PERFECTOID SHIMURA VARIETIES

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Abstract. This is an expanded version of the lecture notes for the minicourse I gave at the 2017 Arizona Winter School. In these notes, I discuss Scholze's construction of Galois representations for torsion classes in the cohomology of locally symmetric spaces for GL_n, with a focus on his proof that Shimura varieties of Hodge type with infinite level at p acquire the structure of perfectoid spaces. I also briefly discuss some recent vanishing results for the cohomology of Shimura varieties with infinite level at p.

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1. Introduction

One of the famous consequences of the Langlands program is the theorem that all elliptic curves over $\mathbb{Q}$ are modular [Wil95, TW95, BCDT01]. The proof of this theorem for semistable elliptic curves led to Wiles's proof of Fermat's last theorem [Wil95] and had an enormous impact on number theory over the decades since.

What does it mean to say that an elliptic curve is modular? It roughly means that the elliptic curve corresponds to a modular form. For example, the elliptic curve $E/\mathbb{Q}$ defined by the equation

$$y^2 + y = x^3 - x^2$$

corresponds to the modular form $f(z)$ with Fourier expansion

$$f(z) = q \cdot \prod_{n=1}^{\infty} (1-q^n)^2(1-q^{11n})^2 = \sum_{n=1}^{\infty} a_n q^n,$$

where $q = e^{2\pi i z}$. The connection between E and f can be made explicit, by relating the number of points of E over finite fields to the Fourier coefficients of f. Concretely, we have

$$\ell + 1 - \#E(\mathbb{F}_\ell) = a_\ell$$

for every prime number ℓ.

The more sophisticated statement that encodes the relationship between E and f says that the p-adic *Galois representations* attached to each of these two objects are isomorphic

$$\rho_E \simeq \rho_f : G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_p),$$

for every prime number p.

We recall that the p-adic Galois representation attached to E arises from the Tate module of E, using the natural $G_{\mathbb{Q}}$-action on the p^n-torsion points of E for every integer $n \geq 1$:

$$\rho_E : G_{\mathbb{Q}} \to \mathrm{GL}(\varprojlim_n E[p^n]) \simeq \mathrm{GL}_2(\varprojlim_n \mathbb{Z}/p^n\mathbb{Z}) \simeq \mathrm{GL}_2(\mathbb{Z}_p).$$

We can rephrase this by saying that the Galois representation arises from the first étale homology of the elliptic curve $E/\mathbb{Q}$. The Galois representation ρ_f satisfies the Eichler–Shimura relation

$$\mathrm{tr}(\rho_f(\mathrm{Frob}_\ell)) = a_\ell,$$

where Frob_ℓ is the geometric Frobenius at the prime number $\ell \neq p, 11$, which determines a conjugacy class in $G_{\mathbb{Q}}$.

The equalities

$$\ell + 1 - \#E(\mathbb{F}_\ell) = a_\ell$$

can be recovered from

$$\rho_E \simeq \rho_f$$

when $\ell \neq p, 11$ by taking the traces of Frob_ℓ on either side, applying the Lefschetz trace formula for the action of Frob_ℓ on the p-adic étale homology of $E/\mathbb{F}_\ell$, and applying the Eichler-Shimura relation for f.

Exercise 1.0.1. *Convince yourself that $\rho_E \simeq \rho_f$ really does recover the relation $\ell + 1 - \#E(\mathbb{F}_\ell) = a_\ell$ for every prime $\ell \neq p, 11$. Of course, we can vary p. What happens for $\ell = 11$?*

l	2	3	5	7	13	17	19	23	29
$\#E(\mathbb{F}_l)$	5	5	5	10	10	20	20	25	30
a_l	-2	-1	1	-2	4	-2	0	-1	0

FIGURE 1. The number of points on the elliptic curve $E : y^2 + y = x^3 - x^2$, and the coefficients of the modular form $\sum a_i q^i = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2$. These satisfy $l + 1 - \#E(\mathbb{F}_l) = a_l$ for primes $l \neq 11$.

These notes are meant to explain how to vastly generalize the construction of the Galois representation ρ_f, so we start by recalling the key elements involved in the construction of ρ_f, going back to Eichler and Shimura. Recall that, under a first approximation, modular forms are holomorphic functions on the upper-half plane

$$\mathbb{H}^2 = \{z \in \mathbb{C} | \text{ Im } z > 0\}$$

which satisfy many symmetries. These symmetries are defined in terms of certain discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$. The upper-half plane has a transitive action of $\mathrm{SL}_2(\mathbb{R})$ by Möbius transformations

$$\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \gamma : z \mapsto \frac{az+b}{cz+d}.$$

The modular form f is a cusp form of weight 2 and level

$$\Gamma_0(11) := \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) | \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod{11}\},$$

a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ defined by congruence conditions. The weight and the level of f specify the symmetries that f must satisfy:

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 f(z).$$

Remark 1.0.2. The Möbius transformations are actually all the holomorphic isometries of $\mathbb{H}^2$ when we endow $\mathbb{H}^2$ with the hyperbolic metric $\frac{(dx)^2+(dy)^2}{y^2}$, where $z = x + iy$. The stabilizer of the point $i \in \mathbb{H}^2$ in $\mathrm{SL}_2(\mathbb{R})$ is $\mathrm{SO}_2(\mathbb{R})$, so we can identify

$$\mathbb{H}^2 \simeq \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}),$$

as smooth real manifolds together with a Riemannian metric. The subgroup $\mathrm{SO}_2(\mathbb{R}) \subset \mathrm{SL}_2(\mathbb{R})$ is maximal compact and SL_2 is semisimple, so we can identify $\mathbb{H}^2$ with the symmetric space for the group SL_2, as defined in Section 2.

In the case of the group SL_2, the symmetric space $\mathbb{H}^2$ has a natural complex structure and, as a result, one can prove that its quotients by congruence subgroups such as $\Gamma_0(11)$ are Riemann surfaces. It turns out that the symmetries that f satisfies allow us to consider instead of f the holomorphic differential $\omega_f := f(z)dz$ on the (non-compact) Riemann surface $\Gamma_0(11) \setminus \mathbb{H}^2$.

Exercise 1.0.3. *Prove that f indeed descends to a well-defined holomorphic differential on the quotient $\Gamma_0(11) \setminus \mathbb{H}^2$.*

The Riemann surface $\Gamma_0(11) \setminus \mathbb{H}^2$ is an example of a *locally symmetric space* for the group SL_2, in the sense of the definition we give in section 2.

Moreover, f is a simultaneous eigenvector for all Hecke operators T_ℓ (with $\ell \neq 11$), i.e. a Hecke eigenform. The ℓth Fourier coefficient a_ℓ can in fact be identified with the eigenvalue of T_ℓ acting on f.[1] (This can be seen by computing the dimension of the space of cusp forms of weight 2 and level $\Gamma_0(11)$, e.g. by computing the dimension of the space of holomorphic differentials on (the compactification of) $\Gamma_0(11) \setminus \mathbb{H}^2$. The space turns out to be one-dimensional and thus generated by f.)

Set $\Gamma := \Gamma_0(11)$. In the special case of the group SL_2, it turns out that the quotients $\Gamma \setminus \mathbb{H}^2$ have even more structure: there exists an algebraic curve Y_Γ defined over $\mathbb{Q}$ such that $\Gamma \setminus \mathbb{H}^2$ can be identified with $Y_\Gamma(\mathbb{C})$. This follows from the fact that $\mathbb{H}^2$ can be interpreted as a moduli of Hodge structures of elliptic curves[2], and, as a result, the quotients $\Gamma \setminus \mathbb{H}^2$ are (coarse) moduli spaces of elliptic curves over $\mathbb{C}$ equipped with certain extra structures. The particular moduli problem for $\Gamma = \Gamma_0(11)$ gives rise to a canonical model Y_Γ over $\mathbb{Q}$. Y_Γ is a smooth, quasi-projective but not projective curve, known as the *modular curve of level* Γ.

The modular form f determines the holomorphic differential $\omega_f \in H^1_{\mathrm{dR}}(\Gamma \setminus \mathbb{H}^2)$. A refinement of Hodge theory for the non-compact Riemann surface $Y_\Gamma(\mathbb{C}) \simeq \Gamma \setminus \mathbb{H}^2$ shows that ω_f determines a system of Hecke eigenvalues in

$$H^1_{\mathrm{Betti}}(Y_\Gamma(\mathbb{C}), \mathbb{C}).$$

This system of Hecke eigenvalues is actually defined over $\mathbb{Q}$ (in this case, the T_ℓ eigenvalues for $\ell \neq 11$ match the Fourier coefficients of f; the system of Hecke eigenvalues will be defined over a number field in general). Now the comparison between the Betti and the étale cohomology of Y_Γ shows that it determines a system of Hecke eigenvalues in

$$H^1_{\text{ét}}(Y_\Gamma \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_p).$$

Eichler and Shimura show that the corresponding eigenspace is two-dimensional (this follows from a refinement of the Hodge decomposition) and the natural Galois action on it is *the Galois representation* ρ_f. By the Cebotarev density theorem, the Galois representation ρ_f (which is absolutely irreducible) is determined by $\rho_f(\mathrm{Frob}_\ell)$ for $\ell \neq 11, p$ and the relationship between ρ_f and f is encoded in the Eichler–Shimura relation

$$\mathrm{tr}(\rho_f(\mathrm{Frob}_\ell)) = a_\ell$$

for all such primes ℓ.

Higher-dimensional analogues of modular forms are *automorphic representations* and they can be associated to any connected reductive group $G/\mathbb{Q}$ (or over a more general number field). Modular forms correspond to the group SL_2 (or GL_2). [3] In order to associate Galois representations to more general automorphic representations, one first relates automorphic representations to systems of Hecke eigenvalues occurring in the Betti cohomology of locally symmetric spaces, as we did above. If the corresponding locally symmetric spaces have the structure of algebraic varieties defined over number fields, as modular curves do, then one can sometimes find the desired Galois representations in their étale cohomology. If the locally symmetric spaces do not have an algebraic structure, the question of constructing Galois representations is much more difficult than in the algebraic case. The question is

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[1] In these notes, we will only be concerned with Hecke eigenforms, not with all modular forms and, more generally, we will be interested in *systems of Hecke eigenvalues*.

[2] We make this precise in section 2, when we discuss Shimura varieties. See Example 2.3.2.

[3] From the representation-theoretic perspective, a modular form is actually a vector inside an automorphic representation of SL_2.

even more difficult if we are interested in understanding *torsion classes* occurring in the Betti cohomology of locally symmetric spaces rather than characteristic 0 classes. Nevertheless, there has been a spectacular amount of progress recently due to Scholze [Sch15], building on work of Harris–Lan–Taylor–Thorne [HLTT16] for the characteristic 0 case.

The goal of these lecture notes is to describe some recent progress in the Langlands program, namely the construction of Galois representations associated to torsion classes in the Betti cohomology of locally symmetric spaces for GL_n/F, where F is a totally real or imaginary CM field. This gives as a corollary the existence of Galois representations for a certain class of automorphic representations of GL_n/F, namely those which are regular and L-algebraic, cf. [BG14]. We will focus on the ingredients coming from the theory of *Shimura varieties*, which are higher-dimensional analogues of modular curves, and from the theory of *perfectoid spaces*, as recently introduced by Scholze [Sch12a]. A central part of these notes concerns Scholze's theorem that the tower of Shimura varieties with increasing level at p has the structure of a perfectoid space and that it admits a period morphism to a flag variety, the *Hodge–Tate period domain*.

Remark 1.0.4. While the focus of these notes is the geometry of Shimura varieties and the construction of Galois representations (thus understanding the automorphic to Galois direction), we started the introduction by mentioning a *modularity* result. The modularity result is proved by the so-called *Taylor–Wiles patching method*, which relies on working in p-adic families, both on the side of the Galois representations (coming from elliptic curves) and on the side of modular forms. The existence of the automorphic to Galois direction, $f \mapsto \rho_f$, is a prerequisite to applying the Taylor–Wiles method. Indeed, modularity is not proved by directly matching ρ_E with ρ_f, but rather by considering a universal Galois deformation ring for the residual representation $\bar{\rho}_E$ and comparing this ring to the Hecke algebra acting on a space of modular forms that contains f. The map from the Galois deformation ring to the Hecke algebra is obtained by interpolating the correspondence $f \mapsto \rho_f$.

In order to prove such modularity results in higher dimensions (or even over imaginary quadratic fields), one needs to understand the automorphic to Galois direction first. Moreover, as the insight of Calegari–Geraghty shows [CG18], one needs to understand Galois representations attached not just to characteristic 0 automorphic representations, but also to classes in the cohomology of locally symmetric spaces with torsion coefficients, which are a reasonable substitute for p-adic and mod p automorphic forms.[4]

1.1. **Organization.**

In Section 2, we first introduce locally symmetric spaces, then we specialize to the case of Shimura varieties. We discuss examples and counterexamples. In Section 3, we recall the necessary background from p-adic Hodge theory on the (relative) Hodge–Tate filtration.[5] In Section 4, we recall the theory of the canonical subgroup and construct the anticanonical tower, which has a perfectoid structure. In Section 5, we show that (many) Shimura varieties with infinite

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[4] In Section 6.2, we explain why torsion classes give a reasonable notion of mod p and p-adic automorphic forms for a general reductive group, by discussing Emerton's notion of *completed cohomology*.

[5] See also the lecture notes of Bhatt for more details on the Hodge–Tate filtration.

level at p are perfectoid and describe the geometry of the Hodge–Tate period morphism. In Section 6, we discuss some conjectures and results about the cohomology of locally symmetric spaces and about the corresponding Galois representations.

1.2. Notation.

If F is a local or global field, we let G_F denote the absolute Galois group of F. If S is a finite set of places of the global field F, we let $G_{F,S}$ denote the Galois group of the maximal extension of F which is unramified at all primes of F not in S.

If F is a number field, we let $\mathbb{A}_F$ denote the adéles of F, $\mathbb{A}_{F,f}$ the finite adéles, $\mathbb{A}_{F,f}^{\mathfrak{p}}$ the finite adéles away from some prime $\mathfrak{p}$ of F, and $\mathbb{A}_{F,f}^{S}$ the finite adéles of F away from some finite set of primes S. If $\mathfrak{p}$ is a prime of F, we let $\mathrm{Frob}_{\mathfrak{p}}$ denote a choice of geometric Frobenius at the prime $\mathfrak{p}$.

We let $\mathbb{Q}_p^{\mathrm{cycl}}$ be the p-adic completion of the field $\mathbb{Q}_p(\mu_{p^\infty})$ obtained by adjoining all the pth power roots of unity to $\mathbb{Q}_p$. We let $\mathbb{Z}_p^{\mathrm{cycl}}$ be the ring of integers inside $\mathbb{Q}_p^{\mathrm{cycl}}$.

If G is a Lie group, we let G° denote the connected component of the identity in G.

If $R \subseteq S$ are rings and V is an R-module, we write $V_S := V \otimes_R S$.

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2. Locally symmetric spaces and Shimura varieties

In this section, we discuss locally symmetric spaces and Shimura varieties. In § 2.1, we introduce locally symmetric spaces for a general connected reductive group over $\mathbb{Q}$ and we give examples of locally symmetric spaces which admit the structure of complex algebraic varieties and which do not. In § 2.2, in preparation for discussing Shimura varieties, we review some necessary background from Hodge theory, in particular the notion of a variation of polarizable Hodge structures. Finally, in § 2.3 we discuss the axioms of a Shimura datum, the corresponding Shimura variety, and give many examples of Shimura varieties.

2.1. Locally symmetric spaces.

Let $G/\mathbb{Q}$ be a connected reductive algebraic group. Let A_G denote the maximal $\mathbb{Q}$-split torus in the center of G. Let $K_\infty \subset G(\mathbb{R})$ denote a maximal compact subgroup and let $A_\infty = A_G(\mathbb{R})$. To G, we can attach a *symmetric space* as follows:

$$X = G(\mathbb{R})/K_\infty^\circ A_\infty^\circ.^{6}$$

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[6] The term A_∞° is included to ensure that the locally symmetric spaces we obtain have finite volume.

This is a disjoint union of smooth real manifolds of some dimension d, it has an induced action of $G(\mathbb{R})$, and it can be endowed with a $G(\mathbb{R})$-invariant Riemannian metric.

Two subgroups Γ_1, Γ_2 of the same group are *commensurable* if the intersection $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2. A subgroup Γ of $G(\mathbb{Q})$ is *arithmetic* if it is commensurable with $G(\mathbb{Q}) \cap \mathrm{GL}_N(\mathbb{Z})$, for some embedding $G \hookrightarrow \mathrm{GL}_N$ of algebraic groups over $\mathbb{Q}$.[7] For an arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$, we can define the *locally symmetric space*

$$X_\Gamma := \Gamma \setminus X.$$

If Γ is torsion-free, the space X_Γ is a smooth real manifold of dimension d endowed with an induced Riemannian metric. (If Γ is not torsion-free, then X_Γ is an *orbifold*.)

Suppose we have a model $\mathcal{G}/\mathbb{Z}$ of G which is a flat affine group scheme of finite type over $\mathbb{Z}$.

Exercise 2.1.1. *Show that a finite index subgroup $\Gamma \subset \mathcal{G}(\mathbb{Z})$ is an arithmetic subgroup of $G(\mathbb{Q})$.*

From now on, we will only consider locally symmetric spaces X_Γ, where $\Gamma \subset \mathcal{G}(\mathbb{Z})$ is a finite-index subgroup. In fact, we will only consider arithmetic subgroups which are *congruence subgroups* of $\mathcal{G}(\mathbb{Z})$, i.e. subgroups which contain

$$\Gamma(N) := \ker\left(\mathcal{G}(\mathbb{Z}) \to \mathcal{G}(\mathbb{Z}/N\mathbb{Z})\right)$$

for some $N \in \mathbb{Z}_{\geq 1}$.[8]

If Γ is a congruence subgroup, the cohomology $H^*_{\mathrm{Betti}}(X_\Gamma, \mathbb{C})$ can be computed in terms of automorphic representations of G [BW00, Fra98]. This is easier to see in the case when the locally symmetric space X_Γ is compact. Then Matsushima's formula expresses $H^*_{\mathrm{Betti}}(X_\Gamma, \mathbb{C})$ in terms of the relative Lie algebra cohomology $H^*(\mathfrak{g}, K_\infty, \pi_\infty)$, where $\pi = \pi_f \otimes \pi_\infty$ runs over automorphic representations of G. The fact that one can express $H^*_{\mathrm{Betti}}(X_\Gamma, \mathbb{C})$ in terms of $(\mathfrak{g}, K_\infty)$-cohomology uses the induced Riemannian structure on X_Γ and Hodge theory for Riemannian manifolds.

We will mostly be interested in the converse direction: realizing certain automorphic representations of G as classes occurring in the Betti cohomology of locally symmetric spaces. Results of Franke guarantee that we can do this, at least for so-called *cohomological* automorphic representations. For GL_n/F, cohomological automorphic representations match regular L-algebraic ones up to a twist, see [BG14] for more details.

Example 2.1.2. (1) If $G = \mathrm{SL}_2$ (and we can take $\mathcal{G} = \mathrm{SL}_2/\mathbb{Z}$), the corresponding symmetric space is the upper-half plane $\mathbb{H}^2$. The locally symmetric spaces are the Riemann surfaces corresponding to modular curves, which are discussed in the introduction. These locally symmetric spaces are non-compact Riemann surfaces.

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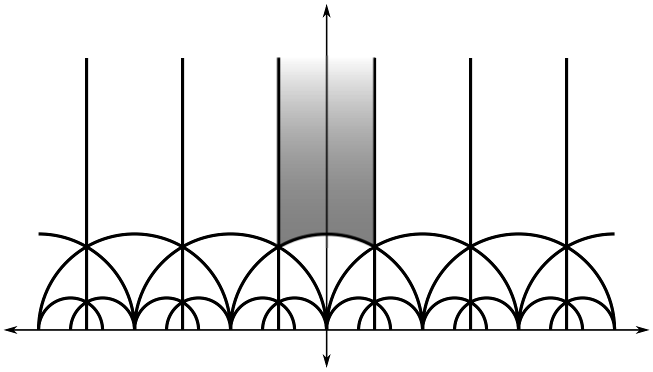
[7] More generally, one can define a *lattice* $\Gamma \subset G(\mathbb{R})$ as a discrete subgroup with finite covolume with respect to the Haar measure on $G(\mathbb{R})$. A remarkable theorem of Margulis shows that, if $G(\mathbb{R})$ is a semisimple Lie group with no factor isogenous to $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$, any lattice $\Gamma \subset G(\mathbb{R})$ is an arithmetic subgroup. See Section 3.3 of [Mil04] for more details on arithmetic subgroups.

[8] It can be shown that $\mathrm{SL}_2(\mathbb{Z})$ contains infinitely many conjugacy classes of finite-index subgroups which are non-congruence, but for $n \geq 3$, every finite-index subgroup of $\mathrm{SL}_n(\mathbb{Z})$ is a congruence subgroup.



If $G = \mathrm{D}^\times$, where $\mathrm{D}/\mathbb{Q}$ is a quaternion algebra which is split at infinity, i.e. $\mathrm{D}(\mathbb{R}) \simeq \mathrm{M}_2(\mathbb{R})$, then the corresponding symmetric domain is again $\mathbb{H}^2$ but the locally symmetric spaces are now compact Riemann surfaces. These correspond to certain so-called Shimura curves, which give another example of Shimura varieties.

(2) If $G = \mathrm{Res}_{\mathbb{Q}[i]/\mathbb{Q}}\mathrm{SL}_2$ (and we can take $\mathcal{G} = \mathrm{Res}_{\mathbb{Z}[i]/\mathbb{Z}}\mathrm{SL}_2$), the corresponding symmetric space can be identified with 3-dimensional hyperbolic space

$$\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2(\mathbb{C}) \simeq \mathbb{H}^3$$

and the locally symmetric spaces are called *Bianchi manifolds*. They are examples of arithmetic hyperbolic 3-manifolds and, since their real dimension is odd, they have no chance of having the structure of algebraic varieties.

(3) If F is a totally real or imaginary CM field with ring of integers $\mathcal{O}$, set $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_n$. In some cases, the corresponding locally symmetric spaces match ones we have already studied. For example, the symmetric space for $\mathrm{GL}_2/\mathbb{Q}$ is

$$\mathrm{GL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})\mathbb{R}^\times_{>0} \simeq \mathbb{H}^{2,\pm},$$

the disjoint union of the upper and lower half complex planes. The corresponding locally symmetric spaces are disjoint unions of finitely many copies of modular curves.

If F is totally real with $[F:\mathbb{Q}] \geq 2$, the locally symmetric spaces for $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ are not complex algebraic varieties. The obstruction comes from the center $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_1$, which is no longer a $\mathbb{Q}$-split torus; by considering a variant of the group we obtain *Hilbert modular varieties.*

If F is totally real and $n \geq 3$, the locally symmetric spaces do not have the structure of complex algebraic varieties. If F is an imaginary CM field and $n \geq 2$, the locally symmetric spaces also do not have the structure of complex algebraic varieties. One way to see this is as follows. Set

$$l_0 := \mathrm{rank}\ G(\mathbb{R}) - \mathrm{rank}\ K_\infty A_\infty.$$

(This is the so-called "defect" of the group G, see [BW00, CG18] for a discussion.) The axioms for a Shimura variety introduced in Section 2.3.1 below imply that $l_0 = 0$ whenever the group G admits a Shimura variety. However, when F is a general number field with r_1 real places and r_2

complex places, one can compute l_0 for $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_n$ to be

$$l_0 = \begin{cases} r_1\left(\frac{n-2}{2}\right) + r_2(n-1) & n \text{ even}, \\ r_1\left(\frac{n-1}{2}\right) + r_2(n-1) & n \text{ odd}. \end{cases}$$

(4) If $G(\mathbb{R})$ is compact (or more generally, if $G(\mathbb{R})/A_\infty$ is compact), then $G(\mathbb{R})/K_\infty^\circ A_\infty^\circ$ is a finite set of points and the locally symmetric spaces attached to G are also just finite sets of points, called Shimura sets. This situation is very favorable for setting up the Taylor–Wiles method, because the cohomology of the locally symmetric space is then concentrated in only one degree, namely in degree 0. This happens, for example, in the case of a *definite* unitary group defined over a totally real field (whose signature at each infinite place is $(0, n)$). However, this situation is not very interesting from the point of view of geometry!

In these notes, we will mostly use the adelic perspective on locally symmetric spaces. Recall that we have chosen a model $\mathcal{G}/\mathbb{Z}$ of $G/\mathbb{Q}$. Let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup of the form $\prod_v K_v$, where v runs over primes of $\mathbb{Q}$ and $K_v \subseteq \mathcal{G}(\mathbb{Z}_v)$, and such that $K_v = \mathcal{G}(\mathbb{Z}_v)$ for all but finitely many primes v. Define the double quotient

$$X_K := G(\mathbb{Q}) \setminus (X \times G(\mathbb{A}_f)/K),$$

where the action of $G(\mathbb{Q})$ on the two factors is via the diagonal embedding. The set $G(\mathbb{Q}) \setminus G(\mathbb{A}_f)/K$ is finite; this follows from [PR94][Thm 5.1].

Exercise 2.1.3. *When $G = \mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}_n$, prove directly that $G(\mathbb{Q}) \setminus G(\mathbb{A}_f)/K$ is finite.*

Let $g_1, \ldots, g_r$ be a set of double coset representatives. For $i = 1, \ldots, r$, let $\Gamma_i := G(\mathbb{Q}) \cap g_i K g_i^{-1}$. This is a discrete subgroup of $G(\mathbb{Q})$ and it is in fact a congruence subgroup of $\mathcal{G}(\mathbb{Z})$. Then we have

$$X_K = G(\mathbb{Q}) \setminus (X \times G(\mathbb{A}_f)/K) = \sqcup_{i=1}^r \Gamma_i \setminus X = \sqcup_{i=1}^r X_{\Gamma_i},$$

so the adelic version of a locally symmetric space is a finite disjoint union of the locally symmetric spaces introduced above.

