial\tilde{\Xi}_\tau \cap \mathbb{R}^k \cap \mathbf{C}_+$ (Lemma 4.4.10) and so we may work with the variety $\tilde{\Xi}$. Suppose $t \in C_\tau^+$, $H \in \tau$ and (x^n, t^n), (γ^n) are sequences in $\tilde{\Xi}_\tau^H$, $N(H)/H$ such that

$$\begin{aligned} \tilde{F}(x^n,t^n) &= \gamma^n x^n \text{ and} \\ ((x^n,t^n),\gamma^n) &\rightarrow ((0,t),I) \in (\{0\} \times \mathbb{R}^k) \times N(H)/H \end{aligned}$$

Denote the Lie algebra of $N(H)/H$ by $\mathfrak{k}$. We may choose a sequence $(k^n) \subset \mathfrak{k}$ so that $k^n \rightarrow 0$ and $\gamma^n = \exp(k^n)$ for sufficiently large n (that is, if $\gamma^n \in (N(H)/H)^0$). It follows from the definition of $\tilde{F}$ and equivariance, that

$$F(x^n, t^n - 1) = \frac{d}{ds}\exp(sk^n)x^n|_{s=0}.$$

Hence $(x^n, t^n) \in \tilde{\Xi}$ if and only if $(x^n, t^n - 1) \in \Sigma$. Letting $n \to \infty$, it follows that $C_\tau^+ - 1 = A_\tau$. In particular, $C_\tau^- + 1$ is conical. Similar arguments apply when (V, Γ) is complex or tangential. □

Remarks 4.4.12. (1) For the last part of (a), it suffices to assume that $F_1, \ldots, F_k$ restrict to odd (possibly zero) maps on V^H. It follows that the restriction of any homogeneous invariant to V^H is either zero or of even degree. Since invariants separate Γ-orbits, it follows that $-I \in N(H)/H$. (2) In general, C_τ^- may be empty. If $C_\tau^- \neq \emptyset$ and the conditions of (a) do not hold, we may ask whether $C_\tau^- + 1$ is a conical subset of $\mathbb{R}^k$. This seems possible, at least if there are no quadratic equivariants. ♢

4.5. **Stability theorems I: Weak regularity.** Suppose $f \in \mathcal{M}(V, \Gamma)$ and write

$$f(x, \lambda) = \sum_{j=1}^{k} f_j(x, \lambda) F_j(x)$$

where $f_j \in C^\infty(V \times \mathbb{R})^\Gamma$, $1 \leq j \leq k$. Define smooth maps $\mathrm{graph}_f : V \times \mathbb{R} \to V \times \mathbb{R}^k$ and $\gamma_f : \mathbb{R} \to \mathbb{R}^k$ by

$$\begin{aligned} \mathrm{graph}_f(x, \lambda) &= (x, (f_1(x, \lambda), \ldots, f_k(x, \lambda)) \\ \gamma_f(\lambda) &= (f_1(0, \lambda), \ldots, f_k(0, \lambda)) \end{aligned}$$

Since $f = F \circ \mathrm{graph}_f$, it follows that $\mathbf{I}(f) = \mathrm{graph}_f^{-1}(\Xi)$.

In the sequel, we frequently talk about transversality of maps to semi-algebraic sets. If no stratification is specified, it will always be understood that transversality is meant with respect to the canonical semi-algebraic stratification of the set. More generally, if a semi-algebraic set C comes with a Whitney stratification $\mathcal{T}$ (not necessarily canonical), we say that a map is transverse to $\mathcal{T}$ if it is transverse to C, given the stratification $\mathcal{T}$. We use the notation $f \pitchfork C$ (or $f \pitchfork \mathcal{T}$) to indicate that f is transverse to C (or $\mathcal{T}$).

Define

$$\begin{aligned} \mathcal{L}_\Gamma(V) &= \{f \in \mathcal{M}(V, \Gamma) \mid \mathrm{graph}_f \pitchfork \Xi \text{ at } \lambda = 0\} \\ \mathcal{L}_\Gamma^\pm(V) &= \{f \in \mathcal{L}_\Gamma(V) \mid f_1(0, 0) = \pm 1\} \\ \mathcal{L}_\Gamma^\theta(V) &= \{f \in \mathcal{L}_\Gamma(V) \mid f_1(0, 0) + \imath f_2(0, 0) = \exp(\imath\theta)\} \end{aligned}$$

The methods of Bierstone [3] and Field [12] may be used to show that $\mathcal{L}_\Gamma(V)$, together with the subsets defined above, are independent of the choices of generators for $P(V)^\Gamma$, $P_\Gamma(V, V)$ and coefficient functions $f_1, \ldots, f_k$ implicit in the definition of graph_f. Just as in [21, §4], it may be shown that the map γ_f is uniquely defined once $\mathcal{F}$ has been chosen.

Proposition 4.5.1. *Let $f \in \mathcal{M}(V, \Gamma)$. The following conditions on f are equivalent.*

(1) $f \in \mathcal{L}_\Gamma(V)$.
(2) $\gamma_f \pitchfork \mathcal{B}$ *at* $\lambda = 0$.
(3) $\gamma_f(0) \in \mathcal{B}_i$, *where* $i \geq k - 1$.

Proof: Since $f \in \mathcal{M}(V, \Gamma)$, $\gamma_f(0) \in \mathbf{C}$ and $\gamma_f \pitchfork \mathbf{C}$ at $\lambda = 0$. We prove the equivalence of (1) and (2) (see [21, Proposition 4.4.3] for details on the analogous results for vector fields). Since $\mathcal{B} \subset \mathcal{S}$, and all strata of $\mathcal{S}$ which meet $\mathbb{R}^k$ lie in $\mathcal{B}$, it follows by the Whitney regularity of the stratification $\mathcal{S}$ that $\text{graph}_f \pitchfork \mathcal{S}$ at $\lambda = 0$ if and only if $\text{graph}_f \pitchfork \mathcal{B}$ at $\lambda = 0$. Noting the definition of graph_f, we see that $\text{graph}_f \pitchfork \mathcal{B}$ at $\lambda = 0$ if and only if $\gamma_f \pitchfork \mathcal{B}$ at $\lambda = 0$. Hence (1) and (2) are equivalent. □

As an immediate consequence of Proposition 4.5.1 and standard properties of maps transversal to Whitney stratified sets we have

Proposition 4.5.2. (1) *$\mathcal{L}_\Gamma(V)$ is an open and dense subset of $\mathcal{M}(V, \Gamma)$.*
(2) *Every $f \in \mathcal{L}_\Gamma(V)$ is weakly regular.*

Theorem 4.5.3. *Let $f \in \mathcal{L}_\Gamma(V)$. Then*
(1) *If codim$(C_\tau) \geq 2$, the germ of $\mathbf{I}(f)$ at zero contains no points of isotropy type τ.*
(2) *If codim$(C_\tau) = 1$ and $\gamma_f(0) \in C_\tau$, there is a branch of invariant group orbits of isotropy type τ for f at zero.*
(3) *The map $\gamma_f : \mathbb{R} \to \mathbb{R}^k$ is transverse to the canonical stratification of C_τ for all $\tau \in \mathcal{O}$.*

Similar results hold if we replace C_τ by $C_\tau^\pm$. Finally, if (V, Γ) is complex irreducible and $C_\tau^\theta \neq \emptyset$ for only finitely many values of θ, then codim$(C_\tau) \geq 2$ and so the germ of $\mathbf{I}(f)$ at zero contains no points of isotropy type τ.