We say that K is *neat* if $G(\mathbb{Q}) \cap gKg^{-1}$ is torsion-free for any $g \in G(\mathbb{A}_f)$, in which case X_K is a smooth real manifold of dimension d. If K is sufficiently small, then it is neat.

As seen in Example 2.1.2 (1) above, the locally symmetric spaces X_K can be non-compact. Borel and Serre [BS73] constructed a compactification of X_K (or rather, of the individual spaces X_Γ), which is a smooth real manifold with corners. If X_K^{BS} denotes the Borel-Serre compactification of X_K, the inclusion

$$X_K \hookrightarrow X_K^{\mathrm{BS}}$$

is a homotopy equivalence. This shows that X_K has the same homotopy type as that of a finite CW complex, so in particular the vector spaces $H^i_{\mathrm{Betti}}(X_K, \mathbb{C})$ are finite-dimensional. Similarly, the cohomology groups $H^i_{\mathrm{Betti}}(X_K, \mathbb{Z}/p^N\mathbb{Z})$ are finite $\mathbb{Z}/p^N\mathbb{Z}$-modules and the groups $H^i_{\mathrm{Betti}}(X_K, \mathbb{Q}_p)$ are finite-dimensional for every prime p.

As K varies, we have a tower of locally symmetric spaces $(X_K)_K$. If K, K' are two compact-open subgroups of $G(\mathbb{A}_f)$ and if $g \in G(\mathbb{A}_f)$ is such that $g^{-1}K'g \subseteq K$, we have a finite étale morphism $c_g : X_{K'} \to X_K$ induced by $hK' \mapsto hgK$ for $h \in G(\mathbb{A}_f)$. If one takes $K' := K \cap gKg^{-1}$, one obtains a correspondence

$$(c_g, c_1) : X_{K'} \to X_K \times X_K,$$

called a *Hecke correspondence*. This correspondence induces an endomorphism of $H^i_{(c)}(X_K)$, where we take the Betti cohomology of the locally symmetric space with coefficients in either $\mathbb{C}, \mathbb{Q}_p, \mathbb{Z}/p^N\mathbb{Z}$ for $N \in \mathbb{Z}_{\geq 1}$ and this endomorphism only depends on the double coset KgK.

2.2. **Review of Hodge structures.**

Roughly speaking, a Shimura variety is an algebraic variety defined over a number field whose underlying complex manifold is a locally symmetric space corresponding to some connected reductive group $G/\mathbb{Q}$. As we have seen in Example 1, this can exist only in special circumstances, for certain groups G. In this section, we recall some notions related to Hodge structures and variations of Hodge structures, which will be useful for explaining the axioms defining a Shimura datum in Section 2.3. For a more in-depth discussion of these notions, see Chapter II of [Mil04].

2.2.1. *Hodge structures.* Recall that a (pure) *Hodge structure* on a finite-dimensional real vector space V is a direct sum decomposition of the complexification $V_\mathbb{C}$ of V of the form

$$V_\mathbb{C} = \oplus_{(i,j)\in\mathbb{Z}^2} V^{i,j}$$

such that the following relation, known as *Hodge symmetry*, holds: for every $(i,j) \in \mathbb{Z}^2$, the complex conjugate of $V^{i,j}$ is $V^{j,i}$. The direct sum decomposition is called the Hodge decomposition. If $V_\mathbb{C} = \oplus_{k\in I} V^{i_k,j_k}$, we say that V has a Hodge structure of *type* $(i_k, j_k)_{k\in I}$. If, moreover, $i_k + j_k = n$ for every $k \in I$ then we say that the Hodge structure on V is pure of *weight* n. The weight decomposition is the direct sum decomposition of V indexed by weight and it is already defined over $\mathbb{R}$. A morphism of Hodge structures is a morphism of real vector spaces which respects the Hodge decomposition of their complexifications.

More generally, one can define *rational* and *integral* Hodge structures. An integral (resp. rational) Hodge structure is a free $\mathbb{Z}$-module of finite rank (resp. finite-dimensional $\mathbb{Q}$-vector space) together with a Hodge decomposition of $V_\mathbb{R}$ such that the weight decomposition is defined over $\mathbb{Q}$.

Example 2.2.2. If $X/\mathbb{C}$ is an smooth projective variety[9], then the Betti cohomology groups $H^n(X(\mathbb{C}), \mathbb{Z})$ are endowed with integral Hodge structures coming from the Hodge decomposition

$$H^n(X(\mathbb{C}), \mathbb{Z}) \otimes_\mathbb{Z} \mathbb{C} = \bigoplus_{i+j=n} H^j(X, \Omega^i_{X/\mathbb{C}});$$

we set $V^{i,j} := H^j(X, \Omega^i_{X/\mathbb{C}})$.

If $X = A$ is an abelian variety over $\mathbb{C}$, the Hodge decomposition is

$$H^1(A(\mathbb{C}), \mathbb{Z}) \otimes_\mathbb{Z} \mathbb{C} = H^0(A, \Omega^1_{A/\mathbb{C}}) \oplus H^1(A, \mathcal{O}_A);$$

---

[9] We could take, more generally, X to be a compact Kähler manifold, in which case the Betti cohomology decomposes as $H^n(X, \mathbb{C}) = \oplus_{i+j=n} H^{i,j}(X)$, where $H^{i,j}(X)$ denotes the space of cohomology classes of type (i,j).

then $H^1(A(\mathbb{C}),\mathbb{Z})$ has an integral Hodge structure of type $(1,0),(0,1)$. The dual $H_1(A(\mathbb{C}),\mathbb{Z})$ has a Hodge structure of type $(-1,0),(0,-1)$. Giving a Hodge structure of this type on $H_1(A(\mathbb{C}),\mathbb{Z})$ is equivalent to giving a complex structure on $H_1(A(\mathbb{C}),\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}$.

The category of integral Hodge structures of type $(-1,0),(0,-1)$ is equivalent to the category of complex tori. (If A is an abelian variety, then $A(\mathbb{C})$ is a complex torus, though not every complex torus arises from an abelian variety.)

Example 2.2.3. If $n\in\mathbb{Z}$, we define the Hodge structure $\mathbb{R}(n)$ to be the unique Hodge structure on $\mathbb{R}$ of type $(-n,-n)$. We define $\mathbb{Q}(n)$ and $\mathbb{Z}(n)$ analogously.

Let $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$; this is a real algebraic group such that $\mathbb{S}(\mathbb{R})=\mathbb{C}^\times$. The group $\mathbb{S}$ is the *Tannakian group* for the category of Hodge structures on real vector spaces.[10] This implies that there is an equivalence of categories between the category of Hodge structures on finite-dimensional real vector spaces and the category of finite-dimensional representations of $\mathbb{S}$ on real vector spaces. We describe the functor in one direction: a representation of $\mathbb{S}$ on a real vector space V determines an action of $\mathbb{C}^\times$ on the complexification $V_{\mathbb{C}}$. Then $V_{\mathbb{C}}$ decomposes as a direct sum of subspaces $V^{i,j}$ with $i,j\in\mathbb{Z}$, such that the action of $\mathbb{C}^\times$ on $V^{i,j}$ is through the cocharacter $z\mapsto z^{-i}\bar{z}^{-j}$. This direct sum decomposition defines a Hodge structure on V. Thus, we can think of a Hodge structure on a real vector space V as a pair (V,h), where $h:\mathbb{S}\to\mathrm{GL}(V)$ is a homomorphism.

A *polarizable* Hodge structure is a Hodge structure which can be equipped with a polarization. A *polarization* on a real Hodge structure (V,h) of weight n is a morphism of Hodge structures

$$\Psi : V\times V\to\mathbb{R}(-n)$$

such that the bilinear form $(v,w)\mapsto\Psi(v,h(i)w)$ is symmetric and positive definite. (One can similarly define polarizable integral and rational Hodge structures.)

Hodge structures coming from algebraic geometry are polarizable.[11] For example, recall Riemann's classification result for abelian varieties over $\mathbb{C}$.

Theorem 2.2.4. *The functor $A\mapsto H_1(A,\mathbb{Z})$ defines an equivalence of categories between the category of abelian varieties over $\mathbb{C}$ and the category of polarizable integral Hodge structures of type* $(-1,0),(0,-1)$.

2.2.5. *Variations of polarizable Hodge structures.* In order to have a Shimura variety, the symmetric space X should be interpreted as a "moduli space" of polarizable Hodge structures. The precise notion of "moduli space" we will use is that of a *variation of Hodge structures.*

For a Hodge structure on V of weight n, we define the associated *Hodge–de Rham filtration*[12] to be the descending filtration given by

$$F^iV := \bigoplus_{i'\geq i} V^{i',j'}\subset V_{\mathbb{C}}.$$

---

[10] See, for example, Chapter I of [Mil90] for a discussion of Tannakian categories and the corresponding Tannakian groups as relevant to Shimura varieties.

[11] More precisely, if $X/\mathbb{C}$ is a smooth projective variety, then its Betti cohomology carries a Hodge structure equipped with a rational polarization. The polarization comes from the hard Lefschetz theorem applied to a rational Kähler cohomology class.

[12] We prefer to refer to the Hodge filtration as the Hodge–de Rham filtration in order to avoid confusion with the Hodge–Tate filtration which will be discussed in Section 3.

Example 2.2.6. If $X/\mathbb{C}$ is a smooth projective variety, the Hodge structure on the Betti cohomology $H^*(X(\mathbb{C}), \mathbb{Z})$ has the Hodge–de Rham filtration

$$F^i\left(H^*(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}\right) = \oplus_{i' \geq i} H^{j'}(X, \Omega^{i'}_{X/\mathbb{C}}).$$

Under the canonical comparison isomorphism between Betti and de Rham cohomology, the Hodge–de Rham filtration on $H^*(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ matches the filtration on the algebraic de Rham cohomology $H^*_{\mathrm{dR}}(X)$ induced from the degeneration of the Hodge–de Rham spectral sequence

$$E_1^{i,j} = H^j(X, \Omega^i_{X/\mathbb{C}}) \Rightarrow H^{i+j}(X, \Omega^{\bullet}_{X/\mathbb{C}}) =: H^{i+j}_{\mathrm{dR}}(X).$$

If $X = A$ is an abelian variety over $\mathbb{C}$, the Hodge–de Rham filtration is determined by $F^1\left(H^1(A(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}\right) = H^0(A, \Omega^1_{A/C})$ (F^0 is everything and F^2 is zero).

We remark that, if X is defined over a number field E, then the algebraic de Rham cohomology $H^{i+j}_{\mathrm{dR}}(X)$ is an E-vector space and the Hodge–de Rham filtration on algebraic de Rham cohomology is also defined over E. This observation, together with standard comparison results between the cohomology of schemes and of the corresponding rigid-analytic varieties, will be used in Section 3. The degeneration of the Hodge–de Rham spectral sequence, which is needed to obtain the Hodge filtration on de Rham cohomology, is a deep result, originally established using analytic techniques (Hodge theory), but it was later on proved purely algebraically in [DI87].

A variation of (pure) Hodge structures should model the Hodge structure on the local system coming from the Betti cohomology of a continuous family of smooth projective varieties over some base. We start with an elementary definition, which we will apply to the case of Shimura varieties, and which can be formulated very concretely. We then give the more general definition.

Let X^+ be a simply-connected connected complex manifold.[13] Fix a real vector space V and a positive integer n. Assume that for each $h \in X^+$ we have a Hodge structure on V of weight n. Let $V_h^{i,j} \subset V_{\mathbb{C}}$ be the subspace of type (i, j) corresponding to the Hodge structure attached to h, and let $F_h^i(V_{\mathbb{C}}) \subset V_C$ be the ith graded piece of the Hodge–de Rham filtration on $V_{\mathbb{C}}$ determined by h.

Definition 2.2.7. *We say that the family of Hodge structures indexed by X^+ is a variation of Hodge structures of weight n if the following conditions are satisfied.*

1. *Firstly, for each (i, j), the subspace $V_h^{i,j}$ varies continuously with $h \in X^+$. This means that the dimension of the subspace $V_h^{i,j}$ is equal to a constant $d(i, j) \in \mathbb{Z}_{\geq 0}$, so there is a natural map to the Grassmannian parametrizing $d(i, j)$-dimensional subspaces of $V_{\mathbb{C}}$*

$$X^+ \to \mathrm{Gr}^{d(i,j)}(V_{\mathbb{C}}).$$

   Moreover, the above map is required to be continuous.
2. *The Hodge filtration $F_h^{\bullet}$ varies holomorphically with $h \in X^+$. More precisely, let $\mathrm{Fl}^{\mathrm{std}}(V_{\mathbb{C}})$ be the flag variety parametrizing descending filtrations on $V_{\mathbb{C}}$ of type $(d(i))_{i \in \mathbb{Z}}$, where $d(i) = \sum_{i' \geq i} d(i', n - i')$. The first condition guarantees that there exists a map*

$$\pi^+_{\mathrm{HdR}} : X^+ \to \mathrm{Fl}^{\mathrm{std}}(V_{\mathbb{C}}), h \mapsto F_h^{\bullet}$$

---

[13] For example, we could take $X^+ = \mathbb{H}^2$, the upper half-plane.

and we require this to be a map of complex manifolds (i.e. holomorphic).

(3) *(Griffiths transversality) The tangent space of* $\mathrm{Fl}^{\mathrm{std}}(V_{\mathbb{C}})$ *at a point corresponding to a filtration* $F^\bullet$ *on* $V_{\mathbb{C}}$ *is contained in* $\oplus_{i\in\mathbb{Z}}\mathrm{Hom}(F^i, V_{\mathbb{C}}/F^i)$. *Let* $h \in X^+$. *The final condition is that we require that the differential* $d\pi^+_{\mathrm{HdR}}$, *which is a map*

$$d\pi^+_{\mathrm{HdR}} : T_h X^+ \to T_{F_h^\bullet}\mathrm{Fl}^{\mathrm{std}}(V_{\mathbb{C}}) \subset \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}(F^i, V_{\mathbb{C}}/F^i),$$

to satisfy the following transversality condition:

$$\mathrm{Im}(d\pi^+_{\mathrm{HdR}}) \subset \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}(F^i, F^{i-1}/F^i) \subset \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}(F^i, V_{\mathbb{C}}/F^i).$$

Remark 2.2.8. In fixed weight n, the Hodge–de Rham filtration determines the Hodge decomposition via $V^{p,q} = F^p(V_{\mathbb{C}}) \cap \overline{F^q(V_{\mathbb{C}})}$. This implies that, if the Hodge structures parametrized by X^+ are all distinct, the holomorphic map

$$\pi^+_{\mathrm{HdR}} : X^+ \hookrightarrow \mathrm{Fl}^{\mathrm{std}}(V_{\mathbb{C}})$$

is injective. We call such a map a *period morphism.* One of the protagonists of these lecture notes is the p-adic analogue of this morphism, called the *Hodge–Tate period morphism.* This will not be injective, in general, but in many situations we will be able to understand its fibers. Note also that, while the Hodge–de Rham filtration varies holomorphically in families, the same is not true for the Hodge decomposition.

Exercise 2.2.9. *Check that the tangent space of* $\mathrm{Fl}^{\mathrm{std}}(V_{\mathbb{C}})$ *at a point corresponding to a filtration* $F^\bullet$ *on* $V_{\mathbb{C}}$ *is indeed contained in* $\oplus_{i\in\mathbb{Z}}\mathrm{Hom}(F^i, V_{\mathbb{C}}/F^i)$.

A variation of polarizable Hodge structures on X^+ is a variation of Hodge structures on X^+ together with a bilinear form

$$\Psi : V \times V \to \mathbb{R}$$

such that Ψ induces for any $h \in X^+$ a polarization on the Hodge structure determined by h.

More generally, let X be a complex manifold. A variation of Hodge structures of some weight $n \in \mathbb{Z}$ on X is a locally constant sheaf of finitely generated $\mathbb{Z}$-modules $\mathcal{V}_{\mathbb{Z}}$ on X (we call such an object a $\mathbb{Z}$-local system on X) together with the following additional structures. Define $\mathcal{E} := \mathcal{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_X$, where $\mathcal{O}_X$ is the sheaf of holomorphic functions on X. Then $\mathcal{E}$ is a holomorphic vector bundle on X; this is equipped with a canonical flat connection

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X,$$

induced from $\partial : \mathcal{O}_X \to \Omega^1_X$ (here, Ω^1_X denotes the sheaf of holomorphic differentials on X). The connection ∇ is called the *Gauss-Manin connection.* The additional structure is a descending filtration $F^\bullet\mathcal{E}$ on $\mathcal{E}$ by holomorphic sub-bundles such that

(1) The filtration $F^\bullet\mathcal{E}$ induces Hodge structures of weight n on the fibers of $\mathcal{E}$.

(2) (Griffiths transversality) For all $i \in \mathbb{Z}$, the Gauss-Manin connection satisfies

$$\nabla : F^i\mathcal{E} \to F^{i-1}\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X \subset \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X.$$

If X is simply-connected, the local system $\mathcal{V}_{\mathbb{Z}}$ on X is trivial. By choosing a trivialization of $\mathcal{V}_{\mathbb{R}}$, we recover Definition 2.2.7. As above, we can also define variations of polarizable Hodge structures. With this more general definition, we have the following example.

Example 2.2.10. Let $f : Y \to X$ be a smooth and projective morphism of complex varieties, such that X is smooth. Let $(R^n f_* \mathbb{Z})_{\mathrm{tf}}$ be the torsion-free part of $R^n f_* \mathbb{Z}$. Then the local system $(R^n f_* \mathbb{Z})_{\mathrm{tf}}$ on $X(\mathbb{C})$ is a variation of polarizable Hodge structures of weight n.

2.3. **Shimura varieties.**

2.3.1. *Definition of a Shimura variety.* Shimura varieties are described by *Shimura data*, which are certain pairs (G, X), consisting of a connected reductive group G defined over $\mathbb{Q}$, and a $G(\mathbb{R})$-conjugacy class X of homomorphisms

$$\mathbb{S} \to G_{\mathbb{R}}.$$

As we saw above that $\mathbb{S}$ is the Tannakian group for the category of real Hodge structures, for any finite-dimensional representation V of G on a real vector space, X parametrizes a family of Hodge structures with underlying vector space V. If we choose an element $h \in X$, we can identify X with $G(\mathbb{R})/K_\infty^h$, where K_∞^h is the stabilizer of h in $G(\mathbb{R})$ under conjugacy. We will impose certain additional conditions on (G, X) which will ensure that X carries a unique complex structure making the family of Hodge structures that X parametrizes a variation of polarizable Hodge structures.

In order for a pair (G, X) as above to be a Shimura datum, it has to also satisfy the following axioms.

1. Let $\mathfrak{g}$ denote the Lie algebra of $G(\mathbb{R})$. For any choice of $h \in X$, the composite
$$h : \mathbb{S} \to G_{\mathbb{R}} \to G_{\mathbb{R}}^{\mathrm{ad}} \to \mathrm{GL}(\mathfrak{g}),$$
i.e. the composite with the adjoint action of $G_{\mathbb{R}}$ on $\mathfrak{g}$, induces a Hodge structure of type $(-1, 1), (0, 0), (1, -1)$ on $\mathfrak{g}$.
2. For any choice of $h \in X$, $h(i)$ is a Cartan involution on $G^{\mathrm{ad}}(\mathbb{R})$.
3. G^{ad} has no factor defined over $\mathbb{Q}$ whose real points form a compact group.

Note that, while the first two conditions are formulated for any choice of $h \in X$, it is enough to check them for one choice of $h \in X$. We discuss the role that each of the three axioms plays below. Assume, for simplicity, that X is connected.

The first axiom implies, in particular, that the Hodge structure on $\mathfrak{g}$ induced by the adjoint representation has weight 0, which in turn implies that $h(\mathbb{R}^\times)$ lies in the center of $G(\mathbb{R})$ for one $h \in X$ (equivalently, for all $h \in X$). Even though a given real representation V of G may not give rise to a family of Hodge structures which are homogeneous of a given weight, the fact that $h(\mathbb{R}^\times)$ is central means that we can write V as a direct sum of G-invariant pieces which do give rise to Hodge structures that are homogeneous of a given weight, independent of the choice of $h \in X$. In other words, the weight decomposition on V is independent of $h \in X$.

We can now ask whether the family of Hodge structures parametrized by X can be made into a variation of polarizable Hodge structures, by endowing X with an appropriate complex structure. Choose V to be the direct sum of the representations in a faithful family of representations of G. The fact that the weight decomposition on V is independent of $h \in X$ is all that is needed to show that X carries a unique complex structure for which the family of Hodge structures varies holomorphically. Indeed, if we let $\mathrm{Fl}^{\mathrm{std}}(V_{\mathbb{C}})$ be the product of the flag varieties defined above for each homogenous piece of V, we have an injection

$$X \hookrightarrow \mathrm{Fl}^{\mathrm{std}}(V_{\mathbb{C}}).$$

The complex structure on X is induced from the natural complex structure on the flag variety $\mathrm{Fl}^{\mathrm{std}}(V_{\mathbb{C}})$. Furthermore, the requirement for the family of Hodge structures on X to satisfy Griffiths transversality is equivalent to $\mathfrak{g} = F^{-1}\mathfrak{g}$. Since the Hodge structure on $\mathfrak{g}$ has weight 0, this is in turn equivalent to asking that the Hodge structure on $\mathfrak{g}$ be of type $(-1, 1), (0, 0), (1, -1)$. See Section 1.1 of [Del79] for more details.

For the second axiom, note that $h(i)$ induces an involution of $G^{\mathrm{ad}}(\mathbb{R})$ because the adjoint action of $h(-1)$ is trivial. The fact that $h(i)$ is a Cartan involution of $G^{\mathrm{ad}}(\mathbb{R})$ means that the inner form over $\mathbb{R}$ of G^{ad} defined by the fixed points of the involution $g \mapsto h(i)\bar{g}h(i)^{-1}$ is compact. It is easy to see now that the second axiom is independent of the choice of conjugacy class of $h(i)$. The second axiom guarantees that the variation of Hodge structures on X (obtained by choosing any V as above) is a variation of polarizable Hodge structures. See Section 1.1 [Del79] for more details. We note that this axiom implies that the stabilizer $K_{\infty}^{h} \subset G(\mathbb{R})$ of any $h \in X$ is compact modulo center.

The third axiom is fairly harmless to assume (since we could replace G with its quotient by a connected normal subgroup whose group of real points is compact), and it allows us to use strong approximation when G is simply-connected.

When (G, X) is a Shimura datum, Deligne proves that X is a finite disjoint union of *Hermitian symmetric domains* in [Del79]. For a neat compact open subgroup $K \subset G(\mathbb{A}_f)$, the double quotient

$$G(\mathbb{Q}) \setminus (X \times G(\mathbb{A}_f)/K)$$

has the structure of an algebraic variety, called a *Shimura variety.* The Shimura variety has a canonical model which is a smooth, quasi-projective variety defined over a number field E, called the *reflex field* of the Shimura datum. Choose a representative $h \in X$. This gives rise to a cocharacter

$$\mu_h := h \times_{\mathbb{R}} \mathbb{C}|_{(\text{1st } \mathbb{G}_m \text{ factor})} : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}.$$

The axioms in the definition of a Shimura datum imply that the cocharacter μ_h is *minuscule*, i.e. its pairing with any root of $G_{\mathbb{C}}$ is in the set $\{-1, 0, 1\}$. The $G(\mathbb{C})$-conjugacy class $\{\mu_h\}$ is independent of h. The reflex field E is the field of definition of the conjugacy class $\{\mu_h\}$ (this may be smaller than the field of definition of the cocharacter μ_h). From now on, we denote by X_K the canonical model of the Shimura variety over E.

Example 2.3.2 (Modular curves)*.* Let V be a 2-dimensional vector space over $\mathbb{Q}$. We consider the algebraic group over $\mathbb{Q}$ given by $G := \mathrm{GL}(V)$. Let X be the set of complex structures on $V \otimes_{\mathbb{Q}} \mathbb{R}$, i.e. of embeddings $\mathbb{C} \subset \mathrm{End}_{\mathbb{R}}(V \otimes_{\mathbb{Q}} \mathbb{R})$. Then X can be identified with a $G(\mathbb{R})$-conjugacy class of homomorphisms

$$h : \mathbb{S} \to G_{\mathbb{R}}$$

via $x \in X \mapsto h_x : \mathbb{S} \to G_{\mathbb{R}}$, where for every $z \in \mathbb{S}(\mathbb{R}) \simeq \mathbb{C}^{\times}$, $h_x(z) \in \mathrm{GL}(V_{\mathbb{R}})$ is identified with $z \in \mathbb{C}^{\times} \subset \mathrm{Aut}_{\mathbb{R}}(V \otimes_{\mathbb{Q}} \mathbb{R})$. One can check that the three axioms for (G, X) to be a Shimura datum are satisfied.

By choosing a basis of V, we can identify G with GL_2 and X with $\mathbb{H}^{\pm}$, the disjoint union of the upper and lower half planes. We see that the symmetric space for $\mathrm{GL}_2/\mathbb{Q}$ can be identified with the conjugacy class X. The corresponding Shimura varieties are disjoint unions of finitely many copies of connected modular curves.

Let Λ be a fixed $\mathbb{Z}$-lattice in V. By Example 2.2.2, we see that X can be identified with the set of integral Hodge structures of type $(-1, 0), (0, -1)$ on Λ. All such Hodge structures are polarizable, so X can be identified with a moduli of Hodge structures of elliptic curves over $\mathbb{C}$. This is the reason for the moduli interpretation of modular curves in terms of elliptic curves together with level structures.

The period morphism taking a Hodge structure to the corresponding Hodge–de Rham filtration can be identified with the natural embedding

$$\mathbb{H}^{\pm} \hookrightarrow \mathbb{P}^1(\mathbb{C})$$

Note that this is equivariant for the action of $\mathrm{GL}_2(\mathbb{R})$ on both sides: given by Möbius transformations on the left hand side and factoring through the usual action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{P}^1(\mathbb{C})$.

Exercise 2.3.3. *Write down the identification* $\mathbb{H}^{\pm} \simeq X$ *such that the usual action of* $\mathrm{GL}_2(\mathbb{R})$ *on* $\mathbb{H}^{\pm}$ *given by the Möbius transformations*

$$\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R}), \gamma : z \mapsto \frac{az + b}{cz + d}$$

can be recovered from the conjugation action of $\mathrm{GL}_2(\mathbb{R})$ *on the set of homomorphisms* $\mathbb{S} \to \mathrm{GL}_{2,\mathbb{R}}$ *from Example 2.3.2.*

We end this section by giving examples of higher-dimensional Shimura varieties. The key examples that we will consider in these lecture notes will be Siegel modular varieties (which are the simplest Shimura varieties from the point of view of the moduli problem that they satisfy) and Shimura varieties for quasi-split unitary groups (which also have an explicit moduli interpretation).

Example 2.3.4 (Siegel modular varieties). Let $n \geq 1$ and let

$$(V, \psi) = \left(\mathbb{Q}^{2n}, \psi((a_i), (b_i)) = \sum_{i=1}^{n} (a_i b_{n+i} - a_{n+i} b_i)\right)$$

be the split symplectic space of dimension $2n$ over $\mathbb{Q}$. Consider the symplectic similitude group $\mathrm{GSp}_{2n} := \mathrm{GSp}(V, \psi)$; this is the algebraic group over $\mathbb{Q}$ defined by

$$\mathrm{GSp}_{2n}(R) = \left\{(g, \lambda) \in \mathrm{GL}(V \otimes_{\mathbb{Q}} R) \times R^{\times} | \psi(gv, gw) = \lambda \cdot \psi(v, w), \forall v, w \in V \otimes_{\mathbb{Q}} R\right\}$$

for any $\mathbb{Q}$-algebra R. In other words, GSp_{2n} is the group of automorphisms of V preserving the symplectic form up to a scalar, called the similitude factor, which is a unit. One can identify the corresponding symmetric space X with the Siegel double space, which has the following explicit description

$$\left\{Z \in \mathrm{M}_n(\mathbb{C}) | Z = Z^t, \mathrm{Im}(Z) \text{ positive or negative definite}\right\},$$

where $\mathrm{Im}(Z)$ denotes the imaginary part of the matrix Z. The Siegel double space has an action of $\mathrm{GSp}_{2n}(\mathbb{R})$, via

$$\Gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GSp}_{2n}(\mathbb{R}), \Gamma : Z \mapsto (AZ + B)(CZ + D)^{-1},^{14}$$

---

[14] These are $n \times n$-matrices, so for $n > 1$ the order of multiplication matters.

which is transitive. The stabilizer in $\mathrm{GSp}_{2n}(\mathbb{R})$ of the matrix $i \cdot \mathrm{Id}_n$ can be identified with $\mathrm{U}(n) \times \mathbb{R}_{>0}$; the unitary group $U(n)$ is the identity component of a maximal compact subgroup of $\mathrm{GSp}_{2n}(\mathbb{R})$. This shows that we do have an identification of the Siegel double space with the symmetric space for GSp_{2n}.

Exercise 2.3.5. *Check that the action of* $\mathrm{GSp}_{2n}(\mathbb{R})$ *described above preserves the Siegel double space, that it is transitive and compute the stabilizer of* $i \cdot \mathrm{Id}_n$.

The Siegel double space X is a disjoint union of two copies of a Hermitian symmetric domain. Using the classification of Hermitian symmetric domains in [Del79], one sees that X can be identified with a conjugacy class of homomorphisms

$$h : \mathbb{S} \to \mathrm{GSp}_{2n,\mathbb{R}}$$

such that the pair (GSp_{2n}, X) satisfies the three axioms in the definition of a Shimura datum. The corresponding Shimura varieties are called *Siegel modular varieties.* When $n = 1$, we have an isomorphism $\mathrm{GSp}_2 \simeq \mathrm{GL}_2$ of algebraic groups over $\mathbb{Q}$, and in this case we recover the modular curves.

Fix the lattice $\Lambda = \mathbb{Z}^{2n}$ in V (which is self-dual under the symplectic form ψ). For every $h \in X$, let

$$\mu_h := h \times_{\mathbb{R}} \mathbb{C}|_{(\text{1st } \mathbb{G}_{m,\mathbb{C}}-\text{factor})};$$

this defines a cocharacter $\mu_h : \mathbb{G}_{m,\mathbb{C}} \to \mathrm{GSp}_{2n,\mathbb{C}}$. For every $h \in X$, the Hodge structure induced by μ_h on V has type $(-1,0),(0,-1)$ and is polarizable by the second axiom in the definition of a Shimura datum. This Hodge structure gives rise by Theorem 2.2.4 to the abelian variety over $\mathbb{C}$ with associated complex torus $V^{(-1,0)}/\Lambda$. This abelian variety has dimension n.

For $K \subset \mathrm{GSp}_{2n}(\mathbb{A}_f)$ a neat compact open subgroup, the corresponding Shimura variety X_K is a moduli space of polarized g-dimensional abelian varieties with level-K-structure. X_K has a canonical model, which is a smooth, quasi-projective variety over the reflex field $\mathbb{Q}$. It carries a universal abelian variety A^{univ} and a natural ample line bundle ω given by the determinant of the sheaf of invariant differentials on A^{univ}.

If p is a good prime for the level K (i.e.), X_K, A^{univ}, ω admit integral models over the localization $\mathbb{Z}_{(p)}$. The integral model X_K is a smooth, quasi-projective but not projective scheme over $\operatorname{Spec} \mathbb{Z}_p$. It admits a minimal (Baily–Borel–Satake) compactification $X_K \hookrightarrow X_K^*$, which is a projective but usually not smooth scheme over $\operatorname{Spec} \mathbb{Z}_{(p)}$. This was constructed by Faltings and Chai in [FC90]. The ample line bundle ω extends canonically to X_K^*.

Example 2.3.6 (Shimura varieties of PEL type). Shimura varieties of PEL type are Shimura varieties which admit a moduli interpretation in terms of abelian varieties equipped with *p*olarizations, *e*ndomorphisms and *l*evel structure. Siegel modular varieties give examples of PEL-type Shimura varieties, since they parametrize abelian varieties equipped with polarizations and level structure. General PEL-type Shimura varieties admit closed embeddings into Siegel modular varieties and they can be studied via these closed embeddings, but they can also be studied directly via their moduli interpretation. One of the key examples of PEL type Shimura varieties that we will consider in these lecture notes will be that of unitary Shimura varieties (and, in particular, those for quasi-split unitary similitude groups).

Let F be an imaginary CM field, with $F^+ \subset F$ maximal totally real subfield. Let $x \mapsto x^*$ denote the non-trivial automorphism in $\mathrm{Gal}(F/F^+)$. Let V be a $2n$-dimensional F-vector space and let

$$\psi(\ ,\ ) : V \times V \to \mathbb{Q}$$

be a non-degenerate alternating $*$-Hermitian form on V. Let $G/\mathbb{Q}$ be the algebraic group of unitary similitudes of (V, ψ): if R is a $\mathbb{Q}$-algebra, then

$$G(R) := \left\{ (g, \lambda) \in \mathrm{GL}(V \otimes_{\mathbb{Q}} R) \times R^\times \mid \psi(gv, gw) = \lambda \cdot \psi(v, w), \forall v, w \in V \otimes_{\mathbb{Q}} R \right\}.$$

The group of real points $G(\mathbb{R})$ can be identified with

$$G\left(\prod_{i=1}^{[F^+:\mathbb{Q}]} U(p_i, q_i) \right),$$

where i indexes embeddings $F^+ \hookrightarrow \mathbb{R}$ and $U(p_i, q_i)$ is the real unitary group of signature (p_i, q_i) with $p_i + q_i = 2n$. (By the notation $G(\ )$, we mean that the similitude factors for all embeddings $F^+ \hookrightarrow \mathbb{R}$ match.)

If $F^+ = \mathbb{Q}$, then we only have one signature (p, q). The corresponding group of real points $G(\mathbb{R})$ can then be identified with $GU(p, q)$, the group of unitary similitudes which preserve up to a scalar the form

$$\langle (a_j), (b_j) \rangle = \sum_{j=1}^{p} a_j \bar{b}_j - \sum_{j=p+1}^{n} a_j \bar{b}_j.$$

We can always arrange that G is a quasi-split group over $\mathbb{Q}$ (this depends on the choice of ψ). Since $2n = \dim_F V$ is even, the quasi-split inner form will have signature (n, n) at every embedding $F^+ \hookrightarrow \mathbb{R}$. (If we had worked V with $\dim_F V = 2n + 1$, then $U(n + 1, n)$ and $U(n, n + 1)$ are isomorphic quasi-split unitary groups over $\mathbb{R}$.)

Remark 2.3.7.
(1) For the purposes of studying the corresponding Shimura varieties, we can assume that the set of signatures $(p_i, q_i)_{i \in \{1, \ldots, [F^+:\mathbb{Q}]\}}$ is arbitrary. We do note that the Hasse principle for unitary groups gives a restriction on whether a unitary group with given signatures at real embeddings and with specific ramification conditions at finite places exists. See [Clo91] for more details; we will not dwell on this aspect since we will ultimately only need to work with the quasi-split group.
(2) A criterion for PEL-type Shimura varieties to be compact can be found in [Lan13a, §5.3.3]. This satisfied, for example, if one works with a unitary similitude group for which one of the signatures is equal to $(0, n)$ or $(n, 0)$. The Shimura varieties attached to the quasi-split unitary similitude group are non-compact.

An example of a *rational PEL datum* is given by a tuple $(F, *, V, \psi, h)$, where $F, *, V, \psi$ are as above and h is an $\mathbb{R}$-algebra homomorphism

$$h : \mathbb{C} \to \mathrm{End}_{F \otimes_{\mathbb{Q}} \mathbb{R}}(V \otimes_{\mathbb{Q}} \mathbb{R}),$$

such that $\psi(h(z)v, w) = \psi(v, h(\bar{z})w)$ for all $z \in \mathbb{C}$ and such that the pairing

$$\langle v, w \rangle := \psi(v, h(i)w)$$

is symmetric and positive definite. Such a homomorphism puts a complex structure on $V \otimes_{\mathbb{Q}} \mathbb{R}$, which is the same as a Hodge structure of type $(-1, 0), (0, -1)$. By restricting h to $\mathbb{C}^\times$ and noticing that it then preserves ψ up to a scalar in $\mathbb{R}^\times$, we get a homomorphism of algebraic groups over $\mathbb{R}$:

$$h|_{\mathbb{C}^\times} : \mathbb{S} \to G_{\mathbb{R}}$$

Let X be the $G(\mathbb{R})$-conjugacy class of $h|_{\mathbb{C}^\times}$.

Exercise 2.3.8. *Assume that the signatures of G at real embeddings are not all $(0,n)$ or $(n,0)$. Check that the pair (G,X) satisfies the axioms in the definition of a Shimura datum.*

Choose a rational PEL datum as above, giving rise to a Shimura datum (G,X). Let $K \subset G(\mathbb{A}_f)$ be a neat compact open subgroup. Let X_K be the corresponding Shimura variety; it is a smooth, quasi-projective scheme over the reflex E, of dimension $\sum_{i=1}^{[F^+:\mathbb{Q}]} p_i \cdot q_i$. It represents the following moduli problem over E. Let S be a connected, locally noetherian, Spec E-scheme and s a geometric point of S. The moduli problem represented by X_K sends the pair (S,s) to the set of isomorphism classes of tuples $(A, \lambda, \iota, \bar{\eta})$, which is described as follows.

1. A is an abelian scheme over S of dimension $n \cdot [F^+ : \mathbb{Q}]$.
2. $\lambda : A \to A^\vee$ (where $A^\vee$ is a dual abelian variety) is a polarization.
3. $\iota : F \hookrightarrow \mathrm{End}^0(A) := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an embedding of $\mathbb{Q}$-algebras giving an action of F on A by quasi-isogenies.[15] This action satisfies the following compatibility with λ: $\lambda \circ \iota(x^*) = \iota(x)^\vee \circ \lambda$ for all $x \in F$.
4. $\bar{\eta}$ is a $\pi_1^{\text{ét}}(S,s)$-invariant K-orbit of $F \otimes_{\mathbb{Q}} \mathbb{A}_f$-equivariant isomorphisms
$$\eta : V \otimes_{\mathbb{Q}} \mathbb{A}_f \xrightarrow{\sim} V_f A_s,$$
where $V_f A_s$ is the rational adelic Tate module of the abelian variety A_s, such that η takes the pairing induced by ψ on $V \otimes_{\mathbb{Q}} \mathbb{A}_f$ to an $\mathbb{A}_f^\times$-multiple of the λ-Weil pairing on $V_f A_s$. [16]

Such a tuple is required to satisfy the following *determinant condition*: the complex structure on $V \otimes_{\mathbb{Q}} \mathbb{R}$ induced by h gives rise to the Hodge decomposition $V \otimes_{\mathbb{Q}} \mathbb{C} = V^{0,-1} \oplus V^{-1,0}$. Explicitly, we must have

$$\det(x|V^{-1,0}) = \det_{\mathcal{O}_S}(x|\mathrm{Lie}\ A), x \in F.$$

This should be understood as an equality of polynomials with $\mathcal{O}_S$-coefficients rather than as an equality of numbers, where we choose a basis for F over $\mathbb{Q}$ and write the indeterminate $x \in F$ in terms of the chosen basis. In characteristic 0, this is just a condition on ranks of $F \otimes_{\mathbb{Q}} \mathcal{O}_S$-modules, but it is more subtle in characteristic p. Intuitively, the determinant condition matches the Hodge structure of the abelian variety A, as it decomposes under the action of F, with the Hodge structures parametrized by the Hermitian symmetric domain X, which are also restricted by the action of F on (V, ψ).

Two such tuples $(A, \lambda, \iota, \bar{\eta})$ and $(A', \lambda', \iota', \bar{\eta'})$, satisfying the determinant condition, are isomorphic if there exists an isogeny $A \to A'$ taking λ to a rational multiple of λ', and taking ι to ι', $\bar{\eta}$ to $\bar{\eta'}$.

If p is a good prime (for the PEL-type Shimura datum and for the level K) then one can also define an integral model of X_K, which is a smooth, quasi-projective

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[15] These are the "endomorphisms" in the PEL-type moduli problem.

[16] This definition can be shown to be independent of the choice of geometric point s and can be extended to non-connected schemes in the obvious way.

scheme over the localization $\mathcal{O}_{E_{(p)}}$. This integral model is also constructed as the universal scheme representing a moduli problem, this time with integral data. For more details on integral models in the case of PEL-type Shimura varieties, see [Kot92b]. There exists a minimal (Baily–Borel) compactification $X_K \hookrightarrow X_K^*$, constructed in this case by Lan in [Lan13b], which is a projective, but not necessarily smooth scheme over $\operatorname{Spec} \mathcal{O}_{E_{(p)}}$.

2.3.9. *Shimura varieties of Hodge type.* Shimura varieties of Hodge type form a class of Shimura varieties which contain the ones of PEL type. To define them, we will first describe morphisms of Shimura data.

Definition 2.3.10. *A morphism of Shimura data $(G, X) \to (G', X')$ is a homomorphism of algebraic groups $G \to G'$ inducing a map $X \to X'$. We call a morphism of Shimura data an embedding if the map $G \to G'$ is injective.*

A Shimura datum of *Hodge type* is a Shimura datum (G, X) which admits an embedding $(G, X) \hookrightarrow (\widetilde{G}, \widetilde{X})$ into some Siegel datum $(\widetilde{G}, \widetilde{X})$. Given a Shimura datum of Hodge type and a neat compact open subgroup $K \subset G(\mathbb{A}_f)$, one can find a neat compact open subgroup $\widetilde{K} \subset \widetilde{G}(\mathbb{A}_f)$ such that we have a closed embedding of Shimura varieties (Proposition 1.15 of [Del71])

$$X_K \hookrightarrow \widetilde{X}_{\widetilde{K}}.$$

The Shimura variety X_K is said to be *of Hodge type.* The universal abelian variety $\widetilde{A}^{\mathrm{univ}}$ over $\widetilde{X}_{\widetilde{K}}$ restricts to an abelian variety A^{univ} over X_K.

Let (V, ψ) be the $2n$-dimensional split symplectic space over $\mathbb{Q}$ as defined above and set $\widetilde{G} = \mathrm{GSp}(V, \psi)$. If $(G, X) \hookrightarrow (\widetilde{G}, \widetilde{X})$ is an embedding of Shimura data, then there exists a finite collection of tensors

$$s_\alpha \subset V^{\otimes} := \oplus_{m,r \in \mathbb{Z}_{\geq 0}} V^{\otimes m} \otimes (V^\vee)^{\otimes r}, m, r \in \mathbb{Z}$$

such that $G = \mathrm{Stab}_{\widetilde{G}}(\{s_\alpha\})$. This holds by Proposition 3.1 of [Del82]. If we consider any choice of $h \in X$ we get an action of $\mathbb{S}$ on V by composing h with $G(\mathbb{R}) \hookrightarrow \widetilde{G}(\mathbb{R}) = \mathrm{GSp}(V_{\mathbb{R}}, \psi)$. Since G stabilizes the collection $\{s_\alpha\}$, we see that the tensors $s_\alpha \otimes 1 \in V_{\mathbb{R}}^{\otimes}$ are also stabilized by $\mathbb{S}$. This can be reformulated to say that the tensors s_α live in Hodge degree $(0, 0)$, i.e. that they are *Hodge tensors.* Once we understand Siegel modular varieties, Shimura varieties of Hodge type can be studied by keeping track of Hodge tensors.

The symplectic form ψ gives rise to a Hodge tensor. In the case of Shimura varieties of PEL type, the additional Hodge tensors one needs to keep track of are particularly simple: they are given by the endomorphisms by the CM field F. Indeed, an endomorphism of a Hodge structure V respecting the Hodge decomposition can be thought of as a degree $(0, 0)$ element in $V \otimes V^\vee$. This explains why Shimura varieties of PEL type are a subclass of Shimura varieties of Hodge type.

Example 2.3.11. A Shimura variety of Hodge type that is not of PEL type is obtained as follows: we consider the same setup as in Example 2.3.6, but rather than taking the group of unitary similitudes we consider the group of unitary isometries:

$$G(R) := \{g \in \mathrm{GL}(V \otimes_{\mathbb{Q}} R) \mid \psi(gv, gw) = \psi(v, w), \forall v, w \in V \otimes_{\mathbb{Q}} R\}.$$

It is not hard to see that, over $\overline{\mathbb{Q}}$, these Shimura varieties are connected components of the corresponding PEL-type Shimura varieties for the similitude group.

Let (G, X) be a Shimura datum of Hodge type and let μ denote the Hodge cocharacter determined by a choice of $h \in X$. Recall that the axioms for (G, X) to be a Shimura datum imply that μ is a minuscule cocharacter. The cocharacter μ determines two opposite parabolic subgroups of G:

$$P_\mu^{\mathrm{std}} := \{g \in G \mid \lim_{t\to\infty} \mathrm{ad}(\mu(t))g \text{ exists}\}, \text{ and}$$

$$P_\mu := \{g \in G \mid \lim_{t\to 0} \mathrm{ad}(\mu(t))g \text{ exists}\}.$$

Remark 2.3.12. From the Tannakian point of view, the first parabolic can be thought of as the "stabilizer of the Hodge–de Rham filtration". Indeed, the Hodge cocharacter μ induces a *grading* on the Tannakian category $\mathrm{Rep}_{\mathbb{C}}(G)$, the category of finite-dimensional representations G on $\mathbb{C}$-vector spaces. This means that for any $(V, \varphi) \in \mathrm{Rep}_{\mathbb{C}}(G)$, the composition $\varphi \circ \mu$ defines an action of $\mathbb{G}_{m,\mathbb{C}}$ on V, which is the same as a grading

$$V = \oplus_{p\in\mathbb{Z}} V^p$$

of the $\mathbb{C}$-vector space V. Note that this is *not* the same as defining a grading on V as a representation of G. The grading depends functorially on V and is compatible with tensor products in $\mathrm{Rep}_{\mathbb{C}}(G)$.

To the grading on $\mathrm{Rep}_{\mathbb{C}}(G)$ one can naturally associate two filtrations. We let $\mathrm{Fil}^\bullet(\mu)$ be the descending filtration on $\mathrm{Rep}_{\mathbb{C}}(G)$ defined by $\mathrm{Fil}^p(V) = \oplus_{p'\geq p} V^{p'}$ for each $(V, \varphi) \in \mathrm{Rep}_{\mathbb{C}}(G)$. The parabolic subgroup P_μ^{std} can be defined as the stabilizer of $\mathrm{Fil}^\bullet(\mu)$ in G. The other filtration is the ascending filtration $\mathrm{Fil}_\bullet(\mu)$ defined by $\mathrm{Fil}_p(V) = \oplus_{p'\leq p} V^{p'}$ for $(V, \varphi) \in \mathrm{Rep}_{\mathbb{C}}(G)$; the parabolic P_μ is the stabilizer of $\mathrm{Fil}_\bullet(\mu)$ in G.

Choose an embedding of Shimura data $(G, X) \hookrightarrow (\widetilde{G}, \widetilde{X})$ with $\widetilde{G} = \mathrm{GSp}(V, \psi)$, and compatible levels $K \subset G(\mathbb{A}_f)$, $\widetilde{K} \subset \widetilde{G}(\mathbb{A}_f)$. The representation V of $G(\mathbb{Q})$ determines a $\mathbb{Q}$-local system on the Shimura variety $X_K(\mathbb{C})$. This local system is the same as the relative rational Betti homology $\mathcal{V}_{\mathrm{B}}$ of abelian variety $\mathcal{A}(\mathbb{C})$ over $X_K(\mathbb{C})$, obtained by restriction from the universal abelian variety over $\widetilde{X}_{\widetilde{K}}(\mathbb{C})$. By considering the relative de Rham homology of $\mathcal{A}$, we also have a vector bundle $\mathcal{V}_{\mathrm{dR}}$ on X_K, equipped with an integrable connection. The filtration $\mathrm{Fil}^\bullet(V_{\mathbb{C}})$ gets identified, under the comparison between relative Betti and de Rham homologies, with the Hodge–de Rham filtration on $\mathcal{V}_{\mathrm{dR}}$. This is the sense in which we mean that P_μ^{std} is the "stabilizer of the Hodge–de Rham filtration".

The conjugacy classes of both parabolics are defined over the reflex field E of the Shimura datum. The two parabolics determine two flag varieties $\mathrm{Fl}_{G,\mu}^{\mathrm{std}}$ and $\mathrm{Fl}_{G,\mu}$ over E, which parametrize parabolic subgroups in the given conjugacy class, or equivalently, filtrations on $\mathrm{Rep}_{\mathbb{C}}(G)$ conjugate to $\mathrm{Fil}^\bullet(\mu)$. There is a holomorphic embedding

$$X \hookrightarrow \mathrm{Fl}_{G,\mu}^{\mathrm{std}}, h \mapsto \mathrm{Fil}^\bullet(\mu_h)^{17}.$$

The map π_{HdR} defined in Section 2.2 factors through the above embedding. The two flag varieties and the embedding π_{HdR} are functorial in the Shimura data.

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[17] There is also an embedding $X \hookrightarrow \mathrm{Fl}_{G,\mu}$, but this is anti-holomorphic.

3. BACKGROUND FROM p-ADIC HODGE THEORY

In this section, we recall the relevant background from p-adic Hodge theory. Let L be a complete, discretely valued field of characteristic 0 with perfect residue field k of characteristic p.[18] Consider a proper smooth morphism $\pi : \mathcal{Y} \to \mathcal{X}$ of smooth rigid analytic spaces over L, considered as adic spaces over $\mathrm{Spa}(L, \mathcal{O}_L)$. In this section, we will:

(1) give a construction of the *relative Hodge–Tate filtration* for $\pi_C : \mathcal{Y}_C \to \mathcal{X}_C$, where C is an algebraically closed perfectoid field extension of L;
(2) explain its relationship to the relative p-adic-de Rham comparison isomorphism and to the relative Hodge–de Rham filtration;
(3) work out the specific example of the morphism
$$\pi : \mathcal{A} \to \mathcal{X}_K$$
obtained by applying the adification functor
$$\{\text{Schemes/Spec } E_{\mathfrak{p}}\} \to \{\text{Adic spaces/Spa}(E_{\mathfrak{p}}, \mathcal{O}_{E,\mathfrak{p}})\}.$$
to $\pi : A^{\mathrm{univ}} \to X_K$, where X_K is a Shimura variety of Hodge type with reflex field E, A^{univ} is the universal abelian variety over X_K and $\mathfrak{p} \mid p$ is a prime of E. If one is merely interested in the form of the relative Hodge–Tate filtration rather than in its construction and relationship to the Hodge–de Rham filtration, one can skip to Example 3.2.6.