Proof: The only result that is not immediate by transversality is (2). That is, we have to prove the branch is C^1. If $H \in \tau$ and $N(H)/H$ is finite, we follow the method used in the proofs of [22, Proposition 5.16] or [21, Theorem 4.4.5]. Otherwise, we use methods based on blowing-up [21, §9]. □

Remark 4.5.4. Just as for vector fields, it follows from our results that if $f \in \mathcal{L}_\Gamma(V)$ and $f(x, \lambda) = \sum_{i=1}^k f_i(x, \lambda) F_i(x)$, then the branching pattern $\Xi(f)$ is determined completely by $(f_1(0,0), f_2(0,0), \ldots, f_k(0,0))$. In particular, the branching pattern will be determined by the d_k-jet of f_0 at the origin. ♢

Definition 4.5.5 (cf. [22, 24, 21])**.** Let $\tau \in \mathcal{O}^\star$. If $\sigma \in \{+, -\}$ and (V, Γ) is absolutely irreducible, we say that τ is *σ-symmetry breaking* (respectively, *generically σ-symmetry breaking*) if there exists a non-empty open (respectively, open and dense) subset U of $\mathcal{L}_\Gamma^\sigma(V)$ such that for every $f \in U$, the germ of $\mathbf{I}(f)$ at zero contains points of isotropy type τ. If (V, Γ) is complex irreducible, we say that τ is *symmetry breaking* (respectively, *generically symmetry breaking*) if there exists a non-empty open (respectively, open and dense) subset U of $\mathcal{L}_\Gamma(V)$ such that for every $f \in U$, the germ of $\mathbf{I}(f)$ at zero contains points of isotropy type τ.

As an immediate consequence of our results so far, we have

Proposition 4.5.6. *Let $\tau \in \mathcal{O}^\star$ and suppose that (V, Γ) is absolutely irreducible and $\sigma \in \{+, -\}$. Then τ is a σ-symmetry breaking isotropy type if and only if codim$(C_\tau^\sigma) = 1$. Moreover, τ is generically σ-symmetry breaking if and only if $C_\tau^\sigma = \mathbf{C}_\sigma$. A similar result holds if (V, Γ) is complex irreducible.*

It follows from Proposition 4.5.2 that $\mathcal{L}_\Gamma(V)$ is an open and dense subset of $\mathcal{M}(V, \Gamma)$ consisting of weakly regular families. The next theorem gives a weak version of stability for families in $\mathcal{L}_\Gamma(V)$.

Theorem 4.5.7. *Let $f \in \mathcal{L}_\Gamma(V)$ and $\{f_t \mid t \in [0,1]\}$ be a continuous path in $\mathcal{L}_\Gamma(V)$ such that $f_0 = f$. Then there exists a compact Γ-invariant neighborhood A of $(0,0) \in V$ and a continuous Γ-equivariant isotopy $K : A \times [0,1] \to V \times \mathbb{R}$ of embeddings such that*

(1) $K_0 = I_V$.
(2) $K_t(A \cap \mathbf{I}(f)) = \mathbf{I}(f_t) \cap K_t(A)$, $t \in [0,1]$.

Proof: The result follows from Thom's first isotopy lemma (see [21, Theorem 4.4.11] for further details). □

As an immediate corollary of Theorem 4.5.7, we have

Proposition 4.5.8. *Let $\tau \in \mathcal{O}^\star$ and suppose that (V, Γ) is absolutely irreducible and $\sigma \in \{+,-\}$. Then τ is σ-symmetry breaking if and only if there exists $f \in \mathcal{L}^\sigma_\Gamma(V)$ such that $\Xi(f)$ contains a branch of isotropy type τ. Further τ is generically σ-symmetry breaking if and only if for every $f \in \mathcal{L}^\sigma_\Gamma(V)$, $\Xi(f)$ contains a branch of isotropy type τ. Similar results hold for the case of complex irreducible representations.*

4.6. **Stability theorems II: Regular families.** In this section we extend results of Field [16, 21] and show that $\mathcal{S}(V,\Gamma)$ is an open and dense subset of $\mathcal{M}(V,\Gamma)$ for all absolutely irreducible and complex irreducible representations (V,Γ). Most of what we say is a straightforward extension of the theory and methods in [16, 21] and so we shall only give brief details of proofs.

Let $\tau \in \mathcal{O}(V,\Gamma)$. Define

$$Z_0(\tau) = \{(x,y) \in V \times V \mid x \in V_\tau,\ y \in N(\Gamma_x)\cdot x\}$$

and set

$$Z_0 = \text{closure}(\bigcup_{\tau \in \mathcal{O}} Z_0(\tau))$$

Lemma 4.6.1 ([21, Lemma 4.5.2])**.** *Each set $Z_0(\tau)$ is a smooth Γ-invariant semi-algebraic subset of $V \times V$, $\tau \in \mathcal{O}$. In particular, Z_0 is semi-algebraic.*

Proof: We prove that $Z_0(\tau)$ is a Γ-invariant semi-algebraic set. Let $H \in \tau$ and set $Z_0(H) = Z_0(\tau) \cap (V^H \times V^H)$. Since $Z_0(\tau) = \Gamma \cdot Z_0(H)$, it suffices to prove that $Z_0(H)$ is a smooth semi-algebraic subset of $V^H \times V^H$. But $Z_0(H) = \{(x,y) \mid Q(x) = Q(y)\}$, where $Q : V^H \to V^H/N(H)$ denotes the orbit map, and so $Z_0(H)$ is semi-algebraic. □

For each $\tau \in \mathcal{O}$, let $L_\tau(V,V)$ denote the semi-algebraic subset of $L(V,V)$ consisting of maps that have at least $g_\tau + 1$ eigenvalues of unit modulus (counting multiplicities).

We define sets $Z_1(\tau) \subset V_\tau \times V_\tau \times L(V,V)$ and $Z_1 \subset V \times V \times L(V,V)$ by

$$\begin{aligned} Z_1(\tau) &= \{((x,y),A) \mid \exists \gamma \in N(\Gamma_x) \text{ such that } y = \gamma x \text{ and } \gamma^{-1}A \in L_\tau(V,V)\} \\ Z_1 &= \overline{(\bigcup_{\tau \in \mathcal{O}} Z_1(\tau))} \end{aligned}$$

Lemma 4.6.2. (1) *For every $\tau \in \mathcal{O}$, $Z_1(\tau)$ is a Γ-invariant semi-algebraic subset of $V \times V \times L(V,V)$.*
(2) *Z_1 is a closed Γ-invariant semi-algebraic subset of $V \times V \times L(V,V)$.*

Proof: Let $H \in \tau$ and define $\tilde{Z}_1(H)$ to be the semi-algebraic subset of $V^H_\tau \times N(H) \times L(V,V)$ consisting of points (x,γ,A) such that $\gamma^{-1}A \in L_\tau(V,V)$. It follows that $\tilde{Z}_1(\tau) = \Gamma \cdot \tilde{Z}_1(H)$ is a Γ-invariant semi-algebraic set. But now there is a natural semi-algebraic map of $\tilde{Z}_1(\tau)$ onto $Z_1(\tau)$ defined by mapping (x,γ,A) to $((x,\gamma x),A)$. Hence $Z_1(\tau)$ is semi-algebraic, proving (1). Statement (2) follows from (1) and the fact that closures and finite unions of semi-algebraic sets are semi-algebraic. □

Let $J^1(V,V)$ denote the space of 1-jets of maps from V to V. Note that $J^1(V,V)$ inherits the structure of a Γ-space from V. Recall that $J^1(V,V) \cong V \times V \times L(V,V)$ and that if $f \in C^\infty_\Gamma(V,V)$ then $j^1 f(x) = (x, f(x), Df(x))$ under this isomorphism.

From now on, we regard Z_1 and $Z_1(\tau)$ as defining semi-algebraic subsets of $J^1(V,V)$.

We turn next to families of maps. The jet space $J^1(V \times \mathbb{R}, V)$ is isomorphic (as a Γ-representation) to $J^1(V,V) \oplus (\mathbb{R} \times V)$. Thus, if $f \in C^\infty_\Gamma(V \times \mathbb{R}, V)$, then

$$\begin{aligned} j^1 f(x,\lambda) &= ((x,\lambda), f(x,\lambda), Df(x,\lambda)) \in J^1(V \times \mathbb{R}, V) \\ &= ([x, f(x,\lambda), D_1 f(x,\lambda)], [\lambda, D_2 f(x,\lambda)]) \in J^1(V,V) \times (\mathbb{R} \times V) \end{aligned}$$

Let $\Pi : J^1(V \times \mathbb{R}, V) \rightarrow J^1(V,V)$ denote the associated projection map. Set $Z_1^1 = \Pi^{-1}(Z_1)$. Since Z_1^1 may be identified with $Z_1 \times (\mathbb{R} \times V)$, Z_1^1 is a closed semi-algebraic subset of $J^1(V \times \mathbb{R}, V)$.