Remark 3.0.1. For this section, we assume as prerequisites: adic spaces, perfectoid spaces, the *flattened pro-étale topology*, i.e the pro-étale topology as used in [Sch13], and the flattened pro-étale site $\mathcal{X}_{\text{proét}}$ of a smooth adic space $\mathcal{X}$ over $\mathrm{Spa}(L, \mathcal{O}_L)$. The proofs of the statements from p-adic Hodge theory are given in Bhatt's lecture notes in this volume, so we contend ourselves to stating the precise results we will use in the study of Shimura varieties. The references we follow are [Sch13], Section 3 of [Sch12b], and Section 2.2 of [CS17].

3.1. **The Hodge–Tate filtration.** We will start with an extended example, where we discuss the Hodge–Tate filtration in the case over points, i.e when $\mathcal{X} = \mathrm{Spa}(L, \mathcal{O}_L)$. Let C be the p-adic completion of an algebraic closure $\overline{L}$ of L, with ring of integers $\mathcal{O}_C$. Let $\mathcal{X}_C := \mathrm{Spa}(C, \mathcal{O}_C)$, with tilt $\mathcal{X}_C^\flat := \mathrm{Spa}(C^\flat, \mathcal{O}_{C^\flat})$. We recall the construction of the ring $B_{\mathrm{dR},C}$, originally due to Fontaine: denote by $B^+_{\mathrm{dR},C}$ the completion of $W(\mathcal{O}_{C^\flat})[1/p]$ along the kernel of the map
$$\theta : W(\mathcal{O}_{C^\flat})[1/p] \to C$$
and then set $B_{\mathrm{dR},C} := B^+_{\mathrm{dR},C}[1/\xi]$ for a generator ξ of $\ker\theta$. The field $B_{\mathrm{dR},C}$ is the field of periods which shows up in the original comparison isomorphism between de Rham and p-adic étale cohomology, i.e. in the setting of schemes. The subring $B^+_{\mathrm{dR},C} \subset B_{\mathrm{dR},C}$ is a complete discrete valuation ring with residue field C and with uniformizer ξ. There is a $\mathrm{Gal}(\overline{L}/L)$-action on ξ, which is via the cyclotomic character. There is a natural decreasing filtration on the ring $B_{\mathrm{dR},C}$ defined by $\mathrm{Fil}^i B_{\mathrm{dR},C} := \xi^i B^+_{dR,C}$ for $i \in \mathbb{Z}$. It has graded pieces $\mathrm{Gr}^i B_{\mathrm{dR},C} \simeq C(i)$.

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[18] Later on, L will be a finite extension of $\mathbb{Q}_p$, more precisely the completion $E_{\mathfrak{p}}$ of the reflex field E at a prime $\mathfrak{p}$ above a good prime p.

We now let $\pi : \mathcal{Y} \to \mathcal{X}$ be a proper smooth rigid analytic variety. Because $\mathcal{Y}$ is defined over L, the Hodge–de Rham spectral sequence

$$E_1^{i-j,j} = H^j(\mathcal{Y}, \Omega_{\mathcal{Y}}^{i-j}) \Rightarrow H^i_{\mathrm{dR}}(\mathcal{Y})$$

degenerates on the first page (cf [Sch13, Thm. 1.6]). The induced filtration on $H^i_{\mathrm{dR}}(\mathcal{Y})$ is called the *Hodge–de Rham filtration*; it has graded pieces $H^j(\mathcal{Y}, \Omega_{\mathcal{Y}}^{i-j})$. There is a natural comparison isomorphism between the p-adic étale cohomology of $\mathcal{Y}$ and the de Rham cohomology of $\mathcal{Y}$

$$H^i(\mathcal{Y}_{\bar{L},\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR},C} \simeq H^i_{\mathrm{dR}}(\mathcal{Y}) \otimes_L B_{\mathrm{dR},C}, \tag{3.1.1}$$

cf. [Sch13, Cor. 1.8]. This isomorphism is $\mathrm{Gal}(\overline{L}/L)$-equivariant and also compatible with the following filtrations on each side: on the LHS, we consider the filtration induced from the natural filtration on $B_{\mathrm{dR},C}$ and on the RHS we consider the convolution of the de Rham filtration on $H^i_{\mathrm{dR}}(\mathcal{Y})$ and the natural filtration on $B_{\mathrm{dR},C}$.

By applying Gr^0 on both sides of (3.1.1), we obtain the direct sum decomposition

$$H^i(\mathcal{Y}_{\bar{L},\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \simeq \oplus_{j=0}^{i} H^{i-j}(\mathcal{Y}, \Omega^j_{\mathcal{Y}}) \otimes_L C(-j),$$

known as the *Hodge–Tate decomposition.* When we consider the analogous p-adic comparison isomorphism in the relative setting, there is no longer a direct sum decomposition. This fact mirrors the complex phenomenon, where only the Hodge–de Rham filtration varies holomorphically, not the Hodge decomposition. It is therefore better to only remember the filtration

$$\mathrm{Fil}^j \left(H^i(\mathcal{Y}_{\bar{L},\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \right) = \oplus_{k=j}^{i} H^{i-k}(\mathcal{Y}, \Omega^k_{\mathcal{Y}}) \otimes_L C(-k)$$

induced by this direct sum decomposition.

The following is the perspective that generalizes to the relative setting: we interpret the comparison isomorphism as saying that the p-adic étale cohomology of $\mathcal{Y}_{\overline{L}}$ and the de Rham cohomology of $\mathcal{Y}$ give rise to two $B^+_{\mathrm{dR},C}$-lattices contained in the same $B_{\mathrm{dR},C}$-vector space. Indeed, we define:

$$M = H^i(\mathcal{Y}_{\overline{L},\text{ét}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B^+_{\mathrm{dR},C}, \text{ and } M_0 = H^i_{\mathrm{dR}}(\mathcal{Y}) \otimes_L B^+_{\mathrm{dR},C}.$$

We can define filtrations on both M, M_0 which measure the relative position of the two lattices. The filtration on $M_0/\xi M_0 = H^i_{\mathrm{dR}}(\mathcal{Y}) \otimes_L C$ induced by the lattice M agrees with the Hodge–de Rham filtration.

On $M/\xi M = H^i(\mathcal{Y}_{C,\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C$, we have the *Hodge–Tate filtration*, defined as follows in [Sch13]. Consider the morphism of sites

$$\nu : \mathcal{Y}_{C,\text{proét}} \to \mathcal{Y}_{C,\text{ét}};$$

this gives rise to a spectral sequence

$$E_2^{i-j,j} = H^{i-j}(\mathcal{Y}_{C,\text{ét}}, R^j\nu_*\widehat{\mathcal{O}}_{\mathcal{Y}_C}) \Rightarrow H^i(\mathcal{Y}_{C,\text{proét}}, \widehat{\mathcal{O}}_{\mathcal{Y}_C}) \xrightarrow{\sim} H^i(\mathcal{Y}_{C,\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C^{19}.$$

In [Sch13], Scholze shows that there are natural isomorphisms

$$\Omega^j_{\mathcal{Y}_C}(-j) \simeq R^j\nu_*\widehat{\mathcal{O}}_{\mathcal{Y}_C}$$

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[19] This last isomorphism is the *primitive comparison theorem* in p-adic Hodge theory. For schemes, the primitive comparison theorem goes back to Faltings, cf. [Fal02], and for rigid-analytic varieties this is [Sch13, Thm 1.3]. The latter result underlies all other p-adic comparison theorems for rigid-analytic varieties.

for all $j \geq 0$. The Hodge–Tate spectral sequence

$$E_2^{i-j,j} = H^{i-j}(\mathcal{Y}, \Omega^j_{\mathcal{Y}}) \otimes_L C(-j) \Rightarrow H^i(\mathcal{Y}_{C,\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C$$

then degenerates on the E_2 page because $\mathcal{Y}$ is defined over the subfield $L \subset C$ and the differentials are $\mathrm{Gal}(\bar{L}/L)$-equivariant. The corresponding filtration on $H^i(\mathcal{Y}_{C,\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C$ is the Hodge–Tate filtration. This is the same as the filtration on $M/\xi M$ induced by the lattice M_0. (In the case when $\mathcal{X}$ is a point, the fact that these two filtrations on $H^i(\mathcal{Y}_{\bar{L},\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C$ agree can be seen from the fact that the Hodge–Tate decomposition is canonical, because it is $\mathrm{Gal}(\overline{L}/L)$-equivariant.)

Example 3.1.1. If we set $i = 1$, we have $\xi M \subset M_0 \subset M$, with $M_0/\xi M \simeq H^1(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \otimes_L C$ and $M/M_0 \simeq H^0(\mathcal{Y}, \Omega^1_{\mathcal{Y}}) \otimes_L C(-1)$. The Hodge–de Rham filtration on $H^i_{\mathrm{dR}}(\mathcal{Y}) \otimes_L C$ is given by

$$0 \to \xi M/\xi M_0 \to M_0/\xi M_0 \to M_0/\xi M \to 0,$$

which becomes

$$0 \to H^0(\mathcal{Y}, \Omega^1_{\mathcal{Y}}) \otimes_L C \to H^1_{\mathrm{dR}}(\mathcal{Y}) \otimes_L C \to H^1(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \otimes_L C \to 0.$$

The Hodge–Tate filtration on $H^i(\mathcal{Y}_{\bar{L},\text{ét}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C$ is given by

$$0 \to M_0/\xi M \to M/\xi M \to M/M_0 \to 0,$$

which becomes

$$0 \to H^1(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to H^1(\mathcal{Y}_{\bar{L},\text{ét}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \to H^0(\mathcal{Y}, \Omega^1_{\mathcal{Y}}) \otimes_L C(-1) \to 0.$$

Note that the graded pieces of these two filtration are isomorphic (up to Tate twists) but the filtrations themselves are not directly related.

3.2. **The relative Hodge–Tate filtration.** We now discuss a relative definition of the Hodge–Tate filtration, which will be crucial to our application to Shimura varieties.

For $\mathcal{X}$ a smooth adic space over $\mathrm{Spa}(L, \mathcal{O}_L)$, we have the following sheaves on $\mathcal{X}_{\text{proét}}$, as defined in [Sch13]: the (integral) completed structure sheaf $\widehat{\mathcal{O}}^{(+)}_{\mathcal{X}}$, the (integral) tilted completed structure sheaf $\widehat{\mathcal{O}}^{(+)}_{\mathcal{X}^\flat}$, the relative period sheaves $\mathbb{B}^{(+)}_{\mathrm{dR},\mathcal{X}}$, and the structural de Rham sheaves $\mathcal{O}\mathbb{B}^{(+)}_{\mathrm{dR},\mathcal{X}}$. We recall the definitions of these sheaves.

Definition 3.2.1.
(1) *The integral completed structure sheaf* $\widehat{\mathcal{O}}^+_{\mathcal{X}}$ *is the inverse limit of the sheaves* $\mathcal{O}^+_{\mathcal{X}}/p^n$ *on* $\mathcal{X}_{\text{proét}}$. *The titled integral structure sheaf* $\widehat{\mathcal{O}}^+_{\mathcal{X}^\flat}$ *is the inverse limit on* $\mathcal{X}_{\text{proét}}$ *of* $\mathcal{O}^+_{\mathcal{X}}/p$ *with respect to the Frobenius morphism.*
(2) *The relative period sheaf* $\mathbb{B}^+_{\mathrm{dR},\mathcal{X}}$ *is the completion of* $W(\widehat{\mathcal{O}}^+_{\mathcal{X}^\flat})[1/p]$ *along the kernel of the natural map*

$$\theta : W(\widehat{\mathcal{O}}^+_{\mathcal{X}^\flat})[1/p] \to \widehat{\mathcal{O}}_{\mathcal{X}}.$$

The relative period sheaf $\mathbb{B}_{\mathrm{dR},\mathcal{X}}$ *is* $\mathbb{B}^+_{\mathrm{dR},\mathcal{X}}[\xi^{-1}]$, *where* ξ *is any element that generates the kernel of* θ. *This is well-defined because such a* ξ *exists proétale locally on* $\mathcal{X}$, *is not a zero divisor, and is unique up to a unit.*

(3) *We now define the sheaf* $\mathcal{O}\mathbb{B}^+_{\mathrm{dR},\mathcal{X}}$ *as the sheafification of the following presheaf. If* $U = \mathrm{Spa}(R, R^+)$ *is affinoid perfectoid, with* (R, R^+) *the completed direct limit of* (R_i, R_i^+)*, the presheaf sends* U *to the direct limit over* i *of the completion of*

$$\left(R_i^+ \hat{\otimes}_{W(k)} W(R^{\flat +})\right)[1/p]$$

along $\ker\theta$*, where*

$$\theta : \left(R_i^+ \hat{\otimes}_{W(k)} W(R^{\flat +})\right)[1/p] \to R$$

is the natural map. We set $\mathcal{O}\mathbb{B}_{\mathrm{dR},\mathcal{X}} := \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\mathcal{X}}[\xi^{-1}]$ *as before.*

These sheaves are equipped with the following structures. The relative period sheaves $\mathbb{B}^{(+)}_{\mathrm{dR},\mathcal{X}}$ are equipped with compatible filtrations: $\mathrm{Fil}^i\mathbb{B}_{\mathrm{dR},\mathcal{X}} := \xi^i\mathbb{B}^+_{\mathrm{dR},\mathcal{X}}$, with $\mathrm{Gr}^0\mathbb{B}_{\mathrm{dR},\mathcal{X}} = \widehat{\mathcal{O}}_{\mathcal{X}}$. The structural de Rham sheaves $\mathcal{O}\mathbb{B}^{(+)}_{\mathrm{dR},\mathcal{X}}$ are equipped with filtrations and connections

$$\nabla : \mathcal{O}\mathbb{B}^{(+)}_{\mathrm{dR},\mathcal{X}} \to \mathcal{O}\mathbb{B}^{(+)}_{\mathrm{dR},\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega^1_{\mathcal{X}}$$

We have a natural identification

$$\left(\mathcal{O}\mathbb{B}^{(+)}_{\mathrm{dR},\mathcal{X}}\right)^{\nabla=0} = \mathbb{B}^{(+)}_{\mathrm{dR},\mathcal{X}}.$$

The following theorem states the relative p-adic-de Rham comparison isomorphism for a proper smooth morphism $\pi : \mathcal{Y} \to \mathcal{X}$ of smooth adic spaces over L. We consider the sheaf $R^i\pi_{\mathrm{dR}*}\mathcal{O}_{\mathcal{Y}}$ on $\mathcal{X}_{\text{proét}}$ obtained by taking the ith cohomology sheaf of the derived pushforward $R\pi_*$ applied to the complex of relative differentials $\Omega^{\cdot}_{\mathcal{Y}/\mathcal{X}}$ on $\mathcal{Y}_{\text{proét}}$. The sheaf $R^i\pi_{\mathrm{dR}*}\mathcal{O}_{\mathcal{Y}}$ is an $\mathcal{O}_{\mathcal{X}}$-module equipped with a filtration (the Hodge–de Rham filtration) and with an integrable connection (the Gauss–Manin connection ∇_{GM}). The Gauss–Manin connection satisfies Griffiths transversality with respect to the Hodge–de Rham filtration.

Theorem 3.2.2. *(Thm 8.8 of* [Sch13]*) For all* $i \geq 0$*, there is a natural isomorphism of sheaves on* $\mathcal{X}_{\text{proét}}$

$$R^i\pi_*\widehat{\mathbb{Z}}_{p,\mathcal{Y}} \otimes_{\widehat{\mathbb{Z}}_{p,\mathcal{X}}} \mathcal{O}\mathbb{B}_{\mathrm{dR},\mathcal{X}} \simeq R^i\pi_{\mathrm{dR}*}\mathcal{O}_{\mathcal{Y}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}\mathbb{B}_{\mathrm{dR},\mathcal{X}},$$

compatible with the filtrations and connections on both sides. [20]

Remark 3.2.3. Note that [Sch13, Thm. 8.8] applies under the assumption that $R^i\pi_{\text{proét}*}\widehat{\mathbb{Z}}_p$ is a lisse $\widehat{\mathbb{Z}}_p$-sheaf on $\mathcal{X}$. This is now guaranteed by [SW17, Thm. 10.5.1].

We will see that the Hodge–de Rham filtration on $R^i\pi_{\mathrm{dR}*}\mathcal{O}_{\mathcal{Y}}$ induces, via the comparison isomorphism in Theorem 3.2.2, a filtration on $R^i\pi_*\widehat{\mathbb{Z}}_{p,\mathcal{Y}}\otimes_{\widehat{\mathbb{Z}}_{p,\mathcal{X}}}\widehat{\mathcal{O}}_{\mathcal{X}}$, which we will call the *relative Hodge–Tate filtration.* To make this precise, we construct two $\mathbb{B}^+_{\mathrm{dR},\mathcal{X}}$-local systems on $\mathcal{X}$. The first one, which is closely related to the relative étale cohomology of $\mathcal{Y}$ is given by

$$\mathbb{M} := R^i\pi_*\widehat{\mathbb{Z}}_{p,\mathcal{Y}} \otimes_{\mathbb{Z}_{p,\mathcal{X}}} \mathbb{B}^+_{\mathrm{dR},\mathcal{X}}.$$

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[20] The filtration and connection on the left hand side are simply induced from the filtration and connection on $\mathcal{O}\mathbb{B}_{\mathrm{dR},\mathcal{X}}$. On the right hand side, one must take the convolution of the Hodge–de Rham filtration with the one on $\mathcal{O}\mathbb{B}_{\mathrm{dR},\mathcal{X}}$ and the connection is $\nabla_{\mathrm{GM}} \otimes 1 + 1 \otimes \nabla$.

The second one, which is closely related to the relative de Rham cohomology of $\mathcal{Y}$, is given by

$$\mathbb{M}_0 := \left(R^i\pi_{\mathrm{dR}*}\mathcal{O}_{\mathcal{Y}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\mathcal{X}}\right)^{\nabla=0}.$$

A consequence of the comparison isomorphism is that $\mathbb{M}$ and $\mathbb{M}_0$ are two “lattices” contained in the same $\mathbb{B}_{\mathrm{dR},\mathcal{X}}$-local system on $\mathcal{X}$.

The following is (a reformulation of) Proposition 7.9 of [Sch13] and Proposition 2.2.3 of [CS17].

Proposition 3.2.4. *There exists a canonical isomorphism*

$$\mathbb{M} \otimes_{\mathbb{B}^+_{\mathrm{dR},\mathcal{X}}} \mathbb{B}_{\mathrm{dR},\mathcal{X}} \simeq \mathbb{M}_0 \otimes_{\mathbb{B}^+_{\mathrm{dR},\mathcal{X}}} \mathbb{B}_{\mathrm{dR},\mathcal{X}}.$$

Consider the descending filtration $\mathrm{Fil}^j\mathbb{M}_{(0)}$ *on* $\mathbb{M}_{(0)}$ *induced by the canonical filtration on* $\mathbb{B}^+_{\mathrm{dR},\mathcal{X}}$*. For any* $j \in \mathbb{Z}$*, there is an identification*

$$(\mathbb{M} \cap \mathrm{Fil}^j\mathbb{M}_0)/(\mathbb{M} \cap \mathrm{Fil}^{j+1}\mathbb{M}_0) = (\mathrm{Fil}^{-j}R^i\pi_{\mathrm{dR}*}\mathcal{O}_{\mathcal{Y}}) \otimes_{\mathcal{O}_{\mathcal{X}}} \widehat{\mathcal{O}}_{\mathcal{X}}(j)$$
$$\subset \mathrm{Gr}^j\mathbb{M}_0 = R^i\pi_{\mathrm{dR}*}\mathcal{O}_{\mathcal{Y}} \otimes_{\mathcal{O}_{\mathcal{X}}} \widehat{\mathcal{O}}_{\mathcal{X}}.$$

In particular, we always have $\mathbb{M}_0 \subset \mathbb{M}$. Moreover, considering the relative position of $\mathbb{M}$ and $\mathbb{M}_0$ induces an ascending filtration on

$$\mathrm{Gr}^0\mathbb{M} = R^i\pi_*\widehat{\mathbb{Z}}_{p,\mathcal{Y}} \otimes_{\widehat{\mathbb{Z}}_{p,\mathcal{X}}} \widehat{\mathcal{O}}_{\mathcal{X}}$$

given by

$$\mathrm{Fil}_{-j}(R^i\pi_*\widehat{\mathbb{Z}}_{p,\mathcal{Y}} \otimes_{\widehat{\mathbb{Z}}_{p,\mathcal{X}}} \widehat{\mathcal{O}}_{\mathcal{X}}) := (\mathbb{M} \cap \mathrm{Fil}^j\mathbb{M}_0)/(\mathrm{Fil}^1\mathbb{M} \cap \mathrm{Fil}^j\mathbb{M}_0).$$

We call this filtration the *relative Hodge–Tate filtration.*

Remark 3.2.5. In this section, we gave the construction of the relative Hodge–Tate filtration via the comparison isomorphism rather than a version of the construction in § 3.1 via the morphism of sites from the proétale to the étale site. For applications to Shimura varieties of Hodge type, we will only need to use the first filtration step on $R^1\pi_*\widehat{\mathbb{Z}}_{p,\mathcal{Y}} \otimes_{\widehat{\mathbb{Z}}_{p,\mathcal{X}}} \widehat{\mathcal{O}}_{\mathcal{X}}$. Proposition 2.2.5 of [CS17], which works in the relative case, shows that the two constructions of the Hodge–Tate filtration agree on the first filtration step.

We made the choice of presenting the construction of the Hodge–Tate filtration via the p-adic comparison isomorphism because this perspective is the one used in constructing the Hodge–Tate period morphism for Shimura varieties of Hodge type in [CS17] (and, as a result, also for Shimura varieties of abelian type in [She17]). We explain this further in section 5. We also chose to present this construction in order to emphasize the close analogy between the period morphisms for Hermitian symmetric domains and the Hodge–Tate period morphism.

Example 3.2.6 (The relative Hodge–Tate filtration for the universal abelian variety). Let (G,X) be a Shimura datum of Hodge type, $K \subset G(\mathbb{A}_f)$ a neat compact open subgroup, and X_K the corresponding Shimura variety over the reflex field E. Since X_K admits a closed embedding into some Siegel modular variety, there exists an abelian scheme $\pi : A^{\mathrm{univ}} \to X_K$.

We let $\mathfrak{p}|p$ be a prime of E, $L := E_{\mathfrak{p}}$, and consider the proper smooth morphism of adic spaces $\pi : \mathcal{A} \to \mathcal{X}_K$ over $\mathrm{Spa}(L, \mathcal{O}_L)$. The relative Hodge–Tate filtration on

$R^1\pi_*\widehat{\mathbb{Z}}_p \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathcal{O}}_{\mathcal{X}_K}$ is encoded in the short-exact sequence of sheaves on $\mathcal{X}_{K,\text{proét}}$

$$0 \to R^1\pi_*\mathcal{O}_{\mathcal{A}} \otimes_{\mathcal{O}_{\mathcal{X}_K}} \widehat{\mathcal{O}}_{\mathcal{X}_K} \to R^1\pi_*\widehat{\mathbb{Z}}_p \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathcal{O}}_{\mathcal{X}_K} \to \pi_*\Omega^1_{\mathcal{A}} \otimes_{\mathcal{O}_{\mathcal{X}_K}} \widehat{\mathcal{O}}_{\mathcal{X}_K}(-1) \to 0.$$

Proposition 2.2.5 of [CS17] shows that the first map in the short exact sequence can be identified with the natural injection

$$R^1\pi_*\mathcal{O}_{\mathcal{A}} \otimes_{\mathcal{O}_{\mathcal{X}_K}} \widehat{\mathcal{O}}_{\mathcal{X}_K} \hookrightarrow R^1\pi_*\widehat{\mathcal{O}}_{\mathcal{A}}$$

of sheaves on $\mathcal{X}_{K,\text{proét}}$, where we have used the primitive relative comparison isomorphism

$$R^1\pi_*\widehat{\mathbb{Z}}_p \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathcal{O}}_{\mathcal{X}_K} \simeq R^1\pi_*\widehat{\mathcal{O}}_{\mathcal{A}}.$$

4. The canonical subgroup and the anticanonical tower

In this section, we describe the theory of the canonical subgroup. We use this theory to explain the construction of the anticanonical tower of formal schemes over the ordinary locus of Siegel modular varieties, which has the following extremely useful properties

1. it overconverges, i.e. it extends to an ε-neighborhood of the ordinary locus;
2. its adic generic fiber gives rise to a perfectoid space.

These two properties, together with the Hodge–Tate period morphism, which is the focus of section 5, are the key ingredients in proving that Siegel modular varieties with infinite level at p are perfectoid. We follow Section 3 of [Sch15], but aim to give more background and fewer technical details.

4.1. **The ordinary locus inside Siegel modular varieties.** In this section, we will only work with the Siegel modular varieties of Example 2.3.4. The same techniques could also be applied directly to the unitary Shimura varieties described in Example 2.3.6, if they are associated to a quasi-split unitary group over $\mathbb{Q}$. We leave this case as an exercise to the reader.[21]

Let $n \geq 1$ and let

$$(V,\psi) = \left(\mathbb{Q}^{2n}, \psi((a_i),(b_i)) = \sum_{i=1}^{n}(a_ib_{n+i} - a_{n+i}b_i)\right)$$

be the split symplectic space of dimension $2n$ over $\mathbb{Q}$. Let $\Lambda = \mathbb{Z}^{2n}$ be the standard lattice in V, which is self-dual under the symplectic form ψ. Consider the group of symplectic similitudes of Λ, $\mathrm{GSp}(\Lambda,\psi)$. This is an algebraic group over $\mathbb{Z}$. Fix a prime number p and a compact open subgroup $K^p \subset \mathrm{GSp}_{2n}(\mathbb{A}_f^p)$ contained in

$$\left\{g \in \mathrm{GSp}_{2n}(\widehat{\mathbb{Z}}^p) \mid g \equiv 1 \pmod N\right\}$$

for some $N \geq 3$ such that $(N,p) = 1$. (This condition is enough to ensure that any level $K = K^pK_p$, with $K_p \subset G(\mathbb{Q}_p)$ compact open is neat.)