Lemma 4.6.3. *Let $f \in C^\infty_\Gamma(V \times \mathbb{R}, V)$. Then*

(1) $\mathbf{I}(f) = \{(x,\lambda) \mid f_\lambda(x) \in Z_0\} = \{(x,\lambda) \mid \mathit{graph}_f(x,\lambda) \in \Xi\}$.
(2) $\mathbf{B}(f) = (j^1 f)^{-1} Z_1^1$.

Proof: Similar to that of [21, Lemma 4.6.9]. □

Just as in [21, §4], [16, Appendix], we may express genericity conditions on maps or families in terms of equivariant jet transversality conditions. We briefly summarize the main results that we shall need on equivariant jet transversality. (We refer the reader to [4], [21, §4] for more details.) Suppose that (V_i, Γ), $i = 1,2$, are Γ-representations and that Q is a Γ-invariant closed semi-algebraic subset of $J^1(V_1, V_2)$. If $f \in C^\infty_\Gamma(V_1, V_2)$, and $A \subset V_1$ is compact and Γ-invariant, we write "$j^1 f \pitchfork_\Gamma Q$ on A" to signify that $j^1 f : V_1 \rightarrow J^1(V_1, V_2)$ is in equivariant general position to Q on A. We recall from Bierstone [4] that $\{f \mid j^1 f \pitchfork_\Gamma Q \text{ on } A\}$ is an open and dense subset of $C^\infty_\Gamma(V_1, V_2)$. Moreover, the usual isotopy and stability theorems hold (for precise statements, see [4, Theorems 7.6, 7.7, 7.8]).

As an immediate consequence of our definitions, we have

Lemma 4.6.4 (cf. [21, Lemma 4.6.10])**.** *Let $f \in C^\infty_\Gamma(V,V)$. The following conditions on f are equivalent:*

(1) *All f-invariant Γ-orbits are normally hyperbolic.*
(2) $j^1 f(V) \cap Z_1 = \emptyset$.
(3) $j^1 f \pitchfork_\Gamma Z_1$ *on V.*

Theorem 4.6.5. *Let*

$$\mathcal{S}_0(V,\Gamma) = \{f \in \mathcal{L}_\Gamma(V) \mid j^1 f \pitchfork_\Gamma Z_1^1 \text{ at } (x,\lambda) = (0,0)\}$$

Then

(a) $\mathcal{S}_0(V,\Gamma)$ *is an open and dense subset of* $\mathcal{M}(V,\Gamma)$.
(b) $\mathcal{S}_0(V,\Gamma) \subset \mathcal{S}(V,\Gamma)$.

Proof: Statement (a) follows from Bierstone's jet transversality theorem. The proof of (b) is very similar to that of the corresponding Theorem 4.6.11 in [21]. We start by observing that if $f \in \mathcal{S}_0(V,\Gamma)$, then we can find a compact Γ-invariant neighborhood A of $(0,0) \in V \times \mathbb{R}$ such that $j^1 f \pitchfork_\Gamma Z_1^1$ on A. Using the definitions of equivariant general position and Z_1^1, it may be shown that $(j^1 f|A)^{-1}(Z_1^1)$ has the structure of a Whitney regular stratified set. It suffices to prove that the origin is an *isolated point* in $(j^1 f|A)^{-1}(Z_1^1)$. If not, there is a continuous non-constant arc in $(j^1 f|A)^{-1}(Z_1^1)$, with initial point at the origin. We derive a contradiction using the openness of equivariant transversality together with Lemma 4.6.4. □

Remark 4.6.6. Using methods similar to those in [4, 12], one can show that $\mathcal{S}_0(V,\Gamma)$ is defined independently of choices of generating sets. ♢

4.7. **Determinacy.** In this section, we indicate how the determinacy theorems of [21, §4], [16, Appendix] can be extended from vector fields to smooth maps. As the methods are completely analogous to those used in [21, 16], we only give very brief details.

We regard Z_0 as a subset of $J^1(V,V)$ and define $Z_0^1 = \Pi^{-1}(Z_0) \subset J^1(V \times \mathbb{R}, V)$. Clearly $Z_1^1 \subset Z_0^1$. It follows that $f \in \mathcal{S}_0(V,\Gamma)$ if and only if

$$j^1 f \pitchfork_\Gamma Z_0^1 \text{ and } j^1 f \pitchfork_\Gamma Z_1^1 \text{ at } (0,0)$$

Just as in [21, 16], we construct a Whitney regular stratification $\mathcal{T}$ of Z_0^1 that induces a Whitney regular stratification of Z_1^1 (we do not know whether this stratification always coincides with the canonical stratification of Z_1^1). We define

$$\mathcal{S}_1(V,\Gamma) = \{f \mid f \pitchfork_\Gamma \mathcal{T} \text{ at } (0,0)\},$$

and note that $\mathcal{S}_1(V,\Gamma) \subset \mathcal{S}(V,\Gamma)$. Just as in [21, Lemma 4.7.9], we may give a simple characterization for membership in $\mathcal{S}_1(V,\Gamma)$. Using this, together with the standard technique for proving density (see the proof of [21, Theorem 4.7.10]) we may prove

Theorem 4.7.1. *Let (V,Γ) be an absolutely irreducible or complex irreducible representation. There exists $q > 0$ such that Γ-equivariant bifurcation problems on V are q-determined.*

4.8. **An invariant sphere theorem and Fiedler's theorem for maps.** Given $f : V \times \mathbb{R} \to V$ and $a \in \mathbb{R}$, we define $f^a(x,t) = f(x,t) + a|x|^2 x$. Using techniques based on the persistence of normally hyperbolic sets, similar to those used in the proof of [16, Theorem 5.2], we may prove the following *invariant sphere theorem* for maps (see also [15, §4]).

Theorem 4.8.1. *Let (V,Γ) be irreducible and suppose $dim_R(V) = m$. Let $r > 0$. Assume that $P^2_\Gamma(V,V) = \{0\}$ and let $f \in \mathcal{M}(V,\Gamma)$. Then we may find $a_0 \in \mathbb{R}$ such that if $a \le a_0$, then there exist $\varepsilon > 0$, a neighborhood U of $0 \in V$, and a continuous family $S_\lambda : S^{m-1} \to U$, $\lambda \in [0,\varepsilon]$, satisfying the following properties.*

1. *$S_\lambda : S^{m-1} \to U$ is a Γ-equivariant C^r-embedding, $\lambda > 0$, and S_0 is the zero map.*
2. *If we set $S(\lambda) = S_\lambda(S^{m-1})$, then $S(\lambda)$ is f^a_λ-invariant.*
3. *If $x \in U \setminus \{0\}$, then the f^a_λ-orbit of x is forward asymptotic to $S(\lambda)$.*

Example 4.8.2 (cf. [10], [21, §11])**.** Suppose (V,Γ) is complex irreducible and let $\dim_R(V) = 2n$. Set $G = \Gamma \times S^1$ and consider the tangential representation (V,G). If $f \in \mathcal{L}_G(V)$, it follows from Theorem 4.5.3 that $f^a \in \mathcal{L}_G(V)$ and $\Xi(f) = \Xi(f^a)$ for all $a \in \mathbb{R}$. Since $P^2_G(V,V) = \{0\}$, we may choose $a_0 \in \mathbb{R}$ so that if $a \le a_0$ then f^a spawns a branch $\{S(\lambda) \mid \lambda \in [0,\varepsilon]\}$ of G- and f^a-invariant C^1-embedded spheres. Set $f^a = g$. Since S^1 acts freely on each $S(\lambda)$, g_λ-induces a C^1 G-equivariant map on the $(2n-2)$-dimensional complex projective space $S(\lambda)/S^1$, $\lambda \in (0,\varepsilon]$. Let $\tau \in \mathcal{O}(V,G)$ be a maximal isotropy type and let $H \in \tau$. For each $\lambda \in (0,\varepsilon]$, g_λ restricts to an $N(H) \times S^1$-equivariant C^1-mapping of $S(\lambda) \cap V^H$ and so induces a

C^1-mapping g_λ^H on the associated projective space. Each g_λ^H is clearly $N(H)\times S^1$-equivariantly homotopic to the identity map. Since the Euler characteristic of every complex projective space is nonzero, it follows that the Lefschetz number $L(g_\lambda^H) \neq 0$. Hence, g_λ^H has at least one fixed point for each $\lambda \in (0,\varepsilon]$. Consequently, g_λ has an invariant G-orbit of isotropy type τ for each $\lambda \in (0,\varepsilon]$. It follows that $C_\tau = \mathbf{C}$. That is, every G-maximal isotropy type is generically symmetry breaking. Using the strong determinacy theorem (see §6), we may ask what happens if we break symmetry from G to Γ. Unlike the case of vector fields [21, §11], complicated dynamics may appear [6]. ♡

5. Examples for Γ finite

In this section, Γ is always assumed finite. We discuss some examples of symmetry breaking closely related to those studied in Field and Richardson [22, 25].