Set $K_p = \mathrm{GSp}_{2n}(\mathbb{Z}_p)$, $K := K^pK_p$ and let X_K be the model over $\mathbb{Z}_{(p)}$ of the corresponding Shimura variety. This is the moduli space of principally polarized n-dimensional abelian varieties with K^p-level structure. Since we will keep the tame level K^p fixed in this section, we denote X_K by X_{K_p}. Over Over X_{K_p}, we have

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[21] In fact, the same techniques should be applicable directly to any Shimura variety of PEL type where the *ordinary locus* is non-empty. The main theorem of [Wed99] shows that the ordinary locus inside the special fiber of the Shimura variety is non-empty if and only if p splits completely in the reflex field E of the Shimura datum.

a natural line bundle ω, given by the top exterior power of the sheaf of invariant differentials on the universal abelian scheme.

Remark 4.1.1. As seen above, the case $n = 1$ corresponds to the group GL_2 and the case of modular curves; the constructions and techniques used in this section will be interesting (and relatively novel) even in this case. One may specialize to the case $n = 1$ on a first reading of this section.

On the level of generic fibers, we will also consider the versions with K_p-level structure for other compact open subgroups $K_p \subset G(\mathbb{Q}_p)$. We will be particularly interested in the case

$$\Gamma_0(p^m) := \left\{ g \in \mathrm{GSp}_{2n}(\mathbb{Z}_p) \mid g \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod{p^m}, \lambda(g) \equiv 1 \pmod{p^m} \right\},$$

where $\lambda(g)$ is the symplectic similitude factor of g. For each $m \in \mathbb{Z}_{\geq 1}$, the Shimura variety $X_{\Gamma_0(p^m)}$ admits a morphism to Spec $\mathbb{Q}_p(\mu_{p^m})$ corresponding to the symplectic similitude factor. We will consider the tower $(X_{\Gamma_0(p^m)})_m$ over the perfectoid field $\mathbb{Q}_p^{\mathrm{cycl}}$, by taking the base change at level m along the natural morphism Spec $\mathbb{Q}_p(\mu_{p^m}) \to$ Spec $\mathbb{Q}_p^{\mathrm{cycl}}$.

We let $\mathfrak{X}_{K_p}$ be the p-adic completion of $X_{K_p} \times_{\mathbb{Z}_{(p)}} \mathbb{Z}_p^{\mathrm{cycl}}$ along its special fiber. This is a formal scheme over $\mathrm{Spf}\, \mathbb{Z}_p^{\mathrm{cycl}}$. We let $\mathcal{X}_{K_p}$ be its adic generic fiber, an analytic adic space over $\mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl}}, \mathbb{Z}_p^{\mathrm{cycl}})$. Then $\mathcal{X}_{K_p}$ is a proper open subset of the analytic adic space $\left(X_{K_p} \times_{\mathbb{Z}_{(p)}} \mathbb{Q}_p^{\mathrm{cycl}}\right)^{\mathrm{ad}}$. The subset $\mathcal{X}_{K_p}$ is referred to in [Sch15] as the *good reduction locus*, i.e. the locus where the universal abelian scheme over $\left(X_{K_p} \times_{\mathbb{Z}_{(p)}} \mathbb{Q}_p^{\mathrm{cycl}}\right)^{\mathrm{ad}}$ has good reduction.

Example 4.1.2. Let $\mathbb{A}^1_{\mathbb{Z}_p} :=$ Spec $\mathbb{Z}_p[x]$ be one-dimensional affine space over $\mathbb{Z}_p$ and $\mathbb{P}^1_{\mathbb{Z}_p}$ be the one-dimensional projective space. The open immersion $\mathbb{A}^1_{\mathbb{Z}_p} \hookrightarrow \mathbb{P}^1_{\mathbb{Z}_p}$ is a toy model for the embedding $X_{K_p} \hookrightarrow X^*_{K_p}$ of the integral model X_{K_p} into its minimal compactification. The formal scheme corresponding to $\mathbb{A}^1_{\mathbb{Z}_p}$ is $\mathrm{Spf}\, \mathbb{Z}_p\langle x \rangle$, where

$$\mathbb{Z}_p\langle x \rangle = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in \mathbb{Z}_p \lim_{i \to \infty} |a_i|_p = 0 \right\}$$

and its adic generic fiber is the closed unit disk $\mathrm{Spa}(\mathbb{Q}_p\langle x \rangle, \mathbb{Z}_p\langle x \rangle)$. On the other hand, the adic space $\mathbb{A}^{1,\mathrm{ad}}_{\mathbb{Q}_p}$ corresponding to the scheme $\mathbb{A}^1_{\mathbb{Q}_p}$ is the increasing union of closed disks

$$\bigcup_{m \geq 0} \mathrm{Spa}(\mathbb{Q}_p\langle p^m x \rangle, \mathbb{Z}_p\langle p^m x \rangle)$$

over $m \geq 0$.

Exercise 4.1.3. *Check that for* $\mathbb{P}^1_{\mathbb{Z}_p}$*, both constructions give rise to the same space* $\mathbb{P}^{1,\mathrm{ad}}_{\mathbb{Q}_p}$.

For any $m \in \mathbb{Z}_{\geq 1}$, we consider the adic space $\left(X_{\Gamma_0(p^m)} \times_{\mathbb{Q}_p(\mu_{p^m})} \mathbb{Q}_p^{\mathrm{cycl}}\right)^{\mathrm{ad}}$, equipped with the natural projection to $\left(X_{K_p} \times_{\mathbb{Z}_{(p)}} \mathbb{Q}_p^{\mathrm{cycl}}\right)^{\mathrm{ad}}$. We define $\mathcal{X}_{\Gamma_0(p^m)}$ to be the inverse image of the good reduction locus $\mathcal{X}_{K_p}$ under this projection.

Remark 4.1.4. The adic space $\mathcal{X}_{\Gamma_0(p^m)}$ parametrizes pairs (A, D), where A is an abelian variety equipped with a principal polarization, K^p-level structure, and having "good reduction" and $D \subset A[p^m]$ is a totally isotropic subgroup scheme of rank p^{nm}.

The special fiber $\overline{X}_{K_p}$ of X_{K_p} (at least after base change to $\overline{\mathbb{F}}_p$) admits a stratification called the *Newton stratification*, which is defined in terms of the p-divisible groups (up to isogeny, and together with their extra structures) of the abelian varieties parametrized by $\bar{X}_{K_p}$. In these notes, we will only describe one Newton stratum, namely the *ordinary locus*. When it is non-empty, which holds for Siegel modular varieties, the ordinary locus is open and dense in $\overline{X}_{K_p}$.

We start by recalling the *Hasse invariant*. Let S be a scheme of characteristic p and let $\pi : A \to S$ be an abelian scheme of dimension n. The sheaf $\pi_*\Omega_{A/S}$ on S is locally free of rank n. We let $\omega_{A/S}$ be its top exterior power; this is a line bundle on S. Let $A^{(p)}$ denote the pullback of A along the absolute Frobenius of S. The Verschiebung isogeny $A^{(p)} \to A$ induces a morphism $\omega_{A/S} \to \omega_{A^{(p)}/S} \simeq \omega_{A/S}^{\otimes p}$ which can be identified with a section $\mathrm{Ha}(A/S) \in \omega_{A/S}^{\otimes(p-1)}$. This section is called the Hasse invariant of A/S.

Definition 4.1.5. *We say that an abelian scheme A/S of dimension n is ordinary if for all geometric points $\bar{s}$ of S, the set $A[p](\bar{s})$ (obtained by evaluating the sheaf $A[p]$ on $S_{\text{ét}}$ on the geometric point $\bar{s}$) has p^n elements.*

This definition only depends on the p-divisible group $\mathcal{G} := A[p^\infty]$.

Exercise 4.1.6. *Prove that A is ordinary if and only if the p-divisible group $\mathcal{G}_{\bar{s}}$ is isomorphic to $(\mu_{p^\infty})^n \times (\mathbb{Q}_p/\mathbb{Z}_p)^n$ for all geometric points $\bar{s}$ of S.*

The following is a well-known result, in the formulation of Lemma 3.2.5 of [Sch15].

Lemma 4.1.7. *The section $\mathrm{Ha}(A/S) \in \omega_{A/S}^{\otimes(p-1)}$ is invertible if and only if A/S is ordinary.*

Proof. The Hasse invariant is the determinant of the map on co-tangent spaces induced by the Verschiebung morphism. Thus, the Hasse invariant is invertible if and only if the Verschiebung $V : A^{(p)} \to A$ induces an isomorphism on tangent spaces. This is equivalent to asking that Verschiebung be finite étale, which is in turn equivalent to asking that ker V has p^n (the degree of V) distinct geometric points above any geometric point $\bar{s}$ of S. If we let $F : A \to A^{(p)}$ be the Frobenius isogeny (i.e. the relative Frobenius of A) then $VF := p : A \to A$ and F is a purely inseparable map. Thus $A[p](\bar{s}) = (\ker\ V)(\bar{s})$ and we get the desired equivalence. □

Now consider $\overline{A}^{\text{univ}}/\overline{X}_{K_p}$. The complement of the vanishing locus of the Hasse invariant $\mathrm{Ha} := \mathrm{Ha}(\overline{A}^{\text{univ}}/\overline{X}_{K_p})$ is called the ordinary locus $\overline{X}_{K_p}^{\text{ord}} \subset \overline{X}_{K_p}$. We also let $\mathfrak{X}_{K_p}(0) \subset \mathfrak{X}_{K_p}$ be the open formal subscheme where Ha is invertible. If we let $\mathcal{X}_{K_p}(0)$ be the adic generic fiber of $\mathfrak{X}_{K_p}(0)$, then $\mathcal{X}_{K_p}(0) \subset \mathcal{X}_{K_p}$ is the open subset cut out by the condition $|\mathrm{Ha}| \geq 1$.

Let $0 \leq \varepsilon < 1/2$ be such that there exists an element $p^\varepsilon \in \mathbb{Z}_p^{\text{cycl}}$ of p-adic valuation ε. Our goal for the rest of this section is to define a tower of formal schemes $\mathfrak{X}_{K_p}(m, \varepsilon)$ over $\mathbb{Z}_p^{\text{cycl}}$ indexed by $m \in \mathbb{Z}_{\geq 0}$ which has the following properties:

1. For $m = 0$ and $\varepsilon = 0$ we recover the formal scheme $\mathfrak{X}_{K_p}(0)$, corresponding to the ordinary locus. For general ε, the formal scheme $\mathfrak{X}_{K_p}(0, \varepsilon)$ will be a neighborhood of the ordinary locus.
2. The transition morphisms $\mathfrak{X}_{K_p}(m+1, \varepsilon) \to \mathfrak{X}_{K_p}(m, \varepsilon)$ reduce modulo $p^{1-\varepsilon}$ to the relative Frobenius morphism.

(3) For each $m \in \mathbb{Z}_{\geq 1}$, there is a compatible system maps

$$\mathcal{X}_{K_p}(m,\varepsilon) \xrightarrow{\sim} \mathcal{X}_{\Gamma_0(p^m)}(\varepsilon)_{\text{anti}} \hookrightarrow \mathcal{X}_{\Gamma_0(p^m)},$$

where the first map is an isomorphism and the second is an open embedding of adic spaces. The adic space $\mathcal{X}_{\Gamma_0(p^m)}(\varepsilon)_{\text{anti}}$ is an "ε-neighborhood" of the so-called anticanonical part of the ordinary locus in $\mathcal{X}_{\Gamma_0(p^m)}$. The inverse system $(\mathcal{X}_{\Gamma_0(p^m)}(\varepsilon)_{\text{anti}})_{m\in\mathbb{Z}_{\geq 1}}$ of adic spaces gives rise to a perfectoid space $\mathcal{X}_{\Gamma_0(p^\infty)}(\varepsilon)_{\text{anti}}$ over $\mathbb{Z}_p^{\text{cycl}}$.

In § 4.2, we explain the construction of the tower $\mathfrak{X}_{K_p}(m,0)$ over $\mathfrak{X}_{K_p}(0)$; this is not a logically necessary step in the argument, but we think it helps clarify the construction of the anticanonical tower. In § 4.3, we use the theory of the canonical subgroup to construct an "ε-neighborhood" $\mathfrak{X}_{K_p}(m,\varepsilon)$ of $\mathfrak{X}_{K_p}(m,0)$. In § 4.4, we use Faltings's almost purity theorem to construct a perfectoid version of the anticanonical tower $\mathcal{X}_{\Gamma(p^\infty)}(\varepsilon)_{\text{anti}}$ at full level $\Gamma(p^\infty)$. Finally, in Remark 4.4.4, we briefly indicate how this construction extends to the boundary of the minimal compactification.

4.2. **The anticanonical tower over the ordinary locus.** Let R be a p-adically complete, flat $\mathbb{Z}_p^{\text{cycl}}$-algebra and let $A \to \text{Spec } R$ be an abelian scheme with reduction $A_0 \to \text{Spec } (R/p)$. A_0 is equipped with the Frobenius $F : A_0 \to A_0^{(p)}$ and the Verschiebung $V : A^{(p)} \to A$ isogenies. For any $m \in \mathbb{Z}_{\geq 1}$, the p^m-torsion $A_0[p^m]$ fits into a short exact sequence of finite locally free group schemes over R/p

$$0 \to \ker\ F^m \to A_0[p^m] \to \mathcal{G}_0 \to 0,$$

where $\mathcal{G}_0 := \ker\ V^m : A_0^{(p^m)} \to A_0$. If A_0 is ordinary, i.e. if $\text{Ha}(A_0/\text{ Spec } (R/p))$ is invertible, then $\mathcal{G}_0$ is a finite étale group scheme, which therefore lifts uniquely to a group scheme $\mathcal{G}$ over Spec R. We get a short exact sequence

$$0 \to C_m \to A[p^m] \to \mathcal{G} \to 0.$$

The subgroup $C_m \subset A[p^m]$ is called the *canonical subgroup* of A of level m.

Exercise 4.2.1. *Let R be a p-adically complete, flat $\mathbb{Z}_p^{\text{cycl}}$-algebra and let $A/$ Spec R be an ordinary abelian variety. Take $C_m \subset A[p^m]$ to be the canonical subgroup of A of level m.*

(1) *Prove that*

$$A' := A/C_m$$

is also an ordinary abelian variety over Spec R.

(2) *Understand the relationship between the canonical subgroup C'_1 of A' and the subgroup $A[p]/C_1 \subset (A/C_1)[p] = A'[p]$.*

For $m = 0$, we take $\mathfrak{X}_{K_p}(0,0) := \mathfrak{X}_{K_p}(0)$. Note that we have an abelian variety $\mathfrak{A}_{K_p}(0,0)$ over $\mathfrak{X}_{K_p}(0,0)$, which is principally polarized, carries level K^p-structure and whose reduction is ordinary.

For $m \in \mathbb{Z}_{\geq 1}$, we define $\mathfrak{X}_{K_p}(m,0)$ to be abstractly isomorphic to $\mathfrak{X}_{K_p}(0)$, but the map to the base of the tower $\mathfrak{X}_{K_p}(m,0) \to \mathfrak{X}_{K_p}(0,0)$ is the canonical lift to characteristic 0 of the mth relative Frobenius morphism

$$F_m : \mathfrak{X}_{K_p}(m,0) \to (\mathfrak{X}_{K_p}(0)/p)^{(p^m)} \simeq \mathfrak{X}_{K_p}(0)/p.$$

We explain how to construct such a characteristic 0 lift: let $\mathfrak{C}_m$ be the canonical subgroup of the abelian variety $\mathfrak{A}_{K_p}(0,0)$. The abelian variety $\mathfrak{A}' := \mathfrak{A}_{K_p}(0,0)/\mathfrak{C}_m$

is also principally polarized and carries a level K^p-structure. By the universal property of $\mathfrak{X}_{K_p}$, $\mathfrak{A}'$ comes by pullback from a morphism

$$\mathfrak{X}_{K_p}(m,0) \to \mathfrak{X}_{K_p},$$

and, since $\mathfrak{A}'$ is ordinary, this morphism lifts uniquely to a morphism

$$\widetilde{F}_m : \mathfrak{X}_{K_p}(m,0) \to \mathfrak{X}_{K_p}(0,0).$$

We call the morphism $\widetilde{F}_m$ a *canonical Frobenius lift.* Modulo p, $\widetilde{F}_m$ agrees with the mth relative Frobenius, up to the isomorphism $(\mathfrak{X}_{K_p}(0)/p)^{(p^m)} \simeq \mathfrak{X}_{K_p}(0)/p$.

For $m' \in \mathbb{Z}$, $m' \geq m$, we obtain in the same way a morphism

$$\mathfrak{X}_{K_p}(m',0) \to \mathfrak{X}_{K_p}(m,0)$$

which is a canonical lift of the $(m-m')$th relative Frobenius, thus we have an inverse system of formal schemes $(\mathfrak{X}_{K_p}(m,0))_{m\in\mathbb{Z}_{\geq 0}}$.

This tower satisfies the first two desired properties by construction. We are left to identify the adic generic fibers $\mathcal{X}_{K_p}(m,0)$ of the formal schemes $\mathfrak{X}_{K_p}(m,0)$ with open adic subspaces of $\mathcal{X}_{\Gamma_0(p^m)}$.

Let $\mathcal{X}_{\Gamma_0(p^m)}(0)_{\mathrm{anti}}$ be the open and closed locus inside the ordinary locus

$$\mathcal{X}_{\Gamma_0(p^m)}(0) \subset \mathcal{X}_{\Gamma_0(p^m)}$$

which parametrizes pairs (A, D) such that

1. A is an ordinary abelian variety equipped with a principal polarization and a K^p-level structure (and with good reduction);
2. $D \subset A[p^m]$ is a totally isotropic subgroup scheme of order p^{mn} such that $D[p] \cap C_1 = \{0\}$, where C_1 is the canonical subgroup of level 1 of A.

We see from the moduli interpretation in 4.1.4 that $\mathcal{X}_{\Gamma_0(p^m)}(0)_{\mathrm{anti}}$ is indeed an open subspace of $\mathcal{X}_{\Gamma_0(p^m)}$. We call $\mathcal{X}_{\Gamma_0(p^m)}(0)_{\mathrm{anti}}$ the *anticanonical* part of the ordinary locus at level m.

Lemma 4.2.2. *For every $m \in \mathbb{Z}_{\geq 1}$, we have a natural isomorphism of adic spaces*

$$\mathcal{X}_{K_p}(m,0) \xrightarrow{\sim} \mathcal{X}_{\Gamma_0(p^m)}(0)_{\mathrm{anti}}.$$

Proof. Over $\mathcal{X}_{K_p}(m,0)$ we have an ordinary abelian variety $\mathcal{A}_{K_p}(m,0)$ together with a canonical subgroup $\mathcal{C}_m$ of level m, which is totally isotropic. The morphism

$$\mathcal{X}_{K_p}(m,0) \to \mathcal{X}_{\Gamma_0(p^m)}$$

is defined to be the one giving rise to the pair $(\mathcal{A}_{K_p}(m,0)/\mathcal{C}_m, \mathcal{A}_{K_p}(m,0)[p^m]/\mathcal{C}_m)$ over $\mathcal{X}_{K_p}(m,0)$, by pullback from the universal objects over $\mathcal{X}_{\Gamma_0(p^m)}$. Using Exercise 4.2.1, we identify the image of this map with $\mathcal{X}_{\Gamma_0(p^m)}(0)_{\mathrm{anti}}$.

Consider also the morphism

$$\mathcal{X}_{\Gamma_0(p^m)} \to \mathcal{X}_{K_p}$$

defined by $(A, D) \mapsto A/D$ (with the canonical principal polarization and level K^p-structure). [22]

The composition of the two morphisms above

$$\mathcal{X}_{K_p}(m,0) \to \mathcal{X}_{\Gamma_0(p^m)} \to \mathcal{X}_{K_p}$$

---

[22] Over $\mathcal{X}_{\Gamma_0(p^m)}(0)_{\mathrm{anti}}$ this has no direct relation to the morphism $\widetilde{F}_m$, we are quotienting out precisely by subgroups $D \subset A[p^m]$ such that $D[p] \cap C_1 = \{0\}$ rather than by the canonical subgroup of level m.

is an open embedding: it corresponds to pulling back the universal abelian variety over $\mathcal{X}_{K_p}$ to $\mathcal{A}_{K_p}(m, 0)$. Furthermore, the second map is étale. With the same proof as in the case of schemes, one deduces that the first map is an open embedding of adic spaces. □

The tower $(\mathcal{X}_{\Gamma_0(p^m)}(0)_{\mathrm{anti}})_m$ is called *the anticanonical tower* over the ordinary locus. It gives rise to a perfectoid space $\mathcal{X}_{\Gamma_0(p^\infty)}(0)_{\mathrm{anti}}$ which lives over the ordinary locus.

4.3. **The overconvergent anti-canonical tower.** We start by showing the existence of a canonical subgroup (of some level m) of an abelian scheme, as long as the valuation of the Hasse invariant of that abelian scheme is not too large (with respect to m). This will generalize the existence of the canonical subgroup in the case where the abelian scheme is ordinary, i.e when the Hasse invariant is invertible, and will follow roughly the same line of argument.

Let $0 < \varepsilon < 1/2$. Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra, and let $A \to \mathrm{Spec}\ R$ be an abelian scheme, with reduction $A_0 \to \mathrm{Spec}\ (R/p)$. Let $m \in \mathbb{Z}_{\geq 1}$. The following is Corollary III.2.6 of [Sch15].

Proposition 4.3.1. *Assume that*

$$(\mathrm{Ha}(A_0/\,\mathrm{Spec}\ (R/p)))^{\frac{p^m-1}{p-1}} \mid p^\varepsilon.$$

Then there exists a unique closed subgroup $C_m \subset A[p^m]$ such that

$$C_m \equiv \ker F^m \subset A_0[p^m] \pmod{p^{1-\varepsilon}}.$$

Proof. We sketch the argument in [Sch15]. As in the ordinary case, the key is to consider the group scheme $\mathcal{G}_0 := A_0[p^m]/\ker\ F^m$. The assumption on the Hasse invariant is made such that p^ε kills the Lie complex of $\mathcal{G}_0$. The results of Illusie's thesis on deformation theory imply that there exists a finite flat group scheme $\mathcal{G}$ over R such that $\mathcal{G}$ and $\mathcal{G}_0$ agree modulo $p^{1-\varepsilon}$. Furthermore, the map $A_0[p^m] \to \mathcal{G}_0$ modulo $p^{1-\varepsilon}$ gives rise to a map $A[p^m] \to \mathcal{G}$ that agrees with the original map modulo $p^{1-2\varepsilon}$. The canonical subgroup C_m is defined as $\ker(A[p^m] \to \mathcal{G})$. □

Now we make the analogous constructions to the ones in Section 4.2 using the fact that the canonical subgroup of any given level m overconverges (as shown above).[23]

We now define a formal scheme $\mathfrak{X}_{K_p}(\varepsilon) \to \mathfrak{X}_{K_p}$ that recovers $\mathfrak{X}_{K_p}(0)$ for $\varepsilon = 0$. First, define the functor $\mathfrak{X}_{K_p}(\varepsilon) \to \mathfrak{X}_{K_p}$ over $\mathbb{Z}_p^{\mathrm{cycl}}$ which sends any p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra S to the set of pairs (f, u) where $f : \mathrm{Spf} S \to \mathfrak{X}_{K_p}$ is a map and $u \in H^0\left(\mathrm{Spf}\ S, f^*\omega^{\otimes(1-p)}\right)$ is a section such that

$$u \cdot \mathrm{Ha}(\bar{f}) = p^\varepsilon \in S/p,$$

up to the equivalence $(f, u) \simeq (f', u')$ if $f = f'$ and there exists some $h \in S$ with $u' = u(1 + p^{1-\varepsilon}h)$. Lemma 3.2.12 of [Sch15] shows that the functor $\mathfrak{X}_{K_p}(\varepsilon)$ is representable by a formal scheme which is flat over $\mathbb{Z}_p^{\mathrm{cycl}}$ and we have an explicit description of this formal scheme over affines $\mathrm{Spf}(R\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p^{\mathrm{cycl}}) \subset \mathfrak{X}_{K_p}$ given by

$$\mathrm{Spf}(R\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p^{\mathrm{cycl}})\langle u\rangle/(u\widetilde{\mathrm{Ha}} - p^\varepsilon)$$

---

[23] While the canonical subgroup of any given level m overconverges, i.e. can be extended to an $\varepsilon = \varepsilon(m)$ neighborhood of the ordinary locus, if we let $m \to \infty$, we get $\varepsilon(m) \to 0$. The *canonical tower* does not overconverge.

for a lift $\widetilde{\mathrm{Ha}}$ of Ha. The adic generic fiber $\mathcal{X}_{K_p}(\varepsilon) \subset \mathcal{X}_{K_p}$ is the open subset defined by $|\mathrm{Ha}| \geq |p^{\varepsilon}|$.

For $m \in \mathbb{Z}_{\geq 1}$, we let the formal scheme at level m in the tower be $\mathfrak{X}_{K_p}(m, \varepsilon) := \mathfrak{X}_{K_p}(p^{-m}\varepsilon)$, with the morphism to the base of the tower $\mathfrak{X}_{K_p}(\varepsilon)$ given by a canonical lift $\widetilde{F}_m$ of the mth relative Frobenius modulo $p^{1-\varepsilon}$. We explain how to do this for $m = 1$. We need to construct a canonical lift of the relative Frobenius, i.e. a map of formal schemes

$$\widetilde{F}_1 : \mathfrak{X}_{K_p}(p^{-1}\varepsilon) \to \mathfrak{X}_{K_p}(\varepsilon)$$

which reduces to the relative Frobenius modulo $p^{1-\varepsilon}$. For this, we simply need to show that the natural map

$$\mathfrak{X}_{K_p}(p^{-1}\varepsilon) \to \mathfrak{X}_{K_p}$$

induced by quotienting out the universal abelian variety by the level 1 canonical subgroup factors through $\mathfrak{X}_{K_p}(\varepsilon)$. The key point is now to observe that quotienting out by the canonical subgroup of level 1 raises Ha to the pth power. Thus, from the initial condition $u \cdot \mathrm{Ha}(A) = p^{\frac{1}{p}\varepsilon}$ on $\mathfrak{X}_{K_p}(p^{-1}\varepsilon)$, we get $u^p \cdot \mathrm{Ha}(A/C_1) = p^{\varepsilon}$, which gives the desired factorization through $\mathfrak{X}_{K_p}(\varepsilon)$.

Using this argument at higher levels, we obtain the tower of formal schemes $(\mathfrak{X}_{K_p}(p^{-m}\varepsilon))_m$, where the transition map at level m is given by the relative Frobenius modulo $p^{1-\varepsilon}$.[24] From this property of the transition morphisms, we can see that the tower of adic generic fibers $(\mathcal{X}_{K_p}(p^{-m}\varepsilon))_m$ gives rise to a perfectoid space.

We are left with one question, namely identify the adic generic fiber $\mathcal{X}_{K_p}(p^{-m}\varepsilon)$ as an open subspace of the Shimura variety $\mathcal{X}_{\Gamma_0(p^m)}$. Let $\mathcal{X}_{\Gamma_0(p^m)}(\varepsilon)$ be the inverse image of $\mathcal{X}_{K_p}(\varepsilon)$ under the map $\mathcal{X}_{\Gamma_0(p^m)} \to \mathcal{X}_{K_p}$.

Lemma 4.3.2. *$\mathcal{X}_{K_p}(p^{-m}\varepsilon)$ is isomorphic to the open and closed locus $\mathcal{X}_{\Gamma_0(p^m)}(\varepsilon)_{\mathrm{anti}}$ in $\mathcal{X}_{K_p}(\varepsilon)$ where the universal totally isotropic subgroup $\mathcal{D} \subset \mathcal{A}(\varepsilon)[p^m]$ satisfies $\mathcal{D}[p] \cap \mathcal{C}_1 = \{0\}$ for $\mathcal{C}_1 \subset \mathcal{A}(\varepsilon)[p]$ the canonical subgroup of level 1.*

We remark that in order to identify $\mathcal{X}_{\Gamma_0(p^m)}(\varepsilon)_{\mathrm{anti}}$ with $\mathcal{X}_{K_p}(p^{-m}\varepsilon)$, we use the map induced by

$$(A, D) \mapsto A/D.$$

When $D[p] \cap C_1 = \{0\}$, this decreases the valuation of the Hasse invariant, so it indeed defines a map

$$\mathcal{X}_{\Gamma_0(p^m)}(\varepsilon)_{\mathrm{anti}} \to \mathcal{X}_{K_p}(p^{-m}\varepsilon).$$

These maps are compatible as m varies. For each $m \in \mathbb{Z}_{\geq 1}$, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{X}_{\Gamma_0(p^{m+1})}(\varepsilon)_{\mathrm{anti}} \arrow[r, "\sim"] \arrow[d] & \mathcal{X}_{K_p}(p^{-m+1}\varepsilon) \arrow[d] \\
\mathcal{X}_{\Gamma_0(p^m)}(\varepsilon)_{\mathrm{anti}} \arrow[r, "\sim"] & \mathcal{X}_{K_p}(p^{-m}\varepsilon)
\end{tikzcd} ,$$

where the horizontal maps are as described above, the left vertical map is the natural projection (i.e. the forgetful map from the moduli-theoretic point of view), and the right vertical map is the canonical lift of relative Frobenius.

Remark 4.3.3. Unlike the canonical tower, the overconvergent anticanonical tower $(\mathcal{X}_{\Gamma_0(p^m)}(\varepsilon))_m$ inside the tower $(\mathcal{X}_{\Gamma_0(p^m)})_m$ has constant radius ε.

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[24] More precisely, the map from level m to the base of the tower agrees with the mth relative Frobenius modulo $p^{1-\frac{\varepsilon}{p^{m-1}}}$, which is a multiple of $p^{1-\varepsilon}$.

4.4. **An application of almost purity.** We have just seen that $\mathcal{X}_{\Gamma_0(p^\infty)}(\varepsilon)_{\mathrm{anti}}$ is perfectoid. For each $m \in \mathbb{Z}_{\geq 1}$, consider the congruence subgroups

$$\Gamma(p^m) := \left\{ g \in \mathrm{GSp}_{2n}(\mathbb{Z}_p) \mid g \equiv \left(\begin{smallmatrix} \mathrm{Id}_n & 0 \\ 0 & \mathrm{Id}_n \end{smallmatrix}\right) \pmod{p^m} \right\}.$$

We will show that there exists a unique perfectoid space over $\mathbb{Q}_p^{\mathrm{cycl}}$ such that

$$\mathcal{X}_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{anti}} \sim \varprojlim_m \mathcal{X}_{\Gamma(p^m)}(\varepsilon)_{\mathrm{anti}}.$$

For this, the key input is Faltings's almost purity theorem, which we now recall.

Theorem 4.4.1. *Let L be a perfectoid field and R a perfectoid L-algebra. Let S/R be finite étale. Then S is a perfectoid L-algebra and S° is almost finite étale over R°.*

For us, the perfectoid field L will be $\mathbb{Q}_p^{\mathrm{cycl}}$ throughout. Note that the projection maps

$$\mathcal{X}_{\Gamma(p^m)} \to \mathcal{X}_{\Gamma_0(p^m)}$$

are finite étale for every $m \in \mathbb{Z}_{\geq 1}$ and therefore the same thing holds true for their restriction to an ε-neighborhood of the anticanonical tower. By combining this observation with Theorem 4.4.1 and the fact that $\mathcal{X}_{\Gamma_0(p^\infty)}(\varepsilon)_{\mathrm{anti}}$ is a perfectoid space over $\mathbb{Q}_p^{\mathrm{cycl}}$, we conclude the following.

Proposition 4.4.2. *For any $m \in \mathbb{Z}_{\geq 1}$, there exists a unique perfectoid space*

$$\mathcal{X}_{\Gamma(p^m)\cap\Gamma_0(p^\infty)}(\varepsilon)_{\mathrm{anti}}$$

over $\mathbb{Q}_p^{\mathrm{cycl}}$ such that

$$\mathcal{X}_{\Gamma(p^m)\cap\Gamma_0(p^\infty)}(\varepsilon)_{\mathrm{anti}} \sim \varprojlim_{m'} \mathcal{X}_{\Gamma(p^m)\cap\Gamma_0(p^{m'})}(\varepsilon)_{\mathrm{anti}}.$$

By varying m, we obtain an inverse system of perfectoid spaces with finite étale transition maps. Take an affinoid perfectoid cover of $\mathcal{X}_{\Gamma(p)\cap\Gamma_0(p^\infty)}(\varepsilon)_{\mathrm{anti}}$. The preimage of any affinoid perfectoid element of the cover in any $\mathcal{X}_{\Gamma(p^m)\cap\Gamma_0(p^\infty)}(\varepsilon)_{\mathrm{anti}}$ is affinoid perfectoid by Theorem 4.4.1. Since inverse limits of affinoid perfectoid spaces are affinoid perfectoid, we obtain an affinoid perfectoid cover of the topological space

$$|\mathcal{X}_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{anti}}| := \varprojlim_m |\mathcal{X}_{\Gamma(p^m)\cap\Gamma_0(p^\infty)}(\varepsilon)_{\mathrm{anti}}|.$$

We thus deduce the following.

Theorem 4.4.3. *There exists a unique perfectoid space $\mathcal{X}_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{anti}}$ over $\mathbb{Q}_p^{\mathrm{cycl}}$ such that*

$$\mathcal{X}_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{anti}} \sim \varprojlim_m \mathcal{X}_{\Gamma(p^m)\cap\Gamma_0(p^\infty)}(\varepsilon)_{\mathrm{anti}}.$$

Remark 4.4.4. We briefly indicate how to extend the results of this section to the minimal compactification of the Siegel modular variety and thus how to construct a perfectoid space $\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{anti}}$. When $K_p = \mathrm{GSp}_{2n}(\mathbb{Z}_p)$, recall that $X^*_{K_p}$ is the minimal (Baily–Borel–Satake) compactification of X_{K_p} over $\mathbb{Z}_{(p)}$ as constructed by Faltings and Chai [CF90]. This is a projective, but not necessarily smooth, scheme over $\mathbb{Z}_{(p)}$ and the line bundle ω extends canonically to an ample line bundle on $X^*_{K_p}$.

Both $\mathrm{Ha} \in \omega^{\otimes(p-1)}/p$ and the ordinary locus (defined as the complement of the vanishing locus of Ha) can be extended to an open dense subscheme of the minimal compactification $\overline{X}^*_{K_p}$. Indeed, the codimension of the boundary $\overline{X}^*_{K_p} \setminus \overline{X}_{K_p}$ of the minimal compactification is n: the boundary can be described in terms of smaller Siegel modular varieties and the relative dimension of the Siegel modular variety for GSp_{2m} over $\mathbb{Z}_{(p)}$ is $\frac{m(m+1)}{2}$. For $n \geq 2$, Koecher's extension principle (see [Lan16] for the most definitive version) guarantees that Ha extends canonically to the whole $\overline{X}^*_{K_p}$. The case $n = 1$, i.e. the case of modular curves, can be done in an ad hoc manner, for example using q-expansions.

The proof that the space $\mathcal{X}_{\Gamma_0(p^\infty)}(\varepsilon)_{\mathrm{anti}}$ is perfectoid extends to the minimal compactification and, in fact, one can prove something even stronger, namely that $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\varepsilon)_{\mathrm{anti}}$ is actually an affinoid perfectoid space. This follows from the fact that ω extends to an ample line bundle, so one can find a global characteristic 0 lift of any sufficiently large pth power of Ha.

Going from level $\Gamma_0(p^\infty)$ to level $\Gamma(p^\infty)$ is much more subtle for the minimal compactification than for the good reduction locus. We cannot simply argue using Theorem 4.4.1, because the maps

$$\mathcal{X}^*_{\Gamma(p^m)}(\varepsilon)_{\mathrm{anti}} \to \mathcal{X}^*_{\Gamma_0(p^m)}(\varepsilon)_{\mathrm{anti}}$$

are ramified along the boundary. Instead, for every $m \in \mathbb{Z}_{\geq 1}$, we consider the congruence subgroups

$$\Gamma_1(p^m) := \left\{ g \in \mathrm{GSp}_{2n}(\mathbb{Z}_p) \mid g \equiv \left(\begin{smallmatrix} \mathrm{Id}_n & * \\ 0 & \mathrm{Id}_n \end{smallmatrix}\right) \pmod{p^m} \right\}$$

One first shows that there exists a unique affinoid perfectoid space $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\varepsilon)_{\mathrm{anti}}$ over $\mathbb{Q}_p^{\mathrm{cycl}}$ such that

$$\mathcal{X}^*_{\Gamma_1(p^\infty)}(\varepsilon)_{\mathrm{anti}} \sim \varprojlim_m \mathcal{X}^*_{\Gamma_1(p^m)}(\varepsilon)_{\mathrm{anti}}.$$

This is the most subtle part of the argument and it uses *Tate's normalized traces* to extend the construction of the anticanonical tower via Theorem 4.4.1 over the boundary of the minimal compactification at level $\Gamma_1(p^m) \cap \Gamma_0(p^\infty)$. See [Sch15, §3.2] for more details.

Finally, the maps

$$\mathcal{X}^*_{\Gamma(p^m)}(\varepsilon)_{\mathrm{anti}} \to \mathcal{X}^*_{\Gamma_1(p^m)}(\varepsilon)_{\mathrm{anti}}$$

are finite étale for every $m \in \mathbb{Z}_{\geq 1}$ even over the boundary. This means that we can go from level $\Gamma_1(p^\infty)$ to level $\Gamma(p^\infty)$ using Theorem 4.4.1 as described above.

One concludes the following strenghtening of Theorem 4.4.3.

Theorem 4.4.5. *There exists a unique affinoid perfectoid space* $\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{anti}}$ *over* $\mathbb{Q}_p^{\mathrm{cycl}}$ *such that*

$$\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{anti}} \sim \varprojlim_m \mathcal{X}^*_{\Gamma(p^m)}(\varepsilon)_{\mathrm{anti}}.$$

5. Perfectoid Shimura varieties and the Hodge–Tate period morphism

In this section, we construct the Hodge–Tate period morphism and use it to show that Siegel modular varieties (and other Shimura varieties) with infinite level at p are perfectoid. In § 5.1, we explain how to use Theorem 4.4.5 to show that there exists a perfectoid space $\mathcal{X}^*_{\Gamma(p^\infty)}$ over $\mathbb{Q}_p^{\mathrm{cycl}}$ such that

$$\mathcal{X}^*_{\Gamma(p^\infty)} \sim \varprojlim_m \mathcal{X}^*_{\Gamma(p^m)}. \tag{5.0.1}$$

In § 5.2, we discuss the geometry of the Hodge–Tate period morphism in the Siegel case. Finally, in § 5.3, we discuss the Hodge–Tate period morphism for Shimura varieties of Hodge type.

5.1. **Siegel modular varieties with infinite level at p are perfectoid.**

Consider the inverse limit of topological spaces

$$|\mathcal{X}^*_{\Gamma(p^\infty)}| = \varprojlim_m |\mathcal{X}^*_{\Gamma(p^m)}|.$$

This must be the underlying topological space of a perfectoid space $\mathcal{X}^*_{\Gamma(p^\infty)}$ satisfying (5.0.1). The topological spaces at finite level $|\mathcal{X}^*_{\Gamma(p^m)}|$ are all *spectral spaces*, as they are the underlying topological spaces of quasi-compact and quasi-separated adic spaces, and the transition maps are *spectral maps*, since they underlie maps of adic spaces.[25] This implies that $|\mathcal{X}^*_{\Gamma(p^\infty)}|$ is itself a spectral topological space. The hard part is endowing this topological space with a perfectoid structure.

The perfectoid space $\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{anti}}$, covers a part of the topological space

$$|\mathcal{X}^*_{\Gamma(p^\infty)}| := \varprojlim_m |\mathcal{X}^*_{\Gamma(p^m)}|.$$

We will show that the entire topological space above underlies a perfectoid space, using the continuous $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$-action on $|\mathcal{X}^*_{\Gamma(p^\infty)}|$[26] and the fact that the translates of $|\mathcal{X}^*_{\Gamma(p^m)}(\varepsilon)_{\mathrm{anti}}|$ under this action cover the entire space $|\mathcal{X}^*_{\Gamma(p^\infty)}|$. The rigorous way of proving this is via *the Hodge–Tate period morphism*, which has as target a flag variety $\mathscr{F}\ell$, which also has an action of $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$. One of the most important properties of the Hodge–Tate period morphism is that it is equivariant for the action of $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$ on both the Shimura variety at infinite level (or, for now, on the corresponding topological space) and on the flag variety $\mathscr{F}\ell$.

Recall that (V, ψ) denotes the split symplectic space of dimension $2n$ over $\mathbb{Q}$. Let $\mathrm{Fl}/\mathbb{Q}$ be the flag variety parametrizing subspaces $W \subset V$ of dimension n which are totally isotropic under ψ. We consider the corresponding a dic space $\mathscr{F}\ell$ over $\mathbb{Q}_p$. The Hodge–Tate period morphism is first defined at the level of topological spaces:

$$|\pi_{\mathrm{HT}}| : |\mathcal{X}^*_{\Gamma(p^\infty)}| \to |\mathscr{F}\ell|$$

For simplicity, in these notes we will only describe the map on the good reduction locus $|\mathcal{X}_{\Gamma(p^\infty)}|$. For each pair (L, L^+), with $L/\mathbb{Q}_p^{\mathrm{cycl}}$ a complete non-archimedean field and $L^+ \subset L$ an open and bounded valuation subring, define

$$\mathcal{X}_{\Gamma(p^\infty)}(L, L^+) := \varprojlim_m \mathcal{X}_{\Gamma(p^m)}(L, L^+).$$

From this definition, one can check that $\mathcal{X}_{\Gamma(p^\infty)}(L, L^+)$ has a moduli interpretation in terms of abelian varieties A/L, equipped with a principal polarization, with a

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[25] We can define a spectral topological space as any topological space that is homeomorphic to the underlying topological space of an affine scheme. For more on spectral spaces and spectral maps, in the context of adic spaces, see, for example, [Wed].

[26] This action can only be seen at level $\Gamma(p^\infty)$; at finite level one only has an action of $\mathrm{GSp}_{2n}(\mathbb{Z}_p)$. To see the action of $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$ on $|\mathcal{X}_{\Gamma(p^\infty)}|$, it is easiest to first redefine the moduli problem in terms of abelian varieties up to isogeny, as in Example 2.3.6; then it is easy to see the group action on the p-part of the level structure.

K^p-level structure $\bar{\eta}^p$, and with a symplectic isomorphism $\eta_p : \mathbb{Z}_p^{2n} \xrightarrow{\sim} T_pA$. We have

$$|\mathcal{X}_{\Gamma(p^\infty)}| = \varinjlim_{(L,L^+)} \mathcal{X}_{\Gamma(p^\infty)}(L, L^+),$$

where the limit on the right hand side is not filtered, but each point comes from a unique minimal pair (L, L^+). The following is a reformulation of Lemma 3.3.4 of [Sch15], restricted to the good reduction locus.

Lemma 5.1.1. *There exists a* $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$*-equivariant, continuous map of topological spaces*

$$|\pi_{\mathrm{HT}}| : |\mathcal{X}_{\Gamma(p^\infty)}| \to |\mathscr{F}\ell|,$$

which is defined at the level of points by sending an abelian variety A/L together with a symplectic isomorphism

$$\eta_p : \mathbb{Z}_p^{2n} \xrightarrow{\sim} T_pA$$

to the (first piece of the) Hodge–Tate filtration Lie $A \subset T_pA \otimes_{\mathbb{Z}_p} L \xrightarrow{\sim} L^{2n}$.

Proof. First, define the map $|\pi_{\mathrm{HT}}|$ on points by the recipe in the statement of the lemma. Since $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$ acts on the level structure η_p, the map $|\pi_{\mathrm{HT}}|$ is $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$-equivariant by definition.

To show that there exists a map of topological spaces which agrees with $|\pi_{\mathrm{HT}}|$ on points, it is enough to work locally on $|\mathcal{X}_{\Gamma(p^\infty)}|$. It will therefore suffice to construct a cover of $|\mathcal{X}_{\Gamma(p^\infty)}|$ which is pulled back from a cover of $|\mathcal{X}_{K_p}|$. We work in the setting of Example 3.2.6, i.e. by considering the proper smooth morphism $\pi : \mathcal{A} \to \mathcal{X}_{K_p}$ of smooth adic spaces over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. The relative Hodge–Tate filtration of the universal abelian variety is encoded by the natural injection

$$R^1\pi_*\mathcal{O}_{\mathcal{A}} \otimes_{\mathcal{O}_{\mathcal{X}_{K_p}}} \widehat{\mathcal{O}}_{\mathcal{X}_{K_p}} \hookrightarrow R^1\pi_*\widehat{\mathcal{O}}_{\mathcal{A}} \simeq R^1\pi_*\widehat{\mathbb{Z}}_p \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathcal{O}}$$

of sheaves on the flattened pro-étale site $(\mathcal{X}_{K_p})_{\text{proét}}$.

Locally on $\mathcal{X}_{K_p}$, one can find a pro-finite étale cover $\widetilde{U} \to \mathcal{X}_{K_p}$ such that $\widetilde{U}$ is affinoid perfectoid. We show that it is possible to pull back $\widetilde{U}$ to an affinoid perfectoid space $\widetilde{U}_\infty$ such that $|\widetilde{U}_\infty|$ covers $|\mathcal{X}_{\Gamma(p^\infty)}|$. For each $m \geq 0$, the map $\mathcal{X}_{\Gamma(p^m)} \to \mathcal{X}_{K_p}$ is finite étale. Thus, we can form the pullback $\widetilde{U}_m := \widetilde{U} \times_{\mathcal{X}_{K_p}} \mathcal{X}_{\Gamma(p^m)}$ and, by Theorem 4.4.1, this is an affinoid perfectoid cover of $\mathcal{X}_{\Gamma(p^\infty)}$. We then take the inverse limit of the $\widetilde{U}_m$ as $m \to \infty$, which we can do for affinoid perfectoid spaces, and we obtain an affinoid perfectoid space $\widetilde{U}_\infty$. This is still an element of the flattened pro-étale site of $\mathcal{X}_{K_p}$.

We now evaluate the injection

$$R^1\pi_*\mathcal{O}_{\mathcal{A}} \otimes_{\mathcal{O}_{\mathcal{X}_{K_p}}} \widehat{\mathcal{O}}_{\mathcal{X}_{K_p}} \hookrightarrow R^1\pi_*\widehat{\mathcal{O}}_{\mathcal{A}} \simeq R^1\pi_*\widehat{\mathbb{Z}}_p \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathcal{O}}$$

on $\widetilde{U}_\infty$. Since $R^1\pi_*\mathcal{O}_{\mathcal{A}}$ can be identified with Lie $\mathcal{A}$, we get a totally isotropic submodule (Lie $\mathcal{A}$) $\otimes_{\mathcal{O}_{cX_{K_p}}} \mathcal{O}_{\widetilde{U}_\infty} \subset \mathcal{O}^{2n}_{\widetilde{U}_\infty}$, which defines a map of adic spaces

$$\widetilde{U}_\infty \to \mathscr{F}\ell.$$

The induced map on topological spaces is automatically continuous. By checking on points, one sees that this map factors through the restriction of $|\pi_{\mathrm{HT}}|$ to $|\widetilde{U}| \times_{|\mathcal{X}_{K_p}|} |\mathcal{X}_{\Gamma(p^\infty)}|$. Moreover, the map

$$|\widetilde{U}_\infty| \to |\widetilde{U}| \times_{|\mathcal{X}_{K_p}|} |\mathcal{X}_{\Gamma(p^\infty)}|$$

is both surjective and open, as it is a pro-finite étale cover and pro-finite étale maps are open. Thus, $|\pi_{\mathrm{HT}}|$ is continous. $\square$

Remark 5.1.2. In fact, if we let $\mathcal{Z}_{\Gamma(p^m)}$ be the boundary of $\mathcal{X}^*_{\Gamma(p^m)}$, we can define the spectral topological space

$$|\mathcal{Z}_{\Gamma(p^\infty)}| := \varprojlim_m |\mathcal{Z}_{\Gamma(p^m)}|$$

and the construction of the map $|\pi_{\mathrm{HT}}|$ in Lemma 5.1.1 extends to the open Shimura variety $|\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}|$ with the same proof, thus we have a continuous, $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$-equivariant map

$$|\pi_{\mathrm{HT}}| : |\mathcal{X}^*_{\Gamma(p^\infty)}| \setminus |\mathcal{Z}_{\Gamma(p^\infty)}| \to |\mathscr{F}\ell|.$$

Let $0 \le \varepsilon < \frac{1}{2}$. Recall that $\mathcal{X}^*_{K_p}(\varepsilon) \subset \mathcal{X}^*_{K_p}$ is the locus where $|\mathrm{Ha}| \ge p^\varepsilon$. Let $|\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)| \subset |\mathcal{X}^*_{\Gamma(p^\infty)}|$ be the preimage of $|\mathcal{X}^*_{K_p}(\varepsilon)|$. We have

$$|\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)| = \mathrm{GSp}_{2n}(\mathbb{Z}_p)|\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{anti}}|.$$

This can be checked at finite level - for example at level $\Gamma_0(p)$, where the ε-neighborhood $\mathcal{X}^*_{\Gamma_0(p)}(\varepsilon)_{\mathrm{anti}} \subset \mathcal{X}^*_{\Gamma_0(p)}$ of the anticanonical locus is defined (recall that everything else is just pulled back from this level). In fact, by doing this, we see that we can replace $\mathrm{GSp}_{2n}(\mathbb{Z}_p)$ by finitely many translates of $|\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{anti}}|$ by elements of $\mathrm{GSp}_{2n}(\mathbb{Z}_p)$; thus, $|\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)|$ is quasi-compact. The key result is now the following (Lemma 3.3.10 of [Sch15]).

Proposition 5.1.3. *There exist finitely many elements $\gamma_1, \dots, \gamma_k \in \mathrm{GSp}_{2n}(\mathbb{Q}_p)$ such that*

$$|\mathcal{X}^*_{\Gamma(p^\infty)}| = \bigcup_{i=1}^k \gamma_i \cdot |\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)|.$$

Proof. We sketch the main steps in the proof.

(1) First, one shows that if $|\pi_{\mathrm{HT}}|$ is the map in Remark 5.1.2, and $\mathscr{F}\ell(\mathbb{Q}_p)$ denotes the $\mathbb{Q}_p$-points of the adic space $\mathscr{F}\ell$, then

$$|\pi_{\mathrm{HT}}|^{-1}\left(\mathscr{F}\ell(\mathbb{Q}_p)\right) = \text{ closure of } |\mathcal{X}^*_{\Gamma(p^\infty)}(0)| \setminus |\mathcal{Z}_{\Gamma(p^\infty)}(0)|.$$

This is Lemma 3.3.6 of [Sch15]. The idea is that for an ordinary abelian variety, the Hodge–Tate filtration is $\mathbb{Q}_p$-rational and measures the relative position of the canonical subgroup.