5.1. **Preliminaries.** Suppose (V,Γ) is complex irreducible. It follows from Theorem 4.5.3 that $\mathrm{codim}(C_\tau) \geq 2$ for all $\tau \in \mathcal{O}$, $\tau \neq (\Gamma)$. Consequently, no branches of invariant orbits bifurcate off the trivial branch for generic maps in $\mathcal{M}(V,\Gamma)$. Hence we may assume (V,Γ) is absolutely irreducible. Since Γ is finite it follows that if $f \in \mathcal{L}^+(V,\Gamma)$ then $\Xi(f)$ consists of branches of *fixed points*. If $f \in \mathcal{L}^-(V,\Gamma)$, then $\Xi(f)$ consists of branches of Γ-orbits consisting of points of prime period 2 for f.

5.2. **Subgroups of the Weyl group of type B_n.** Assume $n \geq 3$. Let S_n denote the group of $n \times n$ permutation matrices and Δ_n denote the group of diagonal matrices with entries ± 1. The Weyl group $W(B_n)$ is the semi-direct product $\Delta_n \rtimes S_n$. Set $W_n = W(B_n)$ and recall that $(\mathbb{R}^n, W_n)$ is absolutely irreducible.

Definition 5.2.1 (cf. [25, Conditions 13.5.1-2])**.** Suppose $n \geq 3$. Let $\mathcal{W}_n$ denote the set of representations $(\mathbb{R}^n,\Gamma)$ satisfying

(1) Γ is a subgroup of W_n.
(2) $(\mathbb{R}^n,\Gamma)$ is absolutely irreducible.
(3) $P^2_\Gamma(\mathbb{R}^n,\mathbb{R}^n) = \{0\}$.

Example 5.2.2. If H is a subgroup of $O(n)$, we let H' denote the determinant one subgroup of H. We recall from [25, §14], that if T is a transitive subgroup of S_n then, with the single exception of the tetrahedral group $\mathbb{T} = \Delta_3' \rtimes \mathbb{Z}_3$, we have

$$(\mathbb{R}^n, \Delta_n \rtimes T),\ (\mathbb{R}^n, \Delta_n' \rtimes T),\ (\mathbb{R}^n, (\Delta_n \rtimes T)') \in \mathcal{W}_n.$$

5.3. **Branches of fixed points.** Let $\mathcal{E}$ denote the set of non-zero vectors $\varepsilon \in \mathbb{R}^n$ such that $\varepsilon_i \in \{0,+1,-1\}$, $1 \leq i \leq n$. Given $(\mathbb{R}^n,\Gamma) \in \mathcal{W}_n$, let $\mathcal{O}_S = \{\iota(\varepsilon) \mid \varepsilon \in \mathcal{E}\}$. It was shown in [25] that an isotropy type τ was symmetry breaking for 1-parameter families of vector fields if and only if $\tau \in \mathcal{O}_S$. In view of this result and Lemma 4.4.11(1), we have the following simple characterization of $+$-symmetry breaking isotropy types for representations in the class $\mathcal{W}_n$.

Proposition 5.3.1. *Let $(\mathbb{R}^n,\Gamma) \in \mathcal{W}_n$. Then $\tau \in \mathcal{O}$ is $+$-symmetry breaking if and only if $\tau \in \mathcal{O}_S$.*

As an easy consequence of results in [24, 25], together with Proposition 4.4.11, we have the following determinacy result.

Lemma 5.3.2. *There is a maximal open and dense semi-algebraic subset $\mathcal{R}$ of $P^3_\Gamma(\mathbb{R}^n,\mathbb{R}^n)$ such that*

(1) *If $f \in \mathcal{M}^+(\mathbb{R}^n, \Gamma)$ and $D^3 f_0(0) \in \mathcal{R}$, then f is stable.*
(2) *If $\mu \in \mathbb{R} \setminus \{0\}$, then $\mu\mathcal{R} = \mathcal{R}$.*

5.4. **Branches of invariant orbits consisting of period two points.**

Lemma 5.4.1. *Let $(\mathbb{R}^n, \Gamma) \in \mathcal{W}_n$.*

(1) *Let $H \in \tau \in \mathcal{O}_S$. Then τ is $-$-symmetry breaking if $-I \in N(H)/H$.*
(2) *If $-I \in \Gamma$, then $\tau \in \mathcal{O}$ is $-$-symmetry breaking if and only if $\tau \in \mathcal{O}_S$.*

Proof: The result follows immediately from Lemma 4.4.6(a). □

Lemma 5.4.2. *Let $(\mathbb{R}^n, \Gamma) \in \mathcal{W}_n$ and suppose $-I \in \Gamma$. Let $\mathcal{R} \subset P^3_\Gamma(\mathbb{R}^n, \mathbb{R}^n)$ be the subset given by Lemma 5.3.2. If $f \in \mathcal{M}(\Gamma, \mathbb{R}^n)$ and $D^3 f_0(0) \in \mathcal{R}$, then f is stable. In particular, Γ-equivariant bifurcation problems on $\mathbb{R}^n$ are 3-determined.*

Proof: It suffices to show that if $f \in \mathcal{M}^-(\mathbb{R}^n, \Gamma)$ and $D^3 f_0(0) \in \mathcal{R}$, then f is stable. Suppose that $F_2, \ldots, F_r$ are the cubic equivariants in $\mathcal{F}$ and set $F_{r+1}(x) = |x|^2 x$. Then $F_2, \ldots, F_{r+1}$ define a basis for $P^3_\Gamma(\mathbb{R}^n, \mathbb{R}^n)$. Suppose $f \in \mathcal{M}^-(\mathbb{R}^n, \Gamma)$. Then

$$D^3 f_0(0)(x) = \sum_{i=2}^{r+1} a_i F_i(x),$$

where $a_2, \ldots, a_{r+1} \in \mathbb{R}$. A simple computation verifies that

$$D^3 f_0^2(0)(x) = \sum_{i=2}^{r+1} -2a_i F_i(x).$$

Hence, by Lemma 5.3.2(2), $D^3 f_0^2(0) \in \mathcal{R}$ if and only if $D^3 f_0(0) \in \mathcal{R}$. □

We extend our results to cover the case where $-I \notin \Gamma$. Let $\mathbb{Z}_2 \subset O(n)$ be the subgroup generated by $-I$ and set $\Gamma^2 = \Gamma \times \mathbb{Z}_2$. Note that if $-I \in \Gamma$, then $(-I, -I) \in \Gamma^2$ fixes every point of $\mathbb{R}^n$. If we identify Γ^2 with its image in $O(n)$, then $(\mathbb{R}^n, \Gamma^2) \in \mathcal{W}_n$. Let $\mathcal{O}_S^2$ be the set of $+$-symmetry breaking isotropy types for Γ^2. If $\tilde{H} \in \tau \in \mathcal{O}_S^2$, then $(\tilde{H} \cap \Gamma) \in \mathcal{O}_S$. This construction defines a natural map $\Pi : \mathcal{O}_S^2 \to \mathcal{O}_S$. Note that if x has Γ^2-isotropy τ, then $\Pi(\tau)$ is the Γ-isotropy of x.

Proposition 5.4.3. *Suppose $(\mathbb{R}^n, \Gamma) \in \mathcal{W}_n$.*

(1) *If τ is $-$-symmetry breaking then $\tau \in \mathcal{O}_S$.*
(2) *If $D^3 f_0(0) \in \mathcal{R}$, then f is stable.*
(3) *If $\tau \in \mathcal{O}_S$, τ will be $-$-symmetry breaking if there exists $\eta \in \Pi^{-1}(\tau)$ such that $\eta \neq \tau$. (That is, if we choose $x \in V_\eta$, then $[\Gamma^2_x : \Gamma_x] = 2$.)*

Proof: If $-I \in \Gamma$, the result follows from Lemma 5.4.2 so we may suppose $-I \notin \Gamma$. Since $\Gamma^2 \in \mathcal{W}_n$ and $-I \in \Gamma^2$, Lemma 5.4.2 applies to $(\mathbb{R}^n, \Gamma^2)$. Since $(\mathbb{R}^n, \Gamma^2)$ and $(\mathbb{R}^n, \Gamma)$ have the *same* cubic equivariants and quadratic equivariants are trivial, (1,2) of the Proposition follow easily. It remains to prove (3). Let τ, η satisfy the conditions of (3). Suppose that $f \in \mathcal{M}^-(\mathbb{R}^n, \Gamma^2)$, $D^3 f_0(0) \in \mathcal{R}$ and f has a curve $\phi = (\rho, \lambda)$ of $-I$-invariant points of prime period two and isotropy type η. Denote the initial direction $\rho'(0)$ of the curve by $\pm u \in S^{n-1}$ (we define the initial direction only up to ± 1 to allow for the reverse parametrization of the branch). Since $D^3 f_0(0) \in \mathcal{R}$, it is easy to verify that the initial direction u depends only on $D^3 f_0(0)$. Consequently, if we take any Γ-equivariant perturbation of f by terms of order at least four, the resulting perturbed curve $\tilde{\phi}$ of period two points will have the *same* initial direction $\pm u$. It follows from our hypotheses on τ, η, that there exists $\gamma \notin \Gamma_u$ such that $-\gamma \in \Gamma_u^2$. By Γ-equivariance, $\gamma\tilde{\phi}$ must also be a branch of points of period two. Since $\gamma\tilde{\phi}$ has the same initial direction $\pm u$ as $\tilde{\phi}$, it follows that $\gamma\tilde{\phi} = \tilde{\phi}$ (with reverse parametrization) and so τ is $-$-symmetry breaking. Conversely, every $-$-symmetry breaking isotropy type arises in this way. □

Using our results in combination with the techniques of [24], we may easily verify strong 3-determinacy for representations in $\mathcal{W}_n$. In summary, we have proved

Theorem 5.4.4. *Let* $(\mathbb{R}^n, \Gamma) \in \mathcal{W}_n$. *Then* Γ*-equivariant bifurcation problems on* $\mathbb{R}^n$ *are strongly* 3*-determined.*

Example 5.4.5. Let $(\mathbb{R}^5, \Gamma) \in \mathcal{W}_5$, where $\Gamma = \Delta_5' \rtimes \mathbb{Z}_5$. It follows from [25, §14] and Proposition 5.3.1 that all isotropy types in $\mathcal{O}^\star(\mathbb{R}^5, \Gamma)$ are $+$-symmetry breaking. It follows either directly or using Proposition 5.4.3 that the maximal isotropy types $\iota(1, \dots, 1, \pm 1)$ are *not* $-$-symmetry breaking. On the other hand, the trivial isotropy type $\iota(1, 1, 1, 1, 0)$ is $-$-symmetry breaking. In this case, branches of invariant orbits will be tangent to the Γ-orbit of the plane $x_5 = 0$. All of the remaining isotropy types satisfy the conditions of Lemma 5.4.1 and so are also $-$-symmetry breaking. ♡

Remark 5.4.6. Additional examples of symmetry breaking for maps for the standard representation of S_{n+1} on $\mathbb{R}^n$ are implicit in [25, §17] and explicit in [1]. ♢

6. Strong determinacy

6.1. **Strong determinacy theorem for maps.** Just as for vector fields, we may prove a strong determinacy theorem for one parameter families of equivariant maps.

Theorem 6.1.1. *Let* (V, Γ) *be either absolutely or complex irreducible. Then* Γ*-equivariant bifurcation problems on* V *are strongly determined. In particular, there exists* $d \in \mathbb{N}$ *and an open and dense semi-analytic subset* $\mathcal{N}(d)$ *of* $P_\Gamma^{(d)}(V, V)$ *such that if* $f \in \mathcal{M}(V, \Gamma)$ *and* $j^d f_0(0) \in \mathcal{N}(d)$ *then*

1. f *is strongly determined.*
2. *If* H *is a closed subgroup of* Γ *then* f *is* (d, H)*-stable.*

6.2. **Proof of the Strong determinacy theorem for maps.** As the proof of Theorem 6.1.1 is similar to that of the corresponding result for vector fields, we shall only sketch the main techniques.

We start by restricting to the set $\mathcal{M}_\omega(V, \Gamma) \subset \mathcal{M}(V, \Gamma)$ of real-analytic families and assume that (V, Γ) is absolutely irreducible or tangential. Using methods based on resolution of singularities, it can be shown that we can find $d, N \in \mathbb{N}$, and an open and dense semi-algebraic subset $\mathcal{R}^1$ of $P_\Gamma^{(d)}(V, V)$ such that if we define

$$\mathcal{M}_1(V, \Gamma) = \{f \in \mathcal{M}_\omega(V, \Gamma) \mid j^d f_0(0) \in \mathcal{R}^1\}$$

then, for all $p \in \mathbb{N}$, the p-jet at zero of solution branches of $f \in \mathcal{M}_1(V, \Gamma)$ depends analytically on $j^{p+N} f(0, 0)$. (Full details of this construction are given in [21, §10].) If Γ is finite, we may use this parametrization theorem, in combination with methods based on Newton-Puiseux series, to obtain estimates on eigenvalues of the linearization along branches of invariant group orbits. A routine application of Tougeron's implicit function theorem [37] then yields strong determinacy for smooth maps. If Γ is not finite, we have to work a little harder. First of all we blow-up along orbit strata using recent results of Schwarz on the coherence of the orbit stratification (see [36] and [21, §9]). In this way, we desingularize the branch.

Next we use the tangential and normal form for the family given by Lemma 3.1.5 and apply the same arguments used for the Γ-finite case to the normal component to obtain eigenvalue estimates along the branch. We obtain strong determinacy using persistence results on families of normally hyperbolic manifolds. (Proofs of these results are in [21, Appendix].) Finally, we extend our strong determinacy result from tangential to complex irreducible representations using equivariant normal forms (see [21, §9.18]). □

6.3. **Applications to normal forms.** We conclude this section by briefly describing how we can use Theorem 6.1.1 to justify normal form computations.

First of all, suppose that (V, Γ) is absolutely irreducible and that $-I \notin \Gamma$. Set $\Gamma^2 = \Gamma \times \mathbb{Z}_2$. It follows from Theorem 6.1.1 that we can find $d \in \mathbb{N}$ such that Γ^2-equivariant bifurcation problems on V are strongly (d, Γ)-determined. Suppose that $f \in \mathcal{M}^-(V, \Gamma^2)$ and f is strongly (d, Γ^2)-determined. Let $f' \in \mathcal{M}^-(V, \Gamma)$ satisfy $j^d f'_0(0) = j^d f_0(0)$. We regard f' as a perturbation of f breaking symmetry from Γ^2 to Γ. It follows from the strong determinacy theorem that each branch of normally hyperbolic invariant Γ^2-orbits in $\Xi^\star(f)$ will persist as a branch of Γ-invariant normally hyperbolic submanifolds for f'. Typically, some of these branches will be branches of Γ-orbits (and so will appear in $\Xi^\star(f')$), others will not be Γ-orbits. If Γ is finite, each branch for f' will consist of hyperbolic points of prime period two.