(2) For $0 < \varepsilon < \frac{1}{2}$, one shows that there exists an open subset $U \subset \mathscr{F}\ell$ containing $\mathscr{F}\ell(\mathbb{Q}_p)$ and such that

$$|\pi_{\mathrm{HT}}|^{-1}(U) \subset |\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)| \setminus |\mathcal{Z}_{\Gamma(p^\infty)}(\varepsilon)|.$$

This is Lemma 3.3.7 of [Sch15]. Using induction, one reduces to the locus of good reduction. The proof then relies on Step 1 and on a compactness argument using the constructible topology on spectral spaces. For the compactness argument, one uses the continuity of the map

$$|\pi_{\mathrm{HT}}| : |\mathcal{X}_{\Gamma(p^\infty)}| \to |\mathscr{F}\ell|$$

and the fact that the space $|\mathcal{X}_{\Gamma(p^\infty)}|$ is spectral with quasi-compact open subset $|\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)|$.

(3) One shows that there exist finitely many elements $\gamma_1, \ldots, \gamma_k \in \mathrm{GSp}_{2n}(\mathbb{Q}_p)$ such that

$$|\mathcal{X}^*_{\Gamma(p^\infty)}| \setminus |\mathcal{Z}_{\Gamma(p^\infty)}| = \bigcup_{i=1}^{k} \gamma_i \cdot (|\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)| \setminus |\mathcal{Z}_{\Gamma(p^\infty)}(\epsilon)|).$$

This is Lemma III.3.9 of [Sch15]. This uses an open subset U as in Step 2; the quasi-compactness of $\mathscr{F}\ell$ implies that finitely many $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$-translates of U cover $\mathscr{F}\ell$. The fact that $|\pi_{\mathrm{HT}}|$ is $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$-equivariant allows one to conclude by taking preimages of everything.

(4) Finally, one shows that with the same $\gamma_1, \ldots, \gamma_k$ as above one has the desired equality

$$|\mathcal{X}^*_{\Gamma(p^\infty)}| = \bigcup_{i=1}^{k} \gamma_i \cdot |\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)|.$$

This again relies on a compactness argument as in Step 2 above. The idea is that the right hand side is a quasi-compact open subset of $|\mathcal{X}^*_{\Gamma(p^\infty)}|$ which contains $|\mathcal{X}^*_{\Gamma(p^\infty)}| \setminus |\mathcal{Z}_{\Gamma(p^\infty)}|$ by Step 3 above. Any such subset must be the whole space. One concludes this by reducing to finite level, considering classical points, and again using a compactness argument for the constructible topology on a spectral space.

□

As a result, we see that $|\mathcal{X}^*_{\Gamma(p^\infty)}|$ is covered by finitely many translates of $|\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{anti}}|$, which is the underlying topological space of an affinoid perfectoid space. This proves the existence of the perfectoid space $\mathcal{X}^*_{\Gamma(p^\infty)}$. With a bit more work, one can also show that there exists a map of adic spaces

$$\pi_{\mathrm{HT}} : \mathcal{X}^*_{\Gamma(p^\infty)} \to \mathscr{F}\ell.$$

which agrees with the previously defined map $|\pi_{\mathrm{HT}}|$ on the underlying topological spaces.

Remark 5.1.4. The closed subset $|\mathcal{Z}_{\Gamma(p^\infty)}| \subset |\mathcal{X}^*_{\Gamma(p^\infty)}|$ has an induced structure of a perfectoid space. If $\mathcal{Z}_{\Gamma(p^\infty)}$ denotes the boundary with the induced perfectoid structure, then the existence of the map of adic spaces

$$\pi_{\mathrm{HT}} : \mathcal{X}^*_{\Gamma(p^\infty)} \setminus \mathcal{Z}_{\Gamma(p^\infty)} \to \mathscr{F}\ell$$

follows by the same argument as in the proof of Lemma 5.1.1, using instead of $\widetilde{U}_\infty$ the affinoid perfectoid cover given by the disjoint union of finitely many copies of $\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{anti}} \setminus \mathcal{Z}_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{anti}}$. The tricky part is to show that the Hodge–Tate period morphism extends to the boundary. For this, one uses a version of Riemann's Hebbarkeitssatz for perfectoid spaces, concerning the extension of bounded functions from complements of Zariski closed subsets. See [Sch15, §2] for more details.

5.2. **The Hodge–Tate period morphism in the Siegel case.** We summarize here some facts about the geometry of $\mathscr{F}\ell$ in the Siegel case. The flag variety admits the Plücker embedding

$$\mathscr{F}\ell \hookrightarrow \mathbb{P}^{\binom{2n}{n}-1}, W \mapsto \wedge^n W.$$

Any subset $J \subset \{1, 2, \ldots, 2n\}$ of cardinality n determines a homogeneous coordinate s_J on $\mathbb{P}^{\binom{2n}{n}-1}$. One can cover $\mathscr{F}\ell$ by open affinoid subsets $\mathscr{F}\ell_J$, which are defined by the conditions $|s_{J'}| \leq |s_J|$ for all $J' \subset \{1, 2, \ldots, 2n\}$ of cardinality n. These affinoid subsets are permuted transitively by the action of $\mathrm{GSp}_{2n}(\mathbb{Z}_p)$. For example, $\mathscr{F}\ell_{\{n+1,\ldots,2n\}}(\mathbb{Q}_p)$ parametrizes those totally isotropic direct summands $M \subset \mathbb{Z}_p^{2n}$ such that $M \oplus (\mathbb{Z}_p^n \oplus 0^n) \simeq \mathbb{Z}_p^{2n}$.

Exercise 5.2.1. *Show that the preimage of* $\mathscr{F}\ell_{\{n+1,\ldots,2n\}}(\mathbb{Q}_p)$ *under* π_{HT} *is given by the closure of* $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_{\mathrm{anti}}$.

Since $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_{\mathrm{anti}}$ is affinoid perfectoid, thus of the form $\mathrm{Spa}(R, R^+)$, and since taking the closure only adds higher rank points, which amounts to only changing the integral structure, i.e R^+, we see that the preimage of $\mathscr{F}\ell_{\{n+1,\ldots,2n\}}(\mathbb{Q}_p)$ under π_{HT} is affinoid perfectoid.

We claim that something stronger holds, namely the preimage of the whole of $\mathscr{F}\ell_{\{n+1,\ldots,2n\}}$ is affinoid perfectoid. To see this, note that the action of the diagonal element γ^{-1}, where $\gamma = (p, \ldots, p, 1, \ldots, 1) \in (\mathbb{Q}_p^\times)^n \times (\mathbb{Q}_p^\times)^n \subset \mathrm{GSp}_{2n}(\mathbb{Q}_p)$, contracts $\mathscr{F}\ell_{\{n+1,\ldots,2n\}}$ towards the point of $\mathscr{F}\ell_{\{n+1,\ldots,2n\}}(\mathbb{Q}_p)$ corresponding to $0^n \oplus \mathbb{Z}_p^n \subset \mathbb{Z}_p^{2n}$. In particular, the action of γ^{-1} contracts $\mathscr{F}\ell_{\{n+1,\ldots,2n\}}$ towards the image of the anticanonical locus $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_{\mathrm{anti}}$ under π_{HT}. To make this precise, for any $0 < \varepsilon \leq \frac{1}{2}$, one can find some large integer N such that

$$\pi_{\mathrm{HT}}^{-1}\left(\gamma^{-N} \cdot \mathscr{F}\ell_{\{n+1,\ldots,2n\}}\right) \subset \mathcal{X}^*_{\Gamma(p^n)}(\varepsilon)_{\mathrm{anti}}$$

is a rational subset. This shows that $\gamma^{-N} \cdot \mathscr{F}\ell_{\{n+1,\ldots,2n\}}$ is affinoid perfectoid and thus that $\mathscr{F}\ell_{\{n+1,\ldots,2n\}}$ is itself affinoid perfectoid. Since the action of $\mathrm{GSp}_{2n}(\mathbb{Z}_p)$ permutes the cardinality n subsets J, we also see that the preimage of any $\mathscr{F}\ell_J$ under π_{HT} is affinoid perfectoid.

Remark 5.2.2. The idea of using an element of $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$ to contract a subset of $\mathcal{X}^*_{\Gamma(p^\infty)}$ towards the anticanonical locus seems quite fruitful. For example, this idea is used in [Lud17] to construct a perfectoid version of the Lubin-Tate tower at level $\Gamma_0(p^\infty)$.

Example 5.2.3. For $n = 1$, the flag variety $\mathscr{F}\ell$ can be identified with the one-dimensional adic projective space $\mathbb{P}^1$. The Plücker embedding is the identity map. If (x_1, x_2) are the usual coordinates on $\mathbb{P}^1$, we see that $\mathscr{F}\ell = \mathbb{P}^1$ has a cover by two affinoid subsets $\mathscr{F}\ell_{\{2\}}$ and $\mathscr{F}\ell_{\{1\}}$, defined by the conditions $|x_1| \leq |x_2|$, respectively $|x_2| \leq |x_1|$. The image of the anticanonical locus under π_{HT} is given by $\{(\frac{x_1}{x_2}, 1) \in \mathbb{P}^1(\mathbb{Q}_p) \mid \frac{x_1}{x_2} \in \mathbb{Z}_p\}$ and the image of the canonical locus is the point $(1, 0) \in \mathbb{P}^1(\mathbb{Q}_p)$. The action of $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_p)$ expands the anticanonical locus towards the complement of the canonical point in $\mathbb{P}^1$.

To summarize the discussion in this section, we have the following result.

Theorem 5.2.4. (1) *For any sufficiently small tame level* $K^p \subset \mathrm{GSp}_{2n}(\mathbb{A}_f^p)$, *there exists a perfectoid space* $\mathcal{X}^*_{\Gamma(p^\infty), K^p}$ *over* $\mathbb{Q}_p^{\mathrm{cycl}}$ *such that*

$$\mathcal{X}^*_{\Gamma(p^\infty), K^p} \sim \varprojlim_m \mathcal{X}^*_{\Gamma(p^m), K^p}.$$

(2) *There exists a* $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$*-equivariant map of adic spaces*

$$\pi_{\mathrm{HT}} : \mathcal{X}^*_{\Gamma(p^\infty), K^p} \to \mathscr{F}\ell$$

which agrees with the map defined explicitly on points in Lemma 5.1.1.

(3) *Let S be a finite set of bad primes for the tame level K^p. The map π_{HT} is equivariant with respect to the natural Hecke action of the abstract spherical Hecke algebra $\mathbb{T}^S$ on $\mathcal{X}^*_{\Gamma(p^\infty),K^p}$ and the trivial action of $\mathbb{T}^S$ on $\mathscr{F}\ell$.*

(4) *The map π_{HT} is "affinoid", in the following sense: for any subset $J \subset \{1, \dots, 2n\}$ of cardinality n, the preimage of $\mathscr{F}\ell_J$ under π_{HT} is affinoid perfectoid.*[27]

(5) *Let $\omega_{\mathscr{F}\ell} := (\wedge^n W_{\mathscr{F}\ell})^\vee$ be the natural ample line bundle on $\mathscr{F}\ell$. Recall that one also has the natural line bundle ω_{K^p} on $\mathcal{X}^*_{\Gamma(p^\infty),K^p}$, obtained by pullback from any finite level. There is a natural,* $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$*-equivariant isomorphism*

$$\omega_{K^p} \simeq \pi^*_{\mathrm{HT}} \omega_{\mathscr{F}\ell}.$$

This isomorphism is also $\mathbb{T}^S$-equivariant.

5.3. **Shimura varieties of Hodge type.** Let (G, X) be a Shimura datum of Hodge type. Let $K^p \subset G(\mathbb{A}_f)$ be a sufficiently small compact open subgroup. For any choice of compact open subgroup $K_p \subset G(\mathbb{Q}_p)$, we let $X_{K^pK_p}$ be the Shimura variety for G, at level K^pK_p, and defined over the reflex field E. Let C be a complete, algebraically closed extension of $\overline{\mathbb{Q}}_p$. We consider the adic space

$$\mathcal{X}_{K^pK_p} := \left(X_{K^pK_p} \times_{\mathrm{Spec}\ E} \mathrm{Spec}\ C\right)^{\mathrm{ad}}.$$

Theorem 5.3.1 (Thm 4.1.1 of [Sch15])**.** *For any sufficiently small tame level K^p, there exists a perfectoid space $\mathcal{X}_{K^p}$ over* $\mathrm{Spa}(C, \mathcal{O}_C)$ *such that*

$$\mathcal{X}_{K^p} \sim \varprojlim_{K_p} \mathcal{X}_{K^pK_p}.$$

The proof goes by embedding the Shimura variety of Hodge type into a Siegel modular variety and using the fact that Siegel modular varieties with infinite level at p are perfectoid spaces, as explained in Section 5.1.

Remark 5.3.2. There is also a version of this result for minimal compactifications. There is one subtlety, having to do with the fact that one does not necessarily have closed embeddings on the level of minimal compactifications. Because of this, one must consider a slightly modified space $X^*_{K^pK_p}$ at finite level, obtained by taking the scheme-theoretic image of $X_{K^pK_p}$ into the corresponding compactification of the Siegel modular variety. However, the map $X^*_{K^pK_p} \to X^*_{K^pK_p}$ is a universal homeomorphism, hence induces an isomorphism of diamonds by [SW17, Prop. 10.2.1] and the remarks following it. In particular, the spaces have the same étale cohomology. Because of this, we write $\mathcal{X}^*_{K^p}$ for the minimal compactification of the perfectoid Shimura variety $\mathcal{X}_{K^p}$. On the level of diamonds, it is the inverse limit of the diamonds corresponding to $\mathcal{X}^*_{K^pK_p}$.

One can define the Hodge–Tate period morphism more generally, for Shimura varieties of Hodge type, as done in Section 2 of [CS17] or even of abelian type [She17]. We contend ourselves here to discussing the Hodge–Tate period morphism for Shimura varieties of Hodge type, in order to give a sense of the role that the Shimura datum plays in the definition of a functorial p-adic period morphism and

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[27] This implies the following, apparently stronger, statement: for any $J \subset \{1, \dots, 2n\}$ the preimage of any rational open $U \subseteq \mathscr{F}\ell_J$ under π_{HT} is an affinoid perfectoid space.

to illustrate the analogy with the complex picture described in Section 2.3. We will use Section 2 of [CS17] as a reference.

Let (G, X) be a Shimura datum of Hodge type, with Hodge cocharacter μ. Recall the parabolic subgroup $P_\mu \subset G \times_{\mathbb{Q}} E$ defined in § 2.3. This parabolic subgroup can be thought of as "the stabilizer of the Hodge–Tate filtration". We have a Hodge–Tate period morphism π_{HT}, which should be thought of as a p-adic analogue of the morphism π_{dR} from § 2.3. The following is part of Theorem 2.1.3 of [CS17].

Theorem 5.3.3. (1) *For any choice of tame level $K^p \subset G(\mathbb{A}_f)$, there is a morphism of adic spaces*

$$\pi_{\mathrm{HT}} : \mathcal{X}_{K^p} \to \mathscr{F}\ell_{G,\mu}.$$

This is functorial in the Shimura datum and agrees with the morphism constructed in Theorem 5.2.4 for Siegel modular varieties.

(2) *The map π_{HT} is equivariant with respect to the Hecke action of $G(\mathbb{Q}_p)$ on $\mathcal{X}_{K^p}$ and the natural action of $G(\mathbb{Q}_p)$ on $\mathscr{F}\ell_{G,\mu}$.*

(3) *The map π_{HT} is equivariant with respect to the action of Hecke operators away from p on $\mathcal{X}_{K^p}$ and the trivial action of these Hecke operators on $\mathscr{F}\ell_{G,\mu}$.*

Proof. We say a few words about the proof. The main idea is to choose a symplectic embedding $(G, X) \hookrightarrow (\widetilde{G}, \widetilde{X})$, and keep track of Hodge tensors, the finite collection of elements $s_\alpha \in V^{\otimes}$ which are stabilized by $G \subset \widetilde{G}$. The relative p-adic étale cohomology

$$\mathcal{V}_p := R^1\pi_{*,\text{ét}}\mathbb{Q}_p$$

of the abelian variety $\pi : \mathcal{A} \to \mathcal{X}_{K^pK_p}$ (restricted from the Siegel modular variety) is trivialized over $\mathcal{X}_{K^p}$. Moreover, under the trivialization, the p-adic realizations of Hodge tensors $s_{\alpha,p} \in \mathcal{V}_p^{\otimes}$ are identified with the $s_\alpha \in V^{\otimes}$. This can be rephrased as saying that the G-torsor of trivializations of $(\mathcal{V}_p, s_{\alpha,p})$ has a section over $\mathcal{X}_{K^p}$, which can be thought of as an object in the flattened pro-étale site of $\mathcal{X}_{K^pK_p}$.

The relative Hodge–Tate filtration gives rise to the Hodge–Tate period morphism; in order to show that this morphism factors through the appropriate flag variety $\mathscr{F}\ell_{G,\mu}$, it is enough to show that the G-torsor described above has a P_μ-structure. This amounts to showing that the p-adic realizations of Hodge tensors respect the Hodge–Tate filtration. The same argument automatically proves that the resulting morphism is independent of the choice of embedding $(G, X) \hookrightarrow (\widetilde{G}, \widetilde{X})$.

The latter statement can be seen as a consequence of the fact that the de Rham realizations of Hodge tensors respect the Hodge de Rham filtration, of the relationship between the Hodge–de Rham and Hodge–Tate filtrations described in Section 3, and of the fact that the de Rham and p-adic realizations of Hodge tensors are matched by the p-adic-de Rham comparison isomorphism. The latter result is known for abelian varieties defined over number fields and is due to Blasius [Bla94]. □

Remark 5.3.4. For Shimura varieties of PEL type, the construction of the map π_{HT} in Theorem 5.3.3 is simpler, as one can keep track of the extra endomorphisms in the moduli problem and cut down to the desired flag variety directly.

Example 5.3.5. Let $F/\mathbb{Q}$ be an imaginary quadratic field. We set $n = 2$ and we consider the corresponding quasi-split unitary similitude group $G/\mathbb{Q}$ as in Example 2.3.6. This has signature $(2, 2)$ at inifinity. The corresponding Shimura variety is of PEL type. Assume that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in F. Let K be a complete nonarchimedean field which is an extension of $\mathbb{Q}_p^{\mathrm{cycl}}$ and $K^+ \subset K$ an open and bounded valuation subring. For any abelian variety A/K parametrized by the Shimura variety for G, we can write its p-divisible group as a direct product

$$A[p^\infty] = A[\mathfrak{p}^\infty] \times A[\bar{\mathfrak{p}}^\infty].$$

The compatibility between the action of F on A by quasi-isogenies and the polarization λ means that conjugation in F is induced by the Rosati involution corresponding to λ. Therefore, $A[\bar{\mathfrak{p}}^\infty]$ is determined by $A[\mathfrak{p}^\infty]$. We understand the latter via the Hodge–Tate period morphism. The target $\mathscr{F}\ell_{G,\mu}$ of this morphism can be identified with the Grassmannian of 2-dimensional subspaces of a 4-dimensional vector space. This space can be described via the Plücker embedding into $\mathbb{P}^5$.

6. The cohomology of locally symmetric spaces: conjectures and results

In this section, we discuss some recent conjectures and results about the cohomology of locally symmetric spaces, including the case of Shimura varieties. This is an active area of research and there are many perspectives on it (coming from number theory, harmonic analysis, algebraic topology, representation theory). We will restrict ourselves to discussing the following two topics: the construction of Galois representations attached to systems of Hecke eigenvalues in the cohomology of locally symmetric spaces for GL_n/F, where F is a CM field, and vanishing conjectures and theorems for the completed cohomology of locally symmetric spaces.

In § 6.1 we state Scholze's main theorem on the existence of Galois representations and we give a brief sketch of the proof, which uses perfectoid Shimura varieties and the Hodge–Tate period morphism. In § 6.2, we define completed cohomology, state a conjecture of Calegari–Emerton, and discuss some recent results towards this in the case of Shimura varieties.

6.1. The construction of Galois representations.

Let F be a CM field. For simplicity, assume that F is imaginary CM. We consider the symmetric space for GL_n/F (in other words, the symmetric space for the group $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_n$) and we let $K \subset \mathrm{GL}_n(\mathbb{A}_{F,f})$ be a neat compact open subgroup. The corresponding locally symmetric space X_K is a smooth orientable Riemannian manifold which does not admit the structure of an algebraic variety.

Let S' be the finite set of primes of F consisting of those primes above any ramified prime of $\mathbb{Q}$ and of those primes v where $K_v \subset \mathrm{GL}_n(F_v)$ is not hyperspecial.[28] Choose a prime p for the coefficients that we will use throughout. Let $S = S' \cup \{v \text{ prime of } F, v \mid p\}$. If $v \notin S$, let

$$\mathbb{T}_v := \mathbb{Z}_p[\mathrm{GL}_n(\mathcal{O}_{F,v})\backslash \mathrm{GL}_n(F_v)/\ \mathrm{GL}_n(\mathcal{O}_{F,v})]$$

be the Hecke algebra of bi-$\mathrm{GL}_n(\mathcal{O}_{F,v})$-invariant, compactly supported, $\mathbb{Z}_p$-valued functions on $\mathrm{GL}_n(\mathcal{O}_{F,v})$. (Recall that this is an algebra under the convolution of

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[28] Recall that a group scheme is reductive if it is smooth and affine, with connected reductive geometric fibers. If G/F_v is a reductive group, a hyperspecial subgroup of $G(F_v)$ is a subgroup that can be identified with the $\mathcal{O}_{F,v}$-points of some reductive model $\mathcal{G}$ of G over Spec $\mathcal{O}_{F,v}$. Such subgroups of $G(F_v)$ are maximal as compact open subgroups of $G(F_v)$.

functions and that it is commutative.) For $v \notin S$ a prime of F and $1 \leq i \leq n$, we consider the double coset operator

$$T_{v,i} := [\mathrm{GL}_n(\mathcal{O}_{F,v})\mathrm{diag}(\varpi_v, \ldots, \varpi_v, 1, \ldots, 1)\,\mathrm{GL}_n(\mathcal{O}_{F,v})]$$

with i occurrences of the uniformizer ϖ_v on the diagonal. Let $\mathbb{T}^S$ be the abstract Hecke algebra over $\mathbb{Z}_p$

$$\mathbb{T}^S := \otimes'_{v \notin S}\mathbb{T}_v,$$

which acts by correspondences on X_K and therefore also on $H^i_{(c)}(X_K, \mathbb{Z}/p^N\mathbb{Z})$.

We will be interested in systems of Hecke eigenvalues occurring in $H^i_{\mathrm{Betti}}(X_K, \mathbb{Z}/p^N\mathbb{Z})$ for some $N \in \mathbb{Z}_{\geq 1}$. Let

$$\mathbb{T}(K, i, N) := \mathrm{Im}\left(\mathbb{T}^S \to \mathrm{End}(H^i_{\mathrm{Betti}}(X_K, \mathbb{Z}/p^N\mathbb{Z}))\right).$$

The goal will be to construct a Galois representation valued in $\mathbb{T}(K, i, n)$; we will not quite do this, but something that is close enough (at least for applications to modularity). The following is Corollary 5.4.3 of [Sch15].

Theorem 6.1.1. *Let $\mathfrak{m} \subset \mathbb{T}(K, i, N)$ be a maximal ideal. Then there exists a unique continuous semisimple Galois representation*

$$\bar{\rho}_{\mathfrak{m}} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$$

such that for all $v \notin S$ $\bar{\rho}_{\mathfrak{m}}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ is unramified and the characteristic polynomial of $\bar{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ is equal to the image of

$$X^n - T_{v,1}X^{n-1} + \cdots + (-1)^i q_v^{i(i-1)/2}T_{v,i}X^{n-i} + \cdots + (-1)^n q_v^{n(n-1)/2}T_{v,n}.$$

in $\mathbb{T}(K, i, N)/\mathfrak{m}$.

One can show that the Galois representation $\bar{\rho}_{\mathfrak{m}}$ is actually valued in $\mathrm{GL}_n(\mathbb{F}_q)$, where $\mathbb{F}_q$ is the finite residue field of $\mathfrak{m}$. We say that $\mathfrak{m}$ is *non-Eisenstein* if $\bar{\rho}_{\mathfrak{m}}$ is (absolutely) irreducible. The following is Corollary 5.4.4 of [Sch15].

Theorem 6.1.2. *Assume that $\mathfrak{m}$ is non-Eisenstein.*

Then there exists a nilpotent ideal $I \subset \mathbb{T}(K, i, N)$, of nilpotence degree bounded only in terms of $[F : \mathbb{Q}]$ and n, and a Galois representation

$$\rho_{\mathfrak{m}} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{T}(K, i, N)_{\mathfrak{m}}/I)$$

such that for all $v \notin S$ a prime of F $\rho_{\mathfrak{m}}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ is unramified and the characteristic polynomial of $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ is equal to the image of

$$X^n - T_{v,1}X^{n-1} + \cdots + (-1)^i q_v^{i(i-1)/2}T_{v,i}X^{n-i} + \cdots + (-1)^n q_v^{n(n-1)/2}T_{v,n}.$$

in $\mathbb{T}(K, I, N)_{\mathfrak{m}}/I$.

Remark 6.1.3. (1) The Galois representations $\bar{\rho}_{\mathfrak{m}}$ and $\rho_{\mathfrak{m}}$ are constructed by *p-adic interpolation* (in other words by keeping track of congruences modulo p^N for $N \in \mathbb{Z}_{\geq 1}$) from the Galois representations associated to (conjugate) self-dual, regular L-algebraic automorphic representations of GL_m/F for some $\mathfrak{m} \in \mathbb{Z}_{\geq 1}$.