Example 6.3.1. Let $\Gamma = \Delta'_5 \rtimes \mathbb{Z}_5$ (Example 5.4.5). Then Γ^2-equivariant bifurcation problems are strongly $(3, \Gamma)$-determined (Theorem 5.4.4). It is easy to verify directly that if $f \in \mathcal{L}^-_\Gamma(\mathbb{R}^5)$, then f has branches of points of prime period two tangent to the axes $\mathbb{R}(1, 1, 1, 1, \pm 1)$. However, the period two points are not related by Γ-symmetries. In this example, $(\mathbb{R}^5, \Gamma)$, $(\mathbb{R}^5, \Gamma^2)$ have the same cubic equivariants and fourth order terms are required to break symmetry from Γ^2 to Γ. ♡

Let $P^{(d)}_\Gamma(V \times \mathbb{R}, V)_0$ denote the subset of $P^{(d)}_\Gamma(V \times \mathbb{R}, V)$ consisting of polynomial maps with linear term $(\lambda - 1)I_V$. We similarly define $P^{(d)}_{\Gamma^2}(V \times \mathbb{R}, V)_0$. The next result follows from the theory of equivariant normal forms [28, Chapter XVI, §5] (see also the proof of [21, Lemma 9.18.3]).

Lemma 6.3.2. *Let $d \in \mathbb{N}$. There is a polynomial submersion*

$$N_d : P^{(d)}_\Gamma(V \times \mathbb{R}, V)_0 \to P^{(d)}_{\Gamma^2}(V \times \mathbb{R}, V)_0,$$

such that if $f \in \mathcal{M}^-(V, \Gamma)$ then $N_d(j^d f(0,0))$ is the Γ^2-equivariant normal form of f to order d. Moreover, if $p > d$, $N_d(N_p(j^p f(0,0))) = N_d(j^d f(0,0))$. In particular, N_d restricts to the identity map on $P^{(d)}_{\Gamma^2}(V \times \mathbb{R}, V)_0$.

Suppose that Γ-equivariant bifurcation problems on V are p-determined, Γ^2-equivariant bifurcation problems on V are q-determined and Γ^2-equivariant bifurcation problems on V are strongly (d, Γ)-determined. It is easy to see that $d \geq p, q$.

Theorem 6.3.3. *We may construct an open and dense semi-analytic subset $\mathcal{N}$ of $P^{(d)}_\Gamma(V \times \mathbb{R}, V)$ such that if $f \in \mathcal{M}^-(V, \Gamma)$ satisfies $j^d f(0,0) \in \mathcal{N}$ then*

(1) $f \in \mathcal{S}(V, \Gamma)$.
(2) $\tilde{f} = N_d(j^d f(0,0)) \in \mathcal{S}(V, \Gamma^2)$.
(3) *Every branch of invariant normally hyperbolic Γ^2-orbits of $\tilde{f}$ persists as a branch of normally hyperbolic Γ-invariant submanifolds manifolds for f and every branch of invariant Γ-orbits of f arises via such a perturbation.*

Proof: Let $\mathcal{R}, \mathcal{R}^2$ be the open and dense semi-algebraic subsets of $P^{(d)}_{\Gamma}(V\times\mathbb{R},V)$, $P^{(d)}_{\Gamma^2}(V\times\mathbb{R},V)$ that respectively determine stable maps for Γ- and Γ^2-equivariant bifurcation problems on V. let $\mathcal{D}$ be the semi-analytic subset of $P^{(d)}_{\Gamma^2}(V\times\mathbb{R},V)$ that determines the strongly (d,Γ)-stable mappings in $\mathcal{M}^-(V,\Gamma^2)$. We define

$$\mathcal{N} = \mathcal{R}\cap N_d^{-1}(\mathcal{R}^2\cap\mathcal{D})$$

Since N_d is a polynomial submersion, $\mathcal{N}$ is an open and dense semi-analytic subset of $P^{(d)}_{\Gamma}(V\times\mathbb{R},V)$. The theorem follows. □

We have somewhat similar results if (V,Γ) is a complex representation such that $S^1\not\subset\Gamma$. In this case, we set $G=\Gamma\times S^1$. Let $d\in\mathbb{N}$. Let $P^{(d)}_{\Gamma}(V\times\mathbb{R},V)_0$ be the set of polynomial maps which have linear term $(1+\lambda)\exp(\imath\omega)I_V$, where $\omega\in[0,2\pi)$. Let $P^{(d)}_{\Gamma}(V\times\mathbb{R},V)_\star$ denote the open and dense semi-algebraic subset of $P^{(d)}_{\Gamma}(V\times\mathbb{R},V)_0$ defined by requiring that $\exp(\imath\omega)$ is not a qth root of unity, $1\le q\le d$. We similarly define $P^{(d)}_{G}(V\times\mathbb{R},V)_\star$.

Lemma 6.3.4. *Let $d\in\mathbb{N}$. There is a polynomial submersion*

$$N_d : P^{(d)}_{\Gamma}(V\times\mathbb{R},V)_\star \rightarrow P^{(d)}_{G}(V\times\mathbb{R},V)_\star,$$

such that if $f\in\mathcal{M}(V,\Gamma)$ then $N_d(j^d f(0,0))$ is the G-equivariant normal form of f to order d. Moreover, if $p>d$, $N_d(N_p(j^p f(0,0))) = N_d(j^d f(0,0))$. In particular, N_d restricts to the identity map on $P^{(d)}_{G}(V\times\mathbb{R},V)_\star$.

Suppose G-equivariant bifurcation problems on V are strongly (d,Γ)-determined. We have

Theorem 6.3.5. *We may construct an open and dense semi-analytic subset $\mathcal{N}$ of $P^{(d)}_{\Gamma}(V\times\mathbb{R},V)$ such that if $f\in\mathcal{M}^-(V,\Gamma)$ satisfies $j^d f(0,0)\in\mathcal{N}$ then*

(1) *$f\in\mathcal{S}(V,\Gamma)$.*
(2) *$\tilde{f} = N_d(j^d f(0,0))\in\mathcal{S}(V,G)$.*
(3) *Every branch of normally hyperbolic G-orbits of $\tilde{f}$ persists as a branch of normally hyperbolic Γ-invariant submanifolds manifolds for f. Moreover, every branch of invariant Γ-orbits of f arises as such a perturbation.*

Remark 6.3.6. The residual dynamics on branches when we break normal form symmetry may, of course, be complicated (see Broer et al. [6]). ♢

7. Equivariant Hopf bifurcation theorem for vector fields

One of the applications of the theory developed in [21] was a proof of a variant of Fiedler's equivariant Hopf bifurcation theorem based on strong determinacy and equivariant normal forms [21, Theorem 11.2.1]. The proof of the general strong determinacy theorem given in [21] depends on rather technical and delicate results on persistence of branches of normally hyperbolic group orbits under symmetry breaking perturbations. It is clear, however, that simpler proofs should be available if we restrict attention to symmetry breaking perturbations which only break normal form symmetry. In this section, we outline a relatively simple proof of a version of the strong determinacy theorem that applies to the normal form analysis of the equivariant Hopf bifurcation. Our proof avoids most normal hyperbolicity issues.

7.1. **Preliminaries.** Suppose that (V,Γ) is a complex irreducible representation. Let X be a smooth Γ-equivariant vector field on V and α be a relative equilibrium of X. We recall [21, §3], [13, 17] that we may define the *reduced Hessian* $\text{HESS}(X,\alpha)$ of X along α and that $\text{HESS}(X,\alpha)$ is a subset of $\mathbb{C}/\imath\mathbb{R}$. The orbit α is normally hyperbolic for X if and only if the multiplicity of 0 in $\text{HESS}(X,\alpha)$ is equal to the dimension of α. If α is an equilibrium orbit, $\text{HESS}(X,\alpha)$ is, up to translations by pure imaginary numbers, the set of eigenvalues (counting multiplicities) of the Hessian of X along α. If $X = X^N + X^T$ is a tangent and normal decomposition of X on a neighborhood of α (see [31, 17]), then $\text{HESS}(X^N,\alpha) = \text{HESS}(X,\alpha)$. In the sequel, we typically work with the tangent and normal decomposition.

Let $\mathcal{V}(V,\Gamma)$ denote the space of normalized Γ-equivariant vector fields on V. We recall that if $f \in \mathcal{V}(V,\Gamma)$ then $Df_\lambda(0) = (\lambda + \imath)I_V$, $\lambda \in \mathbb{R}$.

The next result follows from [21, Theorem 9.18.1] (cf. [28, Chapter XVI, §11]).

Proposition 7.1.1 (Estimates on eigenvalues)**.** *There exist $d \in \mathbb{N}$, $\nu > 0$ and an open and dense semi-analytic subset $\mathcal{R}$ of $P^{(d)}_\Gamma(V,V)$ such that if $f \in \mathcal{V}(V,G)$ and $j^d f_0(0) \in \mathcal{R}$ then*

1. *$f \in \mathcal{S}(V,\Gamma)$.*
2. *If $\mathfrak{b} \in \Sigma(f)$ is a branch of relative equilibria of isotropy type τ, there exists a parametrization $\Psi = (\phi,\lambda) : [0,\delta] \times \Delta_\tau \to V \times \mathbb{R}$ of $\mathfrak{b}$ and $C = C(f) > 0$ such that if $t \in (0,\delta]$ and $\mu(t) \in HESS(f_{\lambda(t)}, \phi(t,\Delta_\tau))$ is nonzero, then*
$$|Re(\mu(t))| \geq Ct^\nu$$

Remark 7.1.2. The first step of the proof of Proposition 7.1.1 depends on choosing $\mathcal{R}$ and $d \in \mathbb{N}$ so that if f is analytic and $j^d f_0(0) \in \mathcal{R}$, then we can choose parametrizations of branches so that initial exponents along the branch are locally constant (as functions of f). This is done in [21, §7] and we may suppose $\mathcal{R}$ is semi-algebraic. In order to obtain estimates on eigenvalues along the branch, we use blowing-up techniques and results based on Newton-Puiseux series. Typically, we now have to allow $\mathcal{R}$ to be semi-analytic rather than semi-algebraic. Finally, using the tangent and normal form, we extend estimates to smooth families (see [21, §8]). ♢

7.2. **Equivariant Hopf bifurcation and normal forms.** Continuing with our assumptions on (V,Γ), set $G = \Gamma \times S^1$ and consider the representation (V,G). Let $d \in \mathbb{N}$, $\mathcal{R} \subset P^{(d)}_G(V,V)$ and ν satisfy the conditions of Proposition 7.1.1 for (V,G).

Theorem 7.2.1. *There exists $\tilde{d} \geq d$ such that (V,G) is strongly $(\tilde{d},\Gamma)$-determined.*

Proof: Let $\mu \in \mathcal{O}(V,G)$ be a symmetry breaking isotropy type. The conjugacy class of Γ_x is constant on V_μ and so μ determines a unique isotropy type $\tau \in \mathcal{O}(V,\Gamma)$. In particular, $V_\mu \subset V_\tau$ (see also [18, §3] and note that in general V_μ may not be an open subset of V_τ). If G-orbits of isotropy type μ are Γ-orbits, it is easy to see that normally hyperbolic branches of relative equilibria of isotropy type μ persist when we break symmetry from G to Γ. Therefore, we assume that G-orbits of isotropy type μ are not Γ-orbits. It follows that if α is a G-orbit of isotropy type μ, then G/Γ is diffeomorphic to S^1 and $G_x/\Gamma_x \cong \mathbb{Z}_p$, for some $p \geq 1$.

Following [21, Lemma 9.3.5], we successively polar blow-up $V \times \mathbb{R}$ along (the strict transforms) of the G-orbit strata $V_\rho \times \mathbb{R}$, $\rho > \mu$. In this way, we obtain an analytic G-equivariant map $\Pi : W \to V \times \mathbb{R}$ such that $W_\mu = \Pi^{-1}(\overline{V}_\mu \times \mathbb{R})$ is a closed submanifold of W. In addition, W_μ will be a submanifold of W_τ – the set of points of Γ-isotropy type τ. Indeed, if we order $\mathcal{O}(V,G)$ so that $\rho > \mu$ if $W_\rho \subset \partial W_\tau$, we may and shall assume that W_τ is a closed submanifold of W. The blowing-down map Π restricts to a local finite-to-one analytic diffeomorphism on the complement of $\Pi^{-1}(\cup_{\rho>\tau}(V\times\mathbb{R})_\rho)$. Every $f \in \mathcal{V}(V,G)$ lifts uniquely to a smooth G-equivariant vector field on $\tilde{f}$ on W. If $j^d f_0(0) \in \mathcal{R}$, then every branch $\mathfrak{b} \in \Sigma(f)$ of isotropy type μ lifts to a branch $\tilde{\mathfrak{b}} \subset W_\mu$ of normally hyperbolic relative equilibria of $\tilde{f}$. If we let $\tilde{\Psi} : [0,\delta]\times\Delta_\mu \to W_\mu \subset W$ denote the lift of the parametrization of $\mathfrak{b}$ given by Proposition 7.1.1, then the estimates of Proposition 7.1.1 hold for $\tilde{\Psi}$ (since Π is a local analytic diffeomorphism off $\Pi^{-1}(\cup_{\rho>\tau}(V\times\mathbb{R})_\rho)$). The map $\tilde{\Psi}$ is a G-equivariant embedding. We set $Z_t = \tilde{\Psi}(t,\Delta_\mu)$, $t\in[0,\delta]$, and note that Z_t will be a smooth family of G-orbits of G-isotropy type μ and Γ-isotropy type τ. We regard the dynamics on Z as Γ-equivariant and define a local Poincaré section $D \subset W$ for Z_0. We recall that D will be a smoothly Γ-equivariantly embedded submanifold of codimension 1 which intersects Z_0 transversally along a Γ-orbit (we refer to [13, 17] for details). It follows by transversality that D will be a Poincaré section for Z_t, $t \in [0,\delta']$, where $0 < \delta' \le \delta$. Choosing a sufficiently small $D' \subset D$, we may define an associated family of Poincaré maps $P_{\lambda(t)} : D' \to D$, $t\in[0,\delta']$, such that $P_{\lambda(t)}$ has an invariant Γ-orbit $Z_t \cap D$ for each $t \in [0,\delta']$. If $z \in D'$, $P_{\lambda(t)}(z)$ will be defined as the first point of intersection of the forward $\tilde{f}_{\lambda(t)}$-trajectory through z with D. If $z \in Z_t\cap D$, then $P_{\lambda(t)}(z) = \exp(2\pi\imath/p)z$. Set $I_t = Z_t \cap D$. It follows from our hyperbolicity conditions on $\tilde{\mathfrak{b}}$ that I_t is a branch of normally hyperbolic invariant Γ-orbits for the family of Γ-equivariant diffeomorphisms $P_{\lambda(t)}$, $t\in[0,\delta']$. Moreover, our estimates on elements of $\mathrm{HESS}(\tilde{f}_{\lambda(t)}, Z_t)$ exponentiate to estimates on elements of $\mathrm{SPEC}(P_t, I_t)$. Specifically, if $\mu(t) \in \mathrm{SPEC}(P_{\lambda(t)}, I_t)$ is not equal to one, then there exists $C > 0$ such that

$$|1-|\mu(t)|| \ge Ct^\nu,\ \ t\in[0,\delta']$$

It is now a straightforward application of the techniques used to study families of Γ-equivariant maps to show that the branch I_t of invariant Γ-orbits will persist, as a family of normally hyperbolic invariant Γ-orbits, under sufficiently small high order Γ-equivariant perturbations of the family P. Moreover, we can choose an S^1-invariant horn neighborhood H of the original branch I_t such that $H \cap \Pi^{-1}(\cup_{\rho>\tau}(V\times\mathbb{R})_\rho) = I_0$ and require that perturbed families lie within H. Finally, we may choose $\tilde{d} \ge d$ (independent of f) such that if $f \in \mathcal{M}(V,\Gamma)$ and $j^{\tilde{d}} f'(0,0) = j^{\tilde{d}} f(0,0)$, then $\tilde{f}'$ defines a family of Poincaré maps $P'_{\lambda(t)}$ with a corresponding branch of invariant normally hyperbolic Γ-orbits contained in H. This branch determines in the usual way a family of Γ- and $\tilde{f}'$-invariant normally hyperbolic submanifolds of W_τ and again we may require that the family is contained in H. Blowing-down by Π we obtain the required family of Γ- and f'-invariant normally hyperbolic submanifolds of f'. □