These Galois representations were constructed in several steps by many people: Kottwitz, Clozel, Harris-Taylor, Shin, Chenevier-Harris [Clo91, Kot92a, HT01, Shi11, CH09], building on fundamental contributions by many others. In almost all cases, one uses a similar method to the one outlined in the introduction in the case of weight 2 modular forms, i.e. one uses the étale cohomology of certain *Shimura varieties*, which are higher-dimensional analogues of modular curves. However, we emphasize that the Galois representations in Theorem 6.1.2 are not "cut out" directly from the étale cohomology of some Shimura variety.

(2) Scholze's result recovers a theorem of Harris–Lan–Taylor–Thorne in characteristic 0, namely the existence of Galois representations for regular L-algebraic automorphic representations of GL_n/F, cf. [HLTT16]. It is possible to give a different proof of Theorem 6.1.2 (even in the torsion setting) as a result of Boxer's thesis [Box15], which uses integral models rather than perfectoid Shimura varieties and understands torsion in the *coherent* cohomology of Shimura varieties.

(3) Newton and Thorne [NT16] improve the bound on the nilpotence degree of I to $I^4 = 0$. In certain instances, it is possible to eliminate the nilpotent ideal completely.

(4) We have stated the main result for the trivial local system on X_K for simplicity. The analogous result also holds with coefficients in a local system on X_K corresponding to some irreducible algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_n$.

Let $\widetilde{G}$ be the unitary group defined in Example 2.3.11.

Exercise 6.1.4. *Show that* $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_n$ *is the Levi subgroup of a maximal parabolic subgroup of* $\widetilde{G}$. *(This is the so-called* Siegel parabolic *of* $\widetilde{G}$ *and, up to conjugacy over* $\mathbb{Q}$, *coincides with the parabolic subgroup* P_μ *defined above.)*

Let $\widetilde{K} \subset \widetilde{G}(\mathbb{A}_f)$. This determines a Shimura variety of Hodge type $\widetilde{X}_{\widetilde{K}}$. There are two key steps for the proof of Theorem 6.1.2.

The first step is to show that any mod p^N system of Hecke eigenvalues in the étale cohomology of $\widetilde{X}_{\widetilde{K}}$ lifts to characteristic 0, to a system of Hecke eigenvalues associated to a cusp form on $\widetilde{G}$. This is proved in [Sch15, Thm. 4.3.1] for any Shimura variety of Hodge type. This can be seen by studying the geometry of the Hodge–Tate period morphism

$$\pi_{\mathrm{HT}} : \widetilde{\mathcal{X}}^*_{\widetilde{K}^p} \to \mathscr{F}\!\ell_{\widetilde{G},\mu}{}^{29}.$$

We briefly mention the key ideas involved in the first step, i.e. in the proof of [Sch15, Thm. 4.3.1]. See *loc. cit.* for more details, as well as the survey [Mor16].

(1) First, Scholze applies a version of the primitive comparison isomorphism for rigid analytic varieties proved in [Sch13]. This reduces [Sch15, Thm. 4.3.1] to constructing Hecke congruences between $H^i_{\text{ét}}(\widetilde{\mathcal{X}}^*_{\widetilde{K}^p}, \mathcal{I}^+/p^N)$ and cusp forms for $\widetilde{G}$, where $\mathcal{I}^+ \subset \mathcal{O}^+_{\widetilde{\mathcal{X}}^*_{\widetilde{K}^p}}$ is the subsheaf of sections which vanish along the boundary.

(2) An important property of the morphism π_{HT} is that it is "affinoid" in the following sense: there exists a cover by affinoid open subsets

$$\mathscr{F}\!\ell_{\widetilde{G},\mu} = \bigcup_{i=1}^{M} U_i$$

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[29] In fact, it is enough to consider a bigger period domain, corresponding to a Siegel embedding $\widetilde{G} \hookrightarrow \mathrm{GSp}_{2n[F^+:\mathbb{Q}]}$.

(defined via an appropriate Plücker embedding) such that each preimage $V_i := \pi_{\mathrm{HT}}^{-1}(U_i)$ is affinoid perfectoid. Moreover, since π_{HT} is Hecke equivariant, the open subsets $V_i \subset \widetilde{\mathcal{X}}^*_{\widetilde{K}^p}$ are stable under the action of Hecke operators away form p. For affinoid perfectoid spaces, the étale cohomology $H^i_{\text{ét}}(V_i, \mathcal{O}^+_{V_i})$ is almost zero for $i > 0$, cf. [Sch13, Lemma 4.10]; a version of this result also holds for sections that vanish along the boundary of $V_i \cap \widetilde{\mathcal{X}}^*_{\widetilde{K}^p}$. Using the open cover $(V_i)_{i=1}^M$ and the above observation, Scholze reduces [Sch15, Thm. 4.3.1] to constructing mod p^N Hecke congruences between the Cech cohomology of the cover $(V_i)_{i=1}^M$ and cusp forms for $\widetilde{G}$.

(3) The key ingredients for constructing these congruences are certain "fake Hasse invariants" pulled back from sections defined over the open cover of $\mathscr{F}\ell_{\widetilde{G},\mu}$. This is inspired by the way the classical Hasse invariant is used to construct Hecke congruences in the setting of modular forms; this method is used in [DS74] to construct Galois representations attached to cusp forms of weight 1 by constructing congruences to cusp forms of higher weight.

The second step is to relate the cohomology of X_K with $\mathbb{Z}/p^N\mathbb{Z}$-coefficients to the cohomology of $\widetilde{X}_{\widetilde{K}}$ with $\mathbb{Z}/p^N\mathbb{Z}$-coefficients, for some neat compact open subgroup $\widetilde{K} \subset \widetilde{G}(\mathbb{A}_f)$. This is done by studying the geometry of the Borel–Serre compactification of $\widetilde{X}_{\widetilde{K}}$, which we now consider simply as a locally symmetric space for $\widetilde{G}$. The rough idea is that the strata in the boundary of the Borel–Serre compactification are generalizations of locally symmetric spaces, more precisely locally symmetric spaces attached to the proper parabolic subgroups of $\widetilde{G}$. In the case of the Siegel parabolic P_μ, the corresponding locally symmetric spaces are torus bundles over the locally symmetric spaces for $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_n$. Since this part of the argument does not involve p-adic geometry, we do not say more about it here; for more details about this step, see [Sch15, §5] and [NT16].

6.2. **Vanishing of cohomology at infinite level.** Completed cohomology, as introduced by Emerton in [Eme06], gives a way of defining p-adic automorphic forms for general reductive groups. Let $G/\mathbb{Q}$ be a connected reductive group with the corresponding tower of locally symmetric spaces $(X_K)_K$. Fix a *tame level*, i.e. a sufficiently small compact open subgroup $K^p \subset G(\mathbb{A}_f^p)$. The *completed cohomology* groups are defined as

$$\widetilde{H}^i(K^p) := \varprojlim_N \left(\varinjlim_{K_p} \left(H^i(X_{K^pK_p}, \mathbb{Z}/p^N\mathbb{Z}) \right) \right)$$

where K_p runs over all compact open subgroups of $G(\mathbb{Q}_p)$. For $N \in \mathbb{Z}_{\geq 1}$ we also define

$$\widetilde{H}^i(K^p, \mathbb{Z}/p^N\mathbb{Z}) := \varinjlim_{K_p} \left(H^i(X_{K^pK_p}, \mathbb{Z}/p^N\mathbb{Z}) \right).$$

The group $\widetilde{H}^i(K^p)$ is a p-adically complete $\mathbb{Z}_p$-module. If S' is the finite set of bad primes determined by the tame level K^p and $S = S' \cup \{p\}$, then $\widetilde{H}^i(K^p)$ has an action of the abstract Hecke algebra $\mathbb{T}^S$. Moreover, $\widetilde{H}^i(K^p)$ also has an action of the full group $G(\mathbb{Q}_p)$. This is induced from the action of c_g^* for $g \in G(\mathbb{Q}_p)$ on the directed system $\left(H^i((X_{K^pK_p}, \mathbb{Z}/p^N\mathbb{Z})\right)_{K_p \subset G(\mathbb{Q}_p)}$, sending a class at level K_p to a class at level $K_p \cap gK_pg^{-1}$. As a representation of $G(\mathbb{Q}_p)$, one can prove that $\widetilde{H}^i(K^p)$ is *p-adically admissible*, which means that

(1) it is p-adically complete and separated, and the $\mathbb{Z}_p$-torsion subspace $\widetilde{H}^i(K^p)[p^\infty]$ is of bounded exponent, which means that $\widetilde{H}^i(K^p)[p^\infty] = \widetilde{H}^i(K^p)[p^M]$ for all sufficiently large integers M.
(2) each $\widetilde{H}^i(K^p)/p^N$, which is a smooth representation of $G(\mathbb{Q}_p)$, is also admissible as a representation of $G(\mathbb{Q}_p)$ (in the usual sense).

Recall that a *smooth* representation of $G(\mathbb{Q}_p)$ is one in which every vector has an open stabilizer in $G(\mathbb{Q}_p)$. It is not hard to show that $\widetilde{H}^i(K^p, \mathbb{Z}/p^N\mathbb{Z})$ are smooth representations of $G(\mathbb{Q}_p)$ for every $N \geq 1$. However, completed cohomology with $\mathbb{Z}_p$-coefficients is *not* a smooth representation of $G(\mathbb{Q}_p)$ - the smooth vectors in completed cohomology correspond to certain classical automorphic forms, which form a much smaller space than the space of all p-adic automorphic forms.

Exercise 6.2.1. *Using the fact that* $\widetilde{H}^i(K^p)$ *is p-adically admissible, show that the inverse system* $\left(\widetilde{H}^i(K^p, \mathbb{Z}/p^N\mathbb{Z})\right)_N$ *satisfies Mittag-Leffler.*

Remark 6.2.2. (1) One can also make the definition for compactly-supported cohomology as well as for homology and Borel-Moore homology. See [CE12, Eme14] for more details on these and the relationships between them. See [Eme14] also for an overview of the role that completed cohomology plays in the p-adic Langlands program, in terms of both local and global aspects.

(2) We can identify the completed cohomology of tame level K^p with the cohomology of the perfectoid Shimura variety of tame level K^p. More precisely, there exists a natural, Hecke-equivariant comparison isomorphism

$$\widetilde{H}^i_{(c)}(K^p, \mathbb{Z}/p^N\mathbb{Z}) \xrightarrow{\sim} H^i_{\text{ét},(c)}(\mathcal{X}_{K^p}, \mathbb{Z}/p^N\mathbb{Z}).$$

(3) One can also make the following definition:

$$\widehat{H}^i(K^p) := \varprojlim_N \left(\varinjlim_{K_p} \left(H^i(X_{K^pK_p}, \mathbb{Z}_p)\right)/p^N \right),$$

which also has an action of the abstract Hecke algebra $\mathbb{T}^S$. Intuitively, the systems of Hecke eigenvalues (i.e. the maximal ideals of $\mathbb{T}^S$) in the support of $\widehat{H}^i(K^p)$ are those which can be p-adically interpolated from systems of Hecke eigenvalues in the support of $H^i(X_{K^pK_p}, \mathbb{Z}_p)$ for some finite level K_p, i.e. systems of Hecke eigenvalues corresponding to classical automorphic forms. The difference between $\widetilde{H}^i(K^p)$ and $\widehat{H}^i(K^p)$ can be expressed as a limit over *torsion classes* occurring in the cohomology of locally symmetric spaces at finite level, as seen in Exercise 6.2.3 below.

Exercise 6.2.3. *Consider the short exact sequence of sheaves*

$$0 \to \mathbb{Z}_p \xrightarrow{\cdot p^N} \mathbb{Z}_p \to \mathbb{Z}/p^N\mathbb{Z} \to 0$$

on $X_{K^pK_p}$ *for every* K_p*. By analyzing the cohomology long exact sequence, prove that we have an injection*

$$\widehat{H}^i(K^p) \hookrightarrow \widetilde{H}^i(K^p)$$

and describe its cokernel in terms of torsion classes, i.e. in terms of the groups $H^i(X_{K^pK_p}, \mathbb{Z}_p)[p^N]$.

Remark 6.2.4. In particular, if the groups $H^i(X_{K^pK_p}, \mathbb{Z}_p)[p^N]$ are zero for all $N \in \mathbb{Z}_{\geq 1}$ and all compact-open K_p, then we have an isomorphism $\widehat{H}^i(K^p) \xrightarrow{\sim} \widetilde{H}^i(K^p)$. This happens, for example, if G is a definite unitary group, so that the locally symmetric spaces are finite sets of points. This also happens in the case of modular curves. However, we will be primarily concerned with a general $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_n$, in which case the groups $H^i(X_{K^pK_p}, \mathbb{Z}_p)$ are known to contain torsion. For example, Bergeron–Venkatesh conjecture in [BV13] that the size of the torsion subgroup in Betti cohomology grows exponentially with the index of K when $l_0 = 1$.

Here are some further important properties of completed cohomology:

(1) The Hochschild–Serre spectral sequence can be used to recover cohomology at finite level from completed cohomology. More precisely, if $K_p \subset G(\mathbb{Q}_p)$ is a compact-open subgroup, then we have spectral sequences

$$E_2^{i,j} = H^i(K_p, \widetilde{H}^j(K^p, \mathbb{Z}/p^N\mathbb{Z})) \Rightarrow H^{i+j}(X_{K^pK_p}, \mathbb{Z}/p^N\mathbb{Z})$$

and

$$E_2^{i,j} = H^i(K_p, \widetilde{H}^j(K^p)) \Rightarrow H^{i+j}(X_{K^pK_p}, \mathbb{Z}_p),$$

where $H^i(K_p, \ )$ denotes the continuous group cohomology of K_p. See [Eme06, Prop. 2.1.11] for a proof, which is slightly subtle with $\mathbb{Z}_p$-coefficients.[30]

(2) One can work with cohomology at finite level with coefficients in a local system $\mathcal{V}_\xi$ corresponding to some algebraic representation ξ of G and the completed cohomology groups one obtains match up. More precisely, assume that ξ is an algebraic representation of G defined over $\mathbb{Q}_p$ (for simplicity, otherwise we would introduce a field of coefficients E which is a finite extension of $\mathbb{Q}_p$). Let $V_\xi^\circ \subset V_\xi$ be a $\mathbb{Z}_p$-lattice stable under the action of $\mathcal{G}(\mathbb{Z}_p)$. The local system $\mathcal{V}_\xi^\circ$ is defined as follows:

$$\mathcal{V}_\xi^\circ := G(\mathbb{Q}) \setminus \left(X \times G(\mathbb{A}_f)/K \times V_\xi^\circ\right).$$

The completed cohomology groups corresponding to the local system $\mathcal{V}_\xi^\circ$ are defined as

$$\widetilde{H}^i(K^p, \mathcal{V}_\xi^\circ) := \varprojlim_N \left(\varinjlim_{K_p} \left(H^i(X_{K^pK_p}, \mathcal{V}_\xi^\circ/p^N) \right) \right),$$

where K_p runs over compact-open subgroups of $\mathcal{G}(\mathbb{Z}_p)$. Then we have a natural, Hecke-equivariant isomorphism of p-adically admissible representations of $\mathcal{G}(\mathbb{Z}_p)$

$$\widetilde{H}^i(K^p, \mathcal{V}_\xi^\circ) \xrightarrow{\sim} V_\xi^\circ \otimes_{\mathbb{Z}_p} \widetilde{H}^i(K^p).$$

(3) Let $(V_\xi^\circ)^\vee$ denote the $\mathbb{Z}_p$-dual of V_ξ°, endowed with the contragredient action of $\mathcal{G}(\mathbb{Z}_p)$. Let $K_p \subset \mathcal{G}(\mathbb{Z}_p)$ be a compact-open subgroup. By combining

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[30] Since K_p is a compact locally $\mathbb{Q}_p$-analytic group, the category of p-adically admissible representations of K_p over $\mathbb{Z}_p$ has enough injectives. Therefore, the continuous cohomology groups $H^i(K_p, \ )$ can be identified with the derived functors of the functor "taking K_p-invariants" on the category of p-adically admissible K_p-representations.

the first two items, one obtains a *control theorem* for completed cohomology in the form of a spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathbb{Z}_p[\![K_p]\!]}\left((V_\xi^\circ)^\vee, \widetilde{H}^i(K^p)\right) \Longrightarrow H^{i+j}(X_{K^pK_p}, \mathcal{V}_\xi^\circ).$$

Motivated by heuristics coming from the Langlands program, Calegari and Emerton made several conjectures about the usual and compactly-supported completed cohomology of general locally symmetric spaces: see [CE12, Conj. 1.5] (which is, however, stated in terms of homology and Borel–Moore homology). In the case of tori, their conjectures are equivalent to Leopoldt's conjecture, cf. [Hil10].

A consequence of the Calegari–Emerton conjectures is the following. Let l_0 be the invariant defined in § 2.1 and set $q_0 := \frac{1}{2}(\dim_{\mathbb{R}} X - l_0)$. Then for every sufficiently small $K^p \subset G(\mathbb{A}_f^p)$ we expect that $\widetilde{H}^i_{(c)}(K^p) = 0$ for $i > q_0 + l_0$. Even this weaker conjecture is wide open for general locally symmetric spaces. Some small-dimensional examples can be studied by hand, such as arithmetic hyperbolic 3-manifolds. In the case of Shimura varieties, where $l_0 = 0$ and q_0 is equal to the dimension of the Shimura variety, Scholze recently proved the following result, cf. [Sch15, Thm. 4.2.2].

Theorem 6.2.5. *Let (G, X) be a Shimura datum of Hodge type. Let $K^p \subset G(\mathbb{A}_f^p)$ be a sufficiently small compact open subgroup and let $N \in \mathbb{Z}_{\geq 1}$. We have*

$$\widetilde{H}^i_c(K^p, \mathbb{Z}/p^N\mathbb{Z}) = 0$$

for all $i > d = \dim_{\mathbb{C}} X$.

Sketch of proof of Theorem 6.2.5. The primitive comparison theorem of [Sch13] (applied at finite level, then taking a direct limit) gives an almost isomorphism between $\widetilde{H}^i(K^p, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p$ and $H^i_{\text{ét}}(\mathcal{X}^*_{K^p}, \mathcal{I}^+/p)$, where $\mathcal{I}^+ \subseteq \mathcal{O}^+_{\mathcal{X}^*_{K^p}}$ is the subsheaf sections which vanish along the boundary. For any affinoid perfectoid cover of $\mathcal{X}^*_{K^p}$, the étale cohomology of the restriction of the sheaf $\mathcal{I}^+$ is almost equal to 0 in degree $i > 0$. This allows us to replace $H^i_{\text{ét}}(\mathcal{X}^*_{K^p}, \mathcal{I}^+/p)$ with $H^i_{\text{an}}(\mathcal{X}^*_{K^p}, \mathcal{I}^+/p)$. The vanishing theorem now follows from a theorem of Scheiderer [Sch92], who shows that the cohomological dimension of a spectral space of Krull dimension d is at most d. □

Remark 6.2.6. The same idea can be used to prove the analogous vanishing theorem for the cohomology $\varinjlim_r H^i(X_r, \mathbb{F}_p)$, where $X \subset \mathbb{P}^N$ is a projective variety, and X_r is the pullback of X under the morphism

$$\mathbb{P}^N \to \mathbb{P}^N, (x_0 : x_1 : \ldots x_N) \mapsto (x_0^{p^r} : x_1^{p^r} : \ldots x_N^{p^r}).$$

A direct geometric proof of this vanishing theorem, which does not use p-adic geometry, was recently given by Esnault in [Esn18] and her result also holds for X of characteristic $\ell \neq p$. It would be interesting to see whether Theorem 6.2.5 or even Theorem 6.2.7 below admit more direct proofs.

We now explain a strenghtening of Theorem 6.2.5 in the particular case of Siegel modular varieties. Let (G, X) denote the Siegel datum in Example 2.3.4. Let $K^p \subset G(\mathbb{A}_f^p)$ be sufficiently small and for every $m \in \mathbb{Z}_{\geq 1}$ recall the compact open subgroups

$$\Gamma_1(p^m) := \left\{ g \in \mathrm{GSp}_{2n}(\mathbb{Z}_p) \mid g \equiv \left(\begin{smallmatrix} \mathrm{Id}_n & * \\ 0 & \mathrm{Id}_n \end{smallmatrix}\right) \pmod{p^m} \right\}.$$

Define $\widetilde{H}^i_{(c)}(K^p, \mathbb{Z}/p^N\mathbb{Z})_{\Gamma_1} := \varinjlim_m H^i_{(c)}(X_{K^p\Gamma_1(p^m)}, \mathbb{Z}/p^N\mathbb{Z})$.

Theorem 6.2.7. *We have* $\widetilde{H}^i_c(K^p, \mathbb{Z}/p^N\mathbb{Z})_{\Gamma_1} = 0$ *for all* $i > d = \dim_{\mathbb{C}} X$.

This is a part of [CGH+18, Thm. 1.1.2]; the result in *loc. cit.* is slightly more general, as it also considers certain Shimura varieties attached to quasi-split unitary groups as in Examples 2.3.6 and 2.3.11. In fact, we expect an even more general statement to be true, but we leave the question of extending Theorem 6.2.7 to other Shimura varieties to be explored in future work.

Sketch of proof of Theorem 6.2.7. Assume for simplicity that $n = 1$, in which case $G = \mathrm{GL}_2$ and we are working with modular curves. Set $N_0 := \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$; this is the group of $\mathbb{Z}_p$-points of the unipotent radical N of the standard (upper-triangular) Borel subgroup B of G. As in the proof of Theorem 6.2.5, we start by applying the primitive comparison theorem of [Sch13]. We then analyze the Leray spectral sequence for the morphism of sites

$$|\pi_{\mathrm{HT}/N_0}| : \left(\mathcal{X}^*_{\Gamma_1(p^\infty)}\right)_{\text{ét}} \to |\mathbb{P}^{1,\mathrm{ad}}|/N_0,$$

using it to compute the cohomology $H^i_{\text{ét}}(\mathcal{X}^*_{\Gamma_1(p^\infty)}, \mathcal{I}^+/p)$.[31] We consider the stratification of $\mathbb{P}^{1,\mathrm{ad}}$ into B-orbits: there is an open cell $\mathbb{A}^{1,\mathrm{ad}}$, containing $\pi_{\mathrm{HT}}(\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{anti}})$ as an open subset, and a closed cell, which consists of one point ∞, the image under π_{HT} of the canonical tower over the ordinary locus. Since $N_0 \subset B(\mathbb{Q}_p)$, this stratification descends to the quotient $|\mathbb{P}^{1,\mathrm{ad}}|/N_0$. We see the following two phenomena:

(1) The fibers of π_{HT/N_0} over rank 1 points of $|\mathbb{A}^{1,\mathrm{ad}}|/N_0$ are affinoid perfectoid spaces. Indeed, the action of $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p)$, expands the anticanonical locus in $\mathbb{P}^{1,\mathrm{ad}}$ to cover all of $\mathbb{A}^{1,\mathrm{ad}}$ and the anticanonical locus is affinoid perfectoid already at level $\Gamma_0(p^\infty)$; see also [Lud17] for a version of this argument.

(2) The fiber of π_{HT/N_0} over ∞/N_0 is not affinoid perfectoid, but it has a "$\mathbb{Z}_p$-cover"[32] which is affinoid perfectoid. Indeed, at full infinite level, the Weyl group element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p)$ takes the anticanonical locus in $\mathbb{P}^{1,\mathrm{ad}}$ to an open neighborhood of ∞. The cohomological dimension of this fiber is then bounded by the cohomological dimension of $\mathbb{Z}_p$ for continuous group cohomology.

While $R^0|\pi_{\mathrm{HT}/N_0}|_*(\mathcal{I}^+/p)$ is supported on the whole of $|\mathbb{P}^{1,\mathrm{ad}}|/N_0$, we deduce that $R^1|\pi_{\mathrm{HT}/N_0}|_*(\mathcal{I}^+/p)$ is almost zero outside a closed spectral subspace of $|\mathbb{P}^{1,\mathrm{ad}}|/N_0$ of dimension 0 and $R^2|\pi_{\mathrm{HT}/N_0}|_*(\mathcal{I}^+/p)$ is almost zero everywhere. We conclude by appealing to [Sch92] to compute the terms in the Leray spectral sequence.

The case of GSp_{2n} for arbitrary n is similar, using the generalized Bruhat stratification of the algebraic flag variety $\mathrm{Fl}_{G,\mu}$ into P_μ-orbits. Setting $N_0 := \cap_m \Gamma_1(p^m)$, we obtain a stratification

$$|\mathscr{F}\ell_{G,\mu}|/N_0 = \bigsqcup |\mathscr{F}\ell^w_{G,\mu}|/N_0$$

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[31] In order to make this rigorous, we need to consider $\mathcal{X}^*_{\Gamma_1(p^\infty)}$ as a diamond and use the étale cohomology of diamonds as developed in [Sch17].

[32] For the precise meaning of "$\mathbb{Z}_p$-cover", see [CGH+18, Thm. 2.2.7].

into locally closed strata, indexed by certain Weyl group elements w. Moreover, for every point $x \in |\mathscr{F}\ell^w_{G,\mu}|/N_0$, we show that

$$R^i \pi_{\mathrm{HT}/N_0,*}(\mathcal{I}^+/p)^a_x = 0 \text{ for } i > d - \dim \mathscr{F}\ell^w_{G,\mu}.$$

This, together with [Sch92], is enough to prove the desired vanishing for Siegel modular varieties. ☐

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