Remarks 7.2.2. (1) Note that it is somewhat easier to prove *persistence* of the branch in W_τ. Once we have defined the Poincaré maps, restrict to the free Γ-manifold W_τ and reduce to the orbit manifold W_τ/Γ. The branch of invariant Γ-orbits drops down to a branch of fixed points. Somewhat similar techniques were used in [7]. (2) Note that the advantage of working with the Poincaré maps is that they are Γ-invariant, even when we break normal form symmetries, and so we can characterize normal hyperbolicity in terms of spectral conditions on eigenvalues of linearizations. This approach is not open to us if we do not work with flow or map invariant Γ-orbits as the behavior *tangent* to the manifold then becomes critical. ♢

References

[1] D. G. Aronson, M. Golubitsky and M. Krupa. 'Coupled arrays of Josephson Junctions and bifurcation of maps with S_N symmetry', *Nonlinearity*, **4** (1991), 861-902.

[2] E. Bierstone. 'Lifting isotopies from orbit spaces', *Topology*, **14** (1975), 245–252.

[3] E. Bierstone. 'General position of equivariant maps', *Trans. Amer. Math. Soc.*, **234** (1977), 447–466.

[4] E. Bierstone. 'Generic equivariant maps', *Real and Complex Singularities, Oslo 1976*, Proc. Nordic Summer School/NAVF Sympos. Math. (Sijthoff and Noordhoff International Publ.) Leyden (1977), 127–161.

[5] T. Bröcker and T. tom Dieck. *Representations of Compact Lie Groups*, (Graduate Texts in Mathematics, Springer, New York, 1985).

[6] H. W. Broer, G. B. Huitema, F. Takens, B. L. T. Braaksma. 'Unfoldings and bifurcation of quasi-periodic tori', *Mem. Amer. Math. Soc.* **83**(421) (1990).

[7] P. Chossat and M. J. Field. 'Geometric analysis of the effect of symmetry breaking perturbations on an $O(2)$ invariant homoclinic cycle', In: Normal forms and Homoclinic Chaos. *Fields Institute Communications* **4** (1995), 21–42.

[8] P. Chossat and M. Golubitsky. 'Iterates of maps with symmetry', *SIAM J. Math. Anal.*, **19**(6) (1988), 1259–1270.

[9] M. Coste. 'Ensembles semi-algébriques', in *Géométrie Algebébrique Réelle et Formes Quadratiques*, Springer Lecture Notes in Math., **959**, 1982, 109–138.

[10] B. Fiedler. *Global Bifurcation of Periodic Solutions with Symmetry*, (Springer Lecture Notes in Math., **1309**, Springer-Verlag, New York-London,1988.)

[11] M. J. Field. 'Stratifications of equivariant varieties', *Bull. Austral. Math. Soc.*, **16**(2) (1977), 279–296.

[12] M. J. Field. 'Transversality in G-manifolds', *Trans. Amer. Math. Soc.*, **231** (1977), 429–450.

[13] M. J. Field. 'Equivariant Dynamical Systems', *Trans. Amer. Math. Soc.*, **259** (1980),185–205. 26(1982), 161–180.

[14] M. J. Field, 'Isotopy and stability of equivariant diffeomorphisms', *Proc. London Math. Soc.*, **46**(3), (1983), 487–516.

[15] M. J. Field. 'Equivariant Dynamics', *Contemp. Math*, **56** (1986), 69–95.

[16] M. J. Field. 'Equivariant Bifurcation Theory and Symmetry Breaking', *J. Dynamics and Diff. Eqns.*, **1**(4) (1989), 369–421.

[17] M. J. Field. 'Local structure of equivariant dynamics', in *Singularity Theory and its Applications, II*, eds. M. Roberts and I. Stewart, Springer Lecture Notes in Math., **1463** (1991), 168–195.

[18] M. J. Field, 'Determinacy and branching patterns for the equivariant Hopf bifurcation', *Nonlinearity*, **7**(1994), 403–415.

[19] M. J. Field, 'Geometric methods in bifurcation theory', In: Pattern formation and symmetry breaking in PDEs. *Fields Institute Communications* **6** (1996), 181–208.

[20] M. J. Field, 'Blowing-up in equivariant bifurcation theory', in *Dynamics, Bifurcation and Symmetries: New Trends and New Tools* (P. Chossat and J.-M. Gambaudo, Eds) NATO ARW Series, Kluwer, Amsterdam (1994), 111–122.

[21] M. J. Field, 'Symmetry breaking for compact Lie groups', *Mem. Amer. Math. Soc.*, **574**, 1996.

[22] M. J. Field and R. W. Richardson. 'Symmetry Breaking and the Maximal Isotropy Subgroup Conjecture for Reflection Groups', *Arch. for Rational Mech. and Anal.*, **105**(1) (1989), 61–94.

[23] M. J. Field and R. W. Richardson. 'Symmetry breaking in equivariant bifurcation problems', *Bull. Amer. Math. Soc.*, **22**(1) (1990), 79–84.

[24] M. J. Field and R. W. Richardson. 'Symmetry breaking and branching patterns in equivariant bifurcation theory I', *Arch. Rational Mech. Anal.* **118** (1992), 297–348.

[25] M. J. Field and R. W. Richardson. 'Symmetry breaking and branching patterns in equivariant bifurcation theory II', *Arch. Rational Mech. Anal.* **120** (1992), 147–190.

[26] C. Gibson, K. Wirthmüller, A. A. du Plessis and E. Looijenga. *Topological Stability of smooth mappings*, (Springer Lecture Notes in Math. **553**, 1976).

[27] M. Golubitsky. 'Genericity, bifurcation and Symmetry', in *Patterns and Dynamics in Reactive Media* (R. Aris, D. G. Aronson and H. L. Swinney, Eds), The IMA Volumes in Math. and its Applic., **37**, Springer-Verlag, New York (1991), 71–87.
[28] M. Golubitsky, D. G. Schaeffer and I. N. Stewart. *Singularities and Groups in Bifurcation Theory, Vol. II*, (Appl. Math. Sci. **69**, Springer-Verlag, New York, 1988).
[29] M. W. Hirsch, C. C. Pugh and M. Shub. *Invariant Manifolds*, (Springer Lect. Notes Math., **583**, 1977).
[30] N. Jacobson. *Basic Algebra I*, (W. H. Freeman, San Francisco, 1974.)
[31] M. Krupa. 'Bifurcations of relative equilibria', *SIAM J. MATH. ANAL.*, **21**(6) (1990), 1453–1486.
[32] J. Mather. 'Stratifications and mappings', *Proceedings of the Dynamical Systems Conference, Salvador, Brazil, ed. M. Peixoto*, (Academic Press, New-York, San Francisco, London, 1973.)
[33] B. B. Peckam and I. G. Kevrekidis. 'Period doubling with higher-order degeneracies', *SIAM J. MATH. ANAL* **22**(6) (1991), 1552–1574.
[34] V. Poenaru. *Singularités C^∞ en Présence de Symétrie*, (Springer Lect. Notes in Math. **510**, Springer-Verlag, New York, 1976.)
[35] D. Ruelle, 'Bifurcation in the presence of a symmetry group', *Arch. Rational Mech. Anal.*, **51**(2) (1973), 136–152.
[36] G. W. Schwarz. 'Algebraic quotients of compact group actions', preprint (1994).
[37] J. C. Tougeron. *Ideaux de fonctions differentiable*, (Erge. der Math. und ihrer Gren., **71**, Springer-Verlag, Berlin, Heidelberg, New-York, 1972.)
[38] A. Vanderbauwhede. 'Equivariant period doubling', in *Advanced Topics in the Theory of Dynamical Systems*, G. Fusco, M. Ianelli, L. Salvadori (eds), Notes Rep. Math. Sci. Engrg. 6 (1989), 235–246.

